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SINGULAR INTEGRAL EQUATIONS – THE CONVERGENCE OF THE NYSTRÖM INTERPOLANT OF THE GAUSS-CHEBYSHEV METHOD

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Abstract.

Nyström's interpolation formula is applied to the numerical solution of singular integral equations. For the Gauss-Chebyshev method, it is shown that this approximation converges uniformly, provided that the kernel and the input functions possess a continuous derivative. Moreover, the error of the Nyström interpolant is bounded from above by the Gaussian quadrature errors and thus convergence is fast, especially for smooth functions. For C^∞ input functions, a sharp upper bound for the error is obtained. Finally numerical examples are considered. It is found that the actual computational error agrees well with the theoretical derived bounds.

1. Introduction.

In this paper we consider the convergence of the natural or Nyström interpolant of the direct Gauss-Chebyshev method, for the numerical solution of Cauchy-type singular integral equations of the form:

$$(1.1)\quad \pi^{-1}\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}(t-s)^{-1}y(t)dt+\lambda\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}K(s,t)y(t)dt = f(s), |s| < 1$$

$$(1.2)\quad \pi^{-1}\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}y(t)dt = N,$$

where $K(s, t)$, $f(s)$ are given input functions and N is a known constant. If we assume that $K(s, t)$ and $f(s)$ are Hölder-continuous functions in $[-1, 1]\times[-1, 1]$ and $[-1, 1]$ respectively, then it is well known [8] that (1.1), (1.2) possess a unique solution $y(t)$ in the space of Hölder-continuous functions.

Erdogan and Gupta [2] proposed a direct method for the solution of (1.1), (1.2), which is based on a quadrature approximation of the integrals. In particular, if the Gauss-Chebyshev quadrature formula is applied on a certain set of points, then (1.1), (1.2) can be reduced to an algebraic system of the form,

$$(1.3)\quad (A_n+\lambda C_n)y_n^* = f$$

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where

$$y_n^* = [y_n^*(t_1), \ldots, y_n^*(t_n)]^T, \qquad f = [f(s_1), \ldots, f(s_{n-1}), N]^T$$

(1.4) $$(A_n)_{i,j} = n^{-1}(t_j - s_i)^{-1}, \qquad (A_n)_{n,j} = n^{-1}$$

(1.5) $$(C_n)_{i,j} = \pi n^{-1} K(s_i, t_j), \qquad (C_n)_{n,j} = 0,$$

and $t_j = \cos[(2j-1)\pi/2n]$, $s_i = \cos[i\pi/n]$, for all $i = 1(1)n-1$, $j = 1(1)n$.

The solution of the linear algebraic system (1.3) is an approximation of the solution of (1.1), (1.2) at a discrete set of points t_j. Since the solution of (1.1) at points different from t_j often represents quantities of great interest in engineering, (e.g. $y(\pm 1)$ represents the stress intensity factor), an interpolation formula is required. Erdogan and Gupta [2] have suggested a quadratic "extrapolation" technique for the evaluation of $y(\pm 1)$. Similarly Krenk [7] has introduced a summation formula which is based on the Lagrange interpolating polynomials $L_n(t)$ at $(t_j, y_n^*(t_j))$. Ioakimidis and Theocaris [6] and Tsamasphyros and Theocaris [12], have considered the convergence of $L_n(t)$. We note that the Lagrange interpolation formula is exact for polynomials of degree $\leqq n$, whereas the Gaussian quadrature that approximates (1.1) and (1.2) is exact for polynomials of degree $\leqq 2n$ and $2n-1$ respectively. Consequently, the use of Lagrange polynomials will result in a significant loss of accuracy when compared to the use of an interpolation formula which is exact for polynomials of higher degree, say $2n$. Until recently ([11], [5]) the natural or Nyström interpolation formula has been completely ignored, although it is well known [1] that for Fredholm integral equations it can yield excellent results. Although some equivalence results for the Nyström interpolant of the direct and indirect Gauss-Chebyshev method have been given [5], the question of convergence and computational efficiency of the Nyström interpolant has not been studied.

In section 2 we introduce the Nyström interpolation formula and give a brief description of some equivalence and existence results of the discrete direct and indirect Gauss-Chebyshev method described in [4].

