BIT 22 (1982), 395–398

GLOBAL ERROR BOUNDS FOR THE CLENSHAW-CURTIS QUADRATURE FORMULA

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Abstract.

The object of this paper is to derive global error bounds for integrals approximated by the Clenshaw-Curtis formula.

Introduction.

Clenshaw and Curtis [2] have described a method for evaluating integrals of the form

$$\int_{-1}^{1} f(x)\,dx$$

in which $f(x)$ is approximated by a finite Chebyshev series.

If we use the notation of [2] and express $f(x)$ by the infinite Chebyshev series

$$f(x) = \sum_{r=0}^{\infty}{}' A_r T_r(x) \tag{1}$$

then the error E_N in the Clenshaw-Curtis quadrature rule is given by (see [2] or [4])

$$E_N = \sum_{R=0}^{N-2} \frac{16(R+1)N}{(N^2-(2R+1)^2)(N^2-(2R+3)^2)} A_{N+2R+2} \tag{2}$$

where N is assumed even and terms beyond A_{3N-2} have been neglected.

Our intention in this paper is to derive global error bounds for the Clenshaw-Curtis quadrature rule. It is interesting to note that Davis [3] has deduced similar bounds for other quadrature formulae by introducing a Hilbert space of analytic functions and using the Riesz representation of bounded linear functionals.

Following Davis, we shall introduce the conformal mapping

$$z = (\zeta + 1/\zeta)/2, \qquad \zeta = \rho \exp(i\theta), \qquad (0 \leqq \theta < 2\pi),$$

Received October 29, 1981. Revised April 27, 1982.

which transforms the circle $|\zeta| = \rho > 1$ onto an ellipse ε_ρ with foci at $z = \pm 1$, semi-axes $(\rho \pm \rho^{-1})/2$ and the interval $[-1, 1]$ deleted.

From (1), whenever $f(z)$ is analytic within and on the ellipse ε_ρ, $\rho > 1$,

$$(3) \qquad A_n = (2/\pi^2 i) \int_{\varepsilon_\rho} f(z) q_n(z) dz$$

where

$$q_n(z) = \tfrac{1}{2} \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_n(x)/(z-x)\, dx.$$

If we further assume

$$(4) \qquad \max |f(z)| \leqq 1, \qquad z \in \varepsilon_\rho$$

then by (2), (3), (4) and [1] (line 36)

$$(5)\ |E_N| \leqq \left(\frac{\rho^2+1}{\rho^2-1}\right) \sum_{R=0}^{N-2} \frac{32(R+1)N}{|(N^2-(2R+1)^2)(N^2-(2R+3)^2)|} \rho^{-(N+2R+2)}$$

$(= \sigma_N(\rho)$, say).

A global error bound now follows from the numerical minimization of the right-hand side of (5) by assuming N (even) to be fixed and ρ to take any value in the interval $(1, \rho^*]$, where ρ^* is constant. In table 1 we list the values of the global error bounds for different values of N and ρ^*. In order to make this table comparable with Davis's table 13.2 (see [3] p. 474) we also list the semi-major axis a of each ellipse ε_{ρ^*}.

It is worth pointing out that global error bounds may also be deduced in the case $\max |f(z)| \leqq 1$, $|z| \leqq R$, $R > 1$. As R increases, these bounds are not dissimilar from those derived from (5) as ρ^* increases.

Finally, for any particular function f, (5) may be replaced by

$$(5') \qquad |E_N(f)| \leqq \min\, (\sigma_N(\rho)|f|_{\varepsilon_\rho}), \qquad 1 < \rho \leqq \rho^*.$$

Since $|f|_{\varepsilon_\rho}$ increases as ρ increases and, for N fixed, $\sigma_N(\rho)$ decreases, an improved error bound may be derived from (5′). (This point is discussed in greater detail by Davis [3].)

Table 1.
Global error bounds for Clenshaw-Curtis quadrature, $|f(z)| \leqq 1$, $z \in \varepsilon_{\rho^*}$.

a	ρ^* \ $N =$	4	8	12	16	20
1.2500	2	6.04(−2)	4.62(−4)	5.26(−6)	1.07(−7)	3.12(−9)
2.1250	4	4.09(−4)	1.03(−7)	9.87(−11)	1.55(−13)	3.04(−16)
2.8409	5.5	5.23(−5)	3.52(−9)	9.96(−13)	4.42(−16)	2.43(−19)
3.0833	6	3.03(−5)	1.43(−9)	2.88(−13)	9.01(−17)	3.50(−20)
4.0625	8	5.05(−6)	7.49(−11)	4.85(−15)	4.83(−19)	5.94(−23)

Numerical examples.

1.
$$I = \int_{-1}^{1} dx/(3+x).$$

O'Hara and Smith [4] chose this integral to compare their methods of bounding the error term E_N in the Clenshaw-Curtis quadrature rule (which, for this example, will give 0.23(−12), 0.41(−9) and 0.36(−11) when $N = 16$) with the error bounds obtained from applying the techniques of Clenshaw and Curtis [2] (which give 0.27(−12) and 0.92(−9) when $N = 16$).

If we use the present method however, with $N = 16$ and $\rho^* = 5.5$ then, because $|1/(3+z)| \leqq 6.286$, $z \in \varepsilon_{5.5}$, we see from table 1 that the error in approximating I by the Clenshaw-Curtis quadrature rule is less than or equal to 6.286 multiplied by the entry in the row $\rho^* = 5.5$ and the column $N = 16$, that is $\leqq 6.286 \times 4.42(-16) \cong 0.278(-14)$. This compares most favourably with the exact error 0.209(−14).

2.
$$J = \int_{-1}^{1} \sin(x^2)dx.$$

This example has been considered by the author [5] in connection with the evaluation of the error term in Gauss-Legendre quadrature.

Suppose we wish the error in the approximation of J to be less than 10^{-15}, say. By letting $\rho^* = 6$ for example, we see

$$|\sin(z^2)| \leqq \cosh(\rho^{*2}/4) = \cosh(9) < 4052, \qquad z \in \varepsilon_{\rho^*}.$$

From table 1, 4052 multiplied by the entry in the row $\rho^* = 6$ for which $N = 20$ is less than 10^{-15} indicating Clenshaw-Curtis quadrature with $N = 20$ would suffice.

Acknowledgement.

The author is extremely grateful to both referees, one of whom drew his attention to the work of P. J. Davis and also suggested a modified form of table 1.

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