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A degenerate parabolic equation in noncylindrical domains

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1 Introduction

In this paper we study the problem

$$\text{(I)}\quad \begin{cases} u_t = \psi(u_x)_x & \text{if } |x| < \zeta(t), -1 < t < 0 \\ u(\pm\zeta(t), t) = 0 & \text{if } \; -1 < t < 0 \\ u(x, -1) = u_0(x) & \text{if } |x| < \zeta(-1), \end{cases}$$

where the functions ψ, ζ and u_0 satisfy the following hypotheses:

H1. $\psi \in C^3(\mathbf{R})$, $0 < \psi' < \gamma$ in $\mathbf{R}$ for some $\gamma > 0$, $\psi(-p) = -\psi(p)$ for $p \in \mathbf{R}$, and

$$\lim_{p\to+\infty} \psi(p) = \psi_\infty < +\infty; \tag{1.1}$$

H2. $\zeta \in C^1([-1,0)) \cap C([-1,0])$, $\zeta(0) = 0$, and $\zeta(t) > 0$ and $\zeta'(t) \leqq 0$ for $t \in [-1, 0)$;

H3. $u_0 \in C^1([-\zeta(-1), \zeta(-1)])$, $u_0 > 0$ in $(-\zeta(-1), \zeta(-1))$ and $u_0(\pm\zeta(-1) = 0$.

In the rest of the paper we shall indicate these assumptions collectively by hypothesis H.

A typical example of a function ψ satisfying H1 is given by

$$\psi(p) = \frac{p}{\sqrt{1+p^2}}, \tag{1.2}$$

which corresponds to the well-known mean curvature operator. The nonlinear diffusion equation

$$u_t = \psi(u_x)_x \tag{1.3}$$

and its generalization $u_t = \psi(u, u_x)_x$ arise in several applications, discussed by Blanc [5] and Rosenau [24].

Because of condition (1.1), equation (1.3) is not uniformly parabolic. Actually the parabolicity of the equation is so strongly degenerate, that, as was noticed by Blanc [4–7] and by Bertsch and Dal Passo [1, 2], solutions may be discontinuous. This rather hyperbolic character of the equation is also reflected in the existence of an entropy condition, which is necessary [1] and sufficient [8] to guarantee uniqueness of "weak" solutions of the corresponding Cauchy problem.

As we shall see in Sect. 2, Problem I possesses a unique solution $u(x,t)$, which is smooth in the set

$$Q = \{(x,t) : |x| < \zeta(t), -1 < t < 0\};$$

in Sect. 3 we shall show that u does not necessarily satisfy the condition $u = 0$ at the lateral boundaries of Q, and that, instead, u_x may be infinite at these boundaries. For the precise definition of a solution we refer to Sect. 2.

The main purpose of this paper is to study the behaviour of u near the vertex $(0,0)$ of Q. More in particular we would like to know for which functions ζ satisfying hypothesis H2, $u(x,t) \to 0$ as $(x,t) \to (0,0)$.

First we establish a class of functions ζ for which all solutions, i.e. independently of u_0, vanish at the vertex:

Theorem A. *Let hypotheses* H1 *and* H2 *be satisfied. If*

$$\int_{-1}^{0} \frac{1}{\zeta(t)} dt = \infty, \tag{1.4}$$

then for any u_0 satisfying hypothesis H3 *the solution u of Problem I satisfies*

$$\lim_{Q \ni (x,t) \to (0,0)} u(x,t) = 0. \tag{1.5}$$

If, in addition, $\zeta'(t)\zeta(t) \to 0$ as $t \to 0$, then there exists a $t_0 \in (-1,0)$ such that

$$\lim_{Q \ni (x,t) \to (\pm\zeta(\tau),\tau)} u(x,t) = 0 \quad \text{for } t_0 < \tau \leqq 0. \tag{1.6}$$

Observe that, generically, $\zeta\zeta'$ vanishes at $t = 0$ as soon as (1.4) is satisfied, and that (1.6) means that u satisfies the boundary condition $u(\pm\zeta(t), t) = 0$ for t close enough to 0.