In section 3, we extend the analysis of [4], and use Nyström's theory [1], to show that, if λ is not an eigenvalue of (1.1), (1.2) then:

(i) The algebraic system (1.3) possesses a unique solution for sufficiently large n.
(ii) The Nyström interpolant converges uniformly to the solution of (1.1), (1.2), provided that $K(s, t) \in C^1([-1, 1] \times [-1, 1])$, $f(s) \in C^1[-1, 1]$.
(iii) The error of the Nyström interpolant is bounded above by Gaussian quadrature errors and thus it is a quickly converging interpolation formula, especially for smooth functions.

In section 4 we solve three integral equations which arise in the solution of elasticity problems, and compare the convergence of the "quadratic" extrapolation technique [2], Krenk's [7] summation formula, and the Lobatto-Chebyshev method [10], with the Nyström interpolant. For these examples, we observe that the Nyström interpolant converges at least as fast or even faster than all of the previously mentioned methods. Finally, for functions $f \in C^{\infty}[-1, 1]$, a sharp upper bound for the error is derived.

2. Regularized equations.

Using the Carleman-Vecua method of reduction [8], equations (1.1), (1.2) are reduced into an equivalent Fredholm Integral equation:

$$y(t)+\lambda\pi^{-1}\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}L(x,t)y(x)dx = F(t). \tag{2.1}$$

where

$$F(t) = -\pi^{-1}\int_{-1}^{1}(1-s^2)^{\frac{1}{2}}(s-t)^{-1}f(s)ds+N, \tag{2.2}$$

$$L(x,t) = -\int_{-1}^{1}(1-s^2)^{\frac{1}{2}}(s-t)^{-1}K(s,x)ds. \tag{2.3}$$

Let us assume that $L(x,t)\in C([-1,1]\times[-1,1])$, $F(t)\in C[-1,1]$ and approximate the integral part of (2.1) using the Gauss-Chebyshev quadrature. Then (2.1) is reduced to a functional equation of the form:

$$y_n(t)+\lambda n^{-1}\sum_{m=1}^{n}L(t_m,t)y_n(t_m) = F(t) \tag{2.4}$$

where $t_m = \cos[(2m-1)\pi/(2n)]$, $m = 1(1)n$. Furthermore, if we set $t = t_i$, $i = i(1)n$, in (2.4) we obtain the algebraic system

$$(I+\lambda Q_n)\boldsymbol{Z} = \boldsymbol{F} \tag{2.5}$$

where

$$(Q_n)_{i,j} = n^{-1}L(t_j,t_i), \qquad i,j = 1(1)n \tag{2.6}$$

and $\boldsymbol{Z} = [z_1,\ldots,z_n]^T$, $\quad \boldsymbol{F} = [F(t_1),\ldots,F(t_n)]^T$.

We can see that there exists a unique correspondence between the solution z_i of (2.5) and the solution $y_n(t_i)$ of (2.4). This implies that if (2.4) possesses a solution then (2.5) possesses a solution and vice versa (for more details see [1], p. 88).

For the remainder of this paper we will assume that $K(s,t)\in C^1([-1,1]\times$

$[-1, 1])$ and $f(s) \in C^1[-1, 1]$. If we define $g(x, t, s)$ and $h(t, s)$ by

(2.7) $$g(x, t, s) = \begin{cases} [K(s, x) - K(t, x)]/(s-t) & \text{if } s \neq t \\ \dfrac{\partial K(s, x)}{\partial s} & \text{if } s = t \end{cases}$$

(2.8) $$h(t, s) = \begin{cases} [f(s) - f(t)]/(s-t) & \text{if } s \neq t \\ f'(s) & \text{if } s = t, \end{cases}$$

then $g(x, t, s) \in C([-1, 1] \times [-1, 1] \times [-1, 1])$ and $h(t, s) \in C([-1, 1] \times [-1, 1])$.

