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AN UPPER BOUND FOR THE GAUSS-LEGENDRE QUADRATURE ERROR FOR ANALYTIC FUNCTIONS

T. H. CHARLES CHEN

Department of Mathematics, The University of Alabama in Huntsville, Huntsville, Alabama 35899, U.S.A.

Abstract

We improve an upper bound for the error of the Gauss-Legendre quadrature for analytic functions previously given by Chawla and Jain.

1. Introduction

It is well-known [3] that the following Gauss-Legendre quadrature has precision $2n-1$,

$$(1)\qquad \int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} w_k f(x_k) + E_{G_n}(f),$$

i.e., $E_{G_n}(f) = 0$ for all $f \in \mathscr{P}_{2n-1}$, the set of all polynomials of degree $2n-1$ or less. Here x_k are the zeros of $P_n(x)$, the Legendre polynomial of degree n, and w_k are defined by

$$w_k = \frac{1}{P_n'(x_k)} \int_{-1}^{1} \frac{P_n(x)}{x-x_k} dx \text{ for } k = 1, 2, \ldots, n.$$

The following contour integral representation for the error term $E_{G_n}(f)$ in (1) can be found in [3]:

$$(2)\qquad E_{G_n}(f) = (2\pi i)^{-1} \int_C [Q_n(z)/P_n(z)] f(z)dz,$$

where C is a simple closed contour which contains the interval $[-1, 1]$ in its interior and the function $f(z)$ is supposed to be analytic inside and on C. The function $Q_n(z)$ is defined by

$$Q_n(z) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(x)}{z-x} dx, \quad \text{for } z \notin [-1, 1].$$

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We introduce the conformal mapping $z = \frac{1}{2}(\xi+\xi^{-1})$ which maps the circle $|\xi| = \rho > 1$ in the ξ-plane onto the ellipse E_ρ with foci at $z = \pm 1$ and semi-axes $\frac{1}{2}(\rho \pm \rho^{-1})$. For the inverse mapping, $\xi = z+(z^2-1)^{\frac{1}{2}}$, we choose the branch of $(z^2-1)^{\frac{1}{2}}$ such that $|\xi| > 1$ for $z \notin [-1, 1]$.

Chawla and Jain [1, p.85] have given the following upper bound for $|E_{G_n}(f)|$:

THEOREM 1. *Let $f(z)$ be analytic on $[-1, 1]$ and continuable analytically so as to be analytic within and on E_ρ, $\rho > 1$. Then, given $\varepsilon > 0$, we have for $n \geqq N(\varepsilon)$,*

$$|E_{G_n}(f)| \leqq 2KM(\rho)(1-\varepsilon/\rho)^{-n}\rho^{-2n}, \tag{3}$$

where $M(\rho) = \max|f(z)|$ on E_ρ, $K = l(E_\rho)/[\pi(\rho-\rho^{-1})]$ and $l(E_\rho)$ represents the length of E_ρ.

We will improve this upper bound for $|E_{G_n}(f)|$ by showing that the constant 2 in (3) can be replaced by $\pi/2$.

2. Upper bounds for $|Q_n(z)|$ and $|E_{G_n}(f)|$

Davis [2, p. 311] gives the following series expansion for $Q_n(z)$,

$$Q_n(z) = \sum_{k=n+1}^{\infty} \sigma_{nk}\xi^{-k}, \qquad \text{for } z \in E_\rho, \tag{4}$$

where $\sigma_{nk} = \int_0^\pi P_n(\cos\theta)\sin(k\theta)d\theta$, $\xi = z+(z^2-1)^{\frac{1}{2}}$ and $|\xi| > 1$.

The following upper bound for $|\sigma_{nk}|$ was given by Davis [2, p. 311]:

LEMMA 1 *For $z \in E_\rho$, $|\sigma_{nk}| \leqq \pi$, $n = 0, 1, 2, \ldots$, $k = n+1, n+2, \ldots$.*

Chawla and Jain [1] have improved this upper bound as follows:

LEMMA 2 *For $z \in E_\rho$, $|\sigma_{nk}| \leqq 2$, $n = 0, 1, 2, \ldots$, $k = n+1, n+3, \ldots$.*

In [1], Chawla and Jain have employed the lemma above and derived the upper bound (3) for the Gauss-Legendre quadrature error. We will sharpen the upper bound for $|\sigma_{nk}|$ in lemma 2 and hence the upper bound for $|E_{G_n}(f)|$ in (3).

LEMMA 3 *For $z \in E_\rho$, $|\sigma_{nk}| \leqq \pi/2$, $n = 0, 1, 2, \ldots$, and $k = n+1, n+3, \ldots$.*

PROOF In [2, p. 308–309], Davis has shown that

$$P_n(\cos \theta) = \sum_{j=0}^{n} a_j a_{n-j} \cos (n-2j)\theta, \tag{5}$$

where $\quad a_j = (2j)!(j!)^{-2}2^{-2j}, \qquad j = 0, 1, 2, \ldots, n.$

It can be shown that

$$0 < a_j \leqq 1, \qquad \text{for } j = 0, 1, 2, \ldots, n.$$

Therefore

$$\begin{aligned} |\sigma_{nk}| &\leqq \int_0^\pi |P_n(\cos \theta)| \cdot |\sin k\theta| d\theta \\ &\leqq \left[\int_0^\pi P_n^2(\cos \theta) d\theta \right]^{\frac{1}{2}} \left[\int_0^\pi \sin^2 k\theta d\theta \right]^{\frac{1}{2}} \\ &= (\pi/2)^{\frac{1}{2}} \left\{ \int_0^\pi \left[\sum_{j=0}^{n} a_j a_{n-j} \cos (n-2j)\theta \right]^2 d\theta \right\}^{\frac{1}{2}} \\ &= (\pi/2)^{\frac{1}{2}} \left[(\pi/2) \sum_{j=0}^{n} a_j^2 a_{n-j}^2 \right]^{\frac{1}{2}} \leqq (\pi/2) \left[\sum_{j=0}^{n} a_j a_{n-j} \right]^{\frac{1}{2}} = \pi/2. \end{aligned}$$

The last equality follows from (5) with $\theta = 0$. ■

Employing the lemma above and following the same procedure as in [1, p. 84–85], we can rewrite (3) as follows:

$$|E_{G_n}(f)| \leqq (\pi/2)KM(\rho)(1-\varepsilon/\rho)^{-n}\rho^{-2n}, \tag{6}$$

where K and $M(\rho)$ are previously defined in theorem 1.

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