We approximate $F(t)$ and $L(x, t)$ by

(2.9) $$F(t) = F_n(t) + r_n(f; t)$$

(2.10) $$L(x, t) = L_n(x, t) + r_n(K; x, t)$$

where

(2.11) $$F_n(t) = -n^{-1} \sum_{k=1}^{n-1} (1-s_k^2)(s_k - t)^{-1} f(s_k) + f(t) T_n(t)/U_{n-1}(t) + N$$

(2.12) $$L_n(x, t) = -n^{-1}\pi \sum_{k=1}^{n-1} (1-s_k^2)(s_k - t)^{-1} K(s_k, x) + \pi K(t, x) T_n(t)/U_{n-1}(t)$$

(2.13) $$r_n(f; t) = -\pi^{-1} \left[\int_{-1}^{1} (1-s^2)^{\frac{1}{2}} h(t, s) ds - \pi n^{-1} \sum_{k=1}^{n-1} (1-s_k^2) h(t, s_k) \right]$$

(2.14) $$r_n(K; x, t) = -\left[\int_{-1}^{1} (1-s^2)^{\frac{1}{2}} g(x, t, s) ds - n^{-1}\pi \sum_{k=1}^{n-1} (1-s_k^2) g(x, t, s_k) \right].$$

Here $T_n(t)$, $U_{n-1}(t)$ are the Chebyshev polynomials of the first and second kind respectively.

We replace $L(t_m, t)$ with $L_n(t_m, t)$, and $F(t)$ with $F_n(t)$ in (2.4) to obtain a new functional equation:

(2.15) $$y_n^*(t) + \lambda n^{-1} \sum_{m=1}^{n} L_n(t_m, t) y_n^*(t_m) = F_n(t),$$

which can be reduced to the following linear algebraic system

(2.16) $$(I + \lambda \tilde{Q}_n) y_n^* = F_n$$

where

(2.17) $$(\tilde{Q})_{i,j} = n^{-1} L_n(t_j, t_i) \qquad i, j = 1(1)n$$

and $F_n = [F_n(t_1), \ldots, F_n(t_n)]^T$.

We are now ready to state the following theorem:

THEOREM 2.1 *The algebraic systems* (2.16) *and* (1.3) *are equivalent.*

PROOF. It has been shown in [4], that $\tilde{Q}_n = A_n^{-1}C_n$ and $F_n = A_n^{-1}f$, where

$$(A_n^{-1})_{i,j} = n^{-1}(1-s_j^2)(t_i-s_j)^{-1}, \quad (A_n^{-1})_{i,n} = 1, \quad i = 1(1)n, \quad j = 1(1)n-1. \tag{2.18}$$

After solving the algebraic system (2.16) or its equivalent (1.3) we obtain $y_n^*(t_m)$, an approximation of the solution at the node points t_m. For other points than t_m, the Nyström interpolation formula (2.15) can be used directly. For points identical to the collocation points s_i, the following modification of (2.15) should be used:

$$y_n^*(s_i) = -\lambda n^{-1} \sum_{m=1}^{n} L_n(t_m, s_i) y_n^*(t_m) + F_n(s_i), \quad i = 1(1)n-1, \tag{2.19}$$

where

$$L_n(t_m, s_i) = -n^{-1}\pi \sum_{\substack{k=1\\k\neq i}}^{n-1} (1-s_k^2)(s_k-s_i)^{-1}[K(s_k, t_m) - K(s_i, t_m)] + \pi s_i K(s_i, t_m) - \pi n^{-1}(1-s_i^2)\frac{\partial K}{\partial s}(s_i, t_m) \tag{2.20}$$

$$F_n(s_i) = -n^{-1} \sum_{\substack{k=1\\k\neq i}}^{n-1} (1-s_k^2)(s_k-s_i)^{-1}[f(s_k)-f(s_i)] + s_i f(s_i) - n^{-1}(1-s_i^2)f'(s_i) + N. \tag{2.21}$$

3. The convergence of the Nyström interpolant.

Let us consider $C[-1, 1]$, the space of all continuous functions in $[-1, 1]$, which is complete with the maximum norm:

$$\|y\|_\infty = \max_{-1 \leqq x \leqq 1} |y(x)|. \tag{3.1}$$

We introduce the following linear operators on $C[-1, 1]$

$$\mathscr{L}y = \pi^{-1} \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} L(x, t) y(x) dx \tag{3.2}$$

$$\mathscr{L}_n y = n^{-1} \sum_{m=1}^{n} L(t_m, t) y(t_m) \tag{3.3}$$

$$\mathscr{L}_n^* y = n^{-1} \sum_{m=1}^{n} L_n(t_m, t) y(t_m), \tag{3.4}$$

and define the norm of a linear operator $\mathscr{T}$ by

$$\|\mathscr{T}\| = \sup\|\mathscr{T}y\|_\infty/\|y\|_\infty \text{ taken over } \|y\| \neq 0, \quad y \in [-1, 1]. \tag{3.5}$$

THEOREM 3.1. *If $f(s) \in C^1[-1, 1]$ and $K(s, t) \in C^1([-1, 1] \times [-1, 1])$, then $\{r_n(f\colon t)\}$ and $\{r_n(K\colon x, t)\}$, defined in* (2.13) *and* (2.14) *respectively, converge uniformly to zero.*

PROOF. From equation (2.14) and the continuity of $g(x, t, s)$ we obtain pointwise convergence of $\{r_n(K; x, t)\}$ (Szegö [9], p. 350). Moreover

$$|r_n(K; x, t)| \leqq (A_1 + C_1) \max_{-1 \leqq x,\, t,\, s \leqq 1} |g(x, t, s)| \tag{3.6}$$

$$|r_n(K; x_1, t_1) - r_n(K; x_2, t_2)| \leqq (A_1 + C_1) \max_{-1 \leqq s \leqq 1} |g(x_1, t_1, s) - g(x_2, t_2, s)|, \tag{3.7}$$

where

$$A_1 = \int_{-1}^{1} (1 - s^2)^{\frac{1}{2}} ds = \pi/2, \qquad C_1 = n^{-1} \sum_{k=1}^{n-1} (1 - s_k^2) = \pi/2. \tag{3.8}$$

The sequence $\{r_n(K; x, t)\}$ is a uniformly bounded equicontinuous family of functions due to the continuity of $g(x, t, s)$ and the inequalities (3.6)–(3.7). The theorem follows by invoking the Arzela-Ascoli lemma (see [1] p. 92 for a similar argument).

The proof for $\{r_n(f; t)\}$ is similar. ∎

THEOREM 3.2. $\|\mathscr{L}_n - \mathscr{L}_n^*\| \to 0$ *as* $n \to \infty$.

PROOF. We have

$$\|\mathscr{L}_n - \mathscr{L}_n^*\| \leqq \max_{-1 \leqq x, t \leqq 1} |L(x, t) - L_n(x, t)| = \max_{-1 \leqq x, t \leqq 1} |r_n(K; x, t)| \tag{3.9}$$

and the theorem follows immediately from theorem 3.1. ∎

THEOREM 3.3. *Under the assumptions of theorem* 3.1 *the sequence $\{\mathscr{L}_n y\}$ converges uniformly to $\mathscr{L}y$, i.e. $\|\mathscr{L}_n y - \mathscr{L}y\|_\infty \to 0$ as $n \to \infty$.*

PROOF. Since

$$\|\mathscr{L}y\|_\infty \leqq \|y\|_\infty \max_{-1 \leqq x, t \leqq 1} |L(x, t)| \tag{3.10}$$

$$|\mathscr{L}y(t_1) - \mathscr{L}y(t_2)| \leqq \|y\|_\infty \max_{-1 \leqq x \leqq 1} |L(x, t_1) - L(x, t_2)|, \tag{3.11}$$

the set $\{\mathscr{L}y \mid \|y\|_\infty \leq 1\}$ is a bounded equicontinuous family of functions on $[-1, 1]$, thus $\mathscr{L}$ is a compact operator from $C[-1, 1]$ to $C[-1, 1]$. Moreover we can easily show that inequalities (3.10), (3.11) hold if we replace $\mathscr{L}$ with $\mathscr{L}_n$. Thus the sequence $\{\mathscr{L}_n y\}$ is a uniformly bounded equicontinuous set of functions. Since $\{\mathscr{L}_n y\}$ converges pointwise to $\mathscr{L}y$ on $[-1, 1]$ (Szegö [9], p. 350), then the Arzela-Ascoli lemma implies that $\{\mathscr{L}_n y\}$ converges uniformly to $\mathscr{L}y$ ([1], p. 91). ∎

REMARK 3.1. The sequence $\{\mathscr{L}_n\}$ is a collectively compact family of operators.

REMARK 3.2. In general $\|\mathscr{L}_n - \mathscr{L}\| \nrightarrow 0$ as $n \to \infty$, in fact $\|\mathscr{L}\| \leq \|\mathscr{L}_n - \mathscr{L}\|$, (see [1], p. 91).

We rewrite equations (2.1), (2.4), (2.15) as follows:

$$(I + \lambda \mathscr{L})y = F \tag{3.12}$$

$$(I + \lambda \mathscr{L}_n)y_n = F \tag{3.13}$$

$$(I + \lambda \mathscr{L}_n^*)y_n^* = F_n \tag{3.14}$$

where I is the identity operator.

We will show that the Nyström interpolant

$$y_n^*(t) = -\lambda \mathscr{L}_n^* y_n^*(t) + F_n(t) \tag{3.15}$$

converges to the solution $y(t)$ of (3.12). To do so we need the following results:

THEOREM 3.4. *If λ is not an eigenvalue of* (3.12), *then $(I + \lambda \mathscr{L}_n)^{-1}$ exists for all $n \geq N(\lambda)$, and it is uniformly bounded by a constant B, i.e. $\|(I + \lambda \mathscr{L}_n)^{-1}\| \leq B$.*

PROOF. It follows directly from Theorem 3.3 (see [1], p. 98 and p. 105 for details).

COROLLARY 3.1. *Under the assumptions of Theorem* 3.4, *$(I + \lambda \mathscr{L}_n^*)^{-1}$ exists and it is uniformly bounded for all $n \geq n_0$.*

PROOF. The identity

$$\begin{aligned}(I + \lambda \mathscr{L}_n^*)^{-1} = (I + \lambda \mathscr{L}_n)^{-1} + [\lambda^{-1} I - (I + \lambda \mathscr{L}_n)^{-1}(\mathscr{L}_n - \mathscr{L}_n^*)]^{-1} \\ \times (I + \lambda \mathscr{L}_n)(\mathscr{L}_n - \mathscr{L}_n^*)(I + \lambda \mathscr{L}_n)^{-1}\end{aligned} \tag{3.16}$$

and Theorem 3.4 shows that $(I + \lambda \mathscr{L}_n^*)^{-1}$ exists whenever

$$[\lambda^{-1} I - (I + \lambda \mathscr{L}_n)^{-1}(\mathscr{L}_n - \mathscr{L}_n^*)]^{-1} \tag{3.17}$$

exists, which is true since Theorem 3.2 implies that

$$\|(I+\lambda \mathscr{L}_n)^{-1}(\mathscr{L}_n-\mathscr{L}_n^*)\| \leqq B\|\mathscr{L}_n-\mathscr{L}_n^*\| < |\lambda|^{-1}, \qquad \text{for } n \geqq n_0. \tag{3.18}$$

The uniform boundedness is obvious.

REMARK 3.3. The last corollary and results given in [1], p. 105 imply that $(I+\lambda\tilde{Q}_n)^{-1}$ exists. Moreover, we observe that the identity $(A_n+\lambda C_n)^{-1} = (I+\lambda A_n^{-1}C_n)^{-1}A_n^{-1} = (I+\lambda\tilde{Q}_n)^{-1}A_n^{-1}$, and the existence of A_n^{-1} (shown in [4]), imply the existence of $(A_n+\lambda C_n)^{-1}$ for sufficiently large n.

We now present the main result of this section.

THEOREM 3.5. *If* $K(s,t)\in C^1([-1,1]\times[-1,1])$, $f(s)\in C^1[-1,1]$ *and* λ *is not an eigenvalue of* (1.1), *then the Nyström interpolant* $y_n^*(t)$ *defined in* (2.15) *converges uniformly to the unique solution* $y(t)$ *of* (1.1), (1.2).

PROOF. For uniform convergence we need to show $\|y_n^*-y\|_\infty \to 0$ as $n\to\infty$. Clearly

$$(I+\lambda\mathscr{L}_n)(y_n-y_n^*) = \lambda(\mathscr{L}_n^*-\mathscr{L}_n)y_n^*+F-F_n \tag{3.19}$$

implies that

$$\|y_n-y_n^*\|_\infty \leqq \|(I+\lambda\mathscr{L}_n)^{-1}\|(|\lambda|\,\|\mathscr{L}_n^*-\mathscr{L}_n\|\,\|y_n^*\|_\infty+\|F-F_n\|_\infty). \tag{3.20}$$

Similarly, we can show that

$$\|y-y_n\|_\infty \leqq \|(I+\lambda\mathscr{L}_n)^{-1}\|\,|\lambda|\,\|(\mathscr{L}-\mathscr{L}_n)y\|_\infty. \tag{3.21}$$

The theorem follows from the inequality

$$\|y-y_n^*\|_\infty \leqq \|y-y_n\|_\infty+\|y_n-y_n^*\|_\infty, \tag{3.22}$$

and theorems 3.1, 3.2, 3.3. ■

Combining (3.20)–(3.22) we have:

(3.23)
$$\|y-y_n^*\|_\infty \leqq \|(I+\lambda\mathscr{L}_n)^{-1}\|\{|\lambda|\,\|(\mathscr{L}-\mathscr{L}_n)y\|_\infty+|\lambda|\,\|\mathscr{L}_n^*-\mathscr{L}_n\|\;\|y_n^*\|_\infty+\|F-F_n\|_\infty\}.$$

This inequality gives us an upper bound for the error of $y_n^*(t)$. Moreover, we can see that the error bound depends on the quadrature error incurred when we approximate $\mathscr{L}y$ with $\mathscr{L}_n y$, $\mathscr{L}_n$ with $\mathscr{L}_n^*$, F with F_n. Since Gaussian-type quadratures have been used for the previously mentioned approximations, (3.23) implies that $\|y-y_n^*\|_\infty$ will converge to zero very fast, expecially if the kernel and the input functions are smooth. We have successfully verified this observation on several numerical examples, three of which are described in the next section.

4. Numerical examples and error bounds.

(i) We first consider the singular integral equation of the form:

$$\pi^{-1}\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}(t-s)^{-1}y(t)dt = f(s), \quad -1 \leqq s \leqq 1 \tag{4.1}$$

$$\pi^{-1}\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}y(t)dt = 0, \tag{4.2}$$

which arises in the stress analysis of a plane crack opened by the load distribution $f(s)$, in an isotropic medium [11]. The solution $y(\pm 1)$ of (4.1), (4.2) represents the stress intensity factor at the tips of the crack. If we assume that $f(s) \in C^\infty[-1, 1]$ then we can obtain an error bound for the Nyström interpolant by using (2.13), (3.23), and the quadrature error formula ((2.12.6.6), p. 75, [3]). Thus we have

(4.3)
$$\|y-y_n^*\|_\infty \leqq \|F-F_n\|_\infty \leqq \max_{-1\leqq t\leqq 1} |\partial^{2n-2}h(t,s)/\partial s^{2n-2}|/[(2n-2)!2^{2n-1}].$$

Using the Taylor series expansion of $h(t, s)$, it can easily be shown that

$$\max_{-1\leqq t\leqq 1} |\partial^{2n-2}h(t,s)/\partial s^{2n-2}| = \|f^{(2n-1)}\|_\infty/(2n-1). \tag{4.4}$$

Finally, by combining (4.3), (4.4)

$$\|y-y_n^*\|_\infty \leqq \|f^{(2n-1)}\|_\infty/[(2n-1)!2^{2n-1}] \tag{4.5}$$

which indicates that Nyström's interpolation formula will yield excellent results, especially for smooth functions.

The errors obtained by the numerical solution of equation (4.1), (4.2) with $f(s) = \cos s$ at the point $t = 1$, by Krenk's [7] and by Nyström's formulae are given in the first and second column of table 1. The maximum error bound (4.5) is listed in the third column. Clearly, the exact solution $y(1)$ is obtained by inverting (4.1), (4.2),

$$y(1) = \pi^{-1}\int_{-1}^{1}(1-s^2)^{-\frac{1}{2}}(1+s)\cos s ds = \pi^{-1}2\int_{0}^{1}(1-s^2)^{-\frac{1}{2}}\cos s\, ds \tag{4.6}$$

which is the Bessel function of order zero, i.e. $y(1) = J_0(1) = 0.7651\ldots$. We observe that for this example (and others not reported here) the theoretical error bound (4.5) agrees extremely well with the actual computational error of Nyström's formula and that the convergence is much faster than Krenk's formula.

Table 1. *The error in the numerical evaluation of the stress intensity factor* $y(1)$ *of eq.* (4.1), (4.2) *with* $f(s) = \cos s$.

n	Error of [7]	Error of (2.15)	Error bound (4.5)
3	$1 \cdot 10^{-1}$	$4 \cdot 10^{-5}$	$2.0 \cdot 10^{-4}$
4	$5 \cdot 10^{-3}$	$2 \cdot 10^{-7}$	$1.6 \cdot 10^{-6}$
5	$2 \cdot 10^{-3}$	$5 \cdot 10^{-10}$	$5.4 \cdot 10^{-9}$
6	$4 \cdot 10^{-5}$	$1 \cdot 10^{-11}$	$1.2 \cdot 10^{-11}$
7	$2 \cdot 10^{-5}$	$1 \cdot 10^{-15}$	$2.0 \cdot 10^{-14}$
8	$2 \cdot 10^{-7}$	$1 \cdot 10^{-18}$	$2.3 \cdot 10^{-17}$

(ii) To provide a further comparison of the different type of interpolation formulas and methods, we consider two integral equations arising in the solution of elasticity problems. The equation,

$$\pi^{-1} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}}(t-s)^{-1} y(t)dt - \lambda \int_{-1}^{s} (1-t^2)^{-\frac{1}{2}} y(t)dt = 1, \quad -1 < s < 1 \tag{4.7}$$

$$\pi^{-1} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} y(t)dt = 0, \tag{4.8}$$

arises in the solution of a cover plate bonded to an elastic half-space [2], and the equation

$$\pi^{-1} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}}(t-s)^{-1} y(t)dt$$

$$+\pi^{-1} \int_{-1}^{1} t(t^2-s^2)(t^2+s^2)^{-2}(1-t^2)^{-\frac{1}{2}} y(t)dt = 1, \quad -1 < s < 1 \tag{4.9}$$

$$\pi^{-1} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} y(t)dt = 0, \tag{4.10}$$

arises in the solution of a cruciform crack [10].

From Tables 2 and 3 we can see that the Nyström interpolant converges at least as fast or even faster than all methods considered, even though the kernels $K(s, t)$, in both cases are not continuous. The reason that the Gauss-Chebyshev method converges here is that the error in the Gaussian quadrature approximating the kernel part of (1.1) depends on the smoothness of $K(s, t)y(t)$, rather than the smoothness of the kernel $K(s, t)$ itself. Although we have considered the Nyström interpolation formula only for the point $t = 1$, the formula can be applied just as easily for all points in $[-1, 1]$.

Table 2. *Equation* (4.7), (4.8). *The strength of Stress Singularity* $y(1)$.

	$\lambda = 10/3$			$\lambda = 1/3$		
n	[2]	[7]	(2.15)	[2]	[7]	(2.15)
20	0.4061	0.4076	0.4133	0.8323	0.8325	0.8336
40	0.4104	0.4108	0.4121	0.8331	0.8332	0.8335
60	0.4115	0.4113	0.4119	0.8340	0.8333	0.8334

Table 3. *The stress intensity factor* $y(1)$ *of Eq.* (4.9), (4.10).

n	Gauss-Chebyshev [7]	Lobatto-Chebyshev [10]	Gauss-Chebyshev (2.15)
6	0.83363	0.85970	0.86261
8	0.87264	0.86387	0.86435
10	0.86289	0.86449	0.86448
12	0.86381	0.86441	0.86433
14	0.86527	0.86424	0.86415
16	0.86281	0.86408	0.86401
18	0.86503	0.86396	0.86391
20	0.86283	0.86387	0.86383
22	0.86463	0.86380	0.86372
40	0.86335		0.86358
60	0.86348		0.86355

Note: All calculations have been performed on DEC-20 in FORTRAN with double precision arithmetic.

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