Condition (1.4) turns out to be necessary in the sense that if it is not satisfied, then Problem I has solutions which do not vanish at the vertex:

Theorem B. *Let hypotheses* H1 *and* H2 *be satisfied. If*

$$\int_{-1}^{0} \frac{1}{\zeta(t)} dt < \infty, \tag{1.7}$$

then there exist initial functions u_0 *which satisfy hypothesis* H3, *such that the corresponding solutions* u *of Problem I satisfy*

$$\limsup_{Q \ni (x,t) \to (0,0)} u(x,t) > 0 . \tag{1.8}$$

Given a function ζ which satisfies condition (1.7), it remains to decide whether there exists initial functions for which the solutions vanish at (0, 0). The following result settles this question almost completely.

Theorem C. *Let hypotheses* H1 *and* H2 *be satisfied, and let* ζ *satisfy condition* (1.7). *There exists a constant* $c^* > 0$ *which only depends on* ψ *such that:*

(i) *if*

$$\liminf_{t \to 0^-} \frac{\zeta(t)}{\sqrt{-t}} > c^* , \tag{1.9}$$

then for all u_0 *satisfying hypothesis* H3 *the corresponding solutions* u *of Problem I satisfy*

$$\limsup_{Q \ni (x,t) \to (0,0)} u(x,t) > 0 ; \tag{1.10}$$

(ii) *if*

$$\limsup_{t \to 0^-} \frac{\zeta(t)}{\sqrt{-t}} < c^* , \tag{1.11}$$

then there exist initial functions u_0 *satisfying hypothesis* H3 *such that the corresponding solutions* u *of Problem I satisfy*

$$\lim_{Q \ni (x,t) \to (0,0)} u(x,t) = 0 . \tag{1.12}$$

We shall prove Theorem A, B and C in, respectively, Sects. 4, 5 and 6. In Sect. 6 we give a precise characterization of the constant c^* of Theorem C.

In the special case that

$$\zeta(t) = c(-t)^\alpha , \qquad (c, \alpha > 0)$$

we may summarize our results as follows:

$$\alpha \geqq 1 \;\Rightarrow\; \text{all solutions vanish at } (0,0) , \tag{1.13}$$

$$\tfrac{1}{2} < \alpha < 1 \text{ or } a = \tfrac{1}{2},\ c < c^* \;\Rightarrow\; \text{some but not all solutions vanish at } (0,0) , \tag{1.14}$$

and

$$0 < \alpha < \tfrac{1}{2} \text{ or } a = \tfrac{1}{2}, c > c^* \;\Rightarrow\; \text{none of the solutions vanish at } (0,0) . \tag{1.15}$$

These results are rather different from the known results for the (non)linear heat equation [11–13, 18, 20–23] and the porous medium equation [9]. For these equations (1.14) never occurs, while (1.13) and (1.15) occur if, respectively, $\alpha \geqq \frac{1}{2}$ and $0 < \alpha < \frac{1}{2}$; in particular the behaviour of the solutions near the vertex depends on the differential operator and the geometry of the boundary, but not on the initial function u_0. In the case of Problem I however, the behaviour may depend on u_0 (cf. (1.14)), which in some sense is another hyperbolic feature of the problem.

2 Existence and uniqueness of a solution

First we define what we mean by a solution of Problem I. We shall use the notation

$$Q^* = \{(x,t) : |x| < \zeta(t), -1 \leqq t < 0\}$$

and, for $T \in (-1,0)$

$$Q_T = \{(x,t) \in Q : -1 < t \leqq T\}.$$

Definition 2.1. *A function $u : Q^* \to \mathbf{R}$ is a solution of Problem I if, for any $T \in (-1,0)$*

(i) $u \in C^{2,1}(Q) \cap C(Q^*) \cap BV(Q_T)$;

(ii) *there exists a function $\overline{\psi} : \overline{Q} \to \mathbf{R}$ which is continuous in $\overline{Q}_T$ such that*

$$\overline{\psi}(x,t) = \psi(u_x(x,t)) \quad \textit{for any } (x,t) \in Q;$$

(iii) $u_t = \psi(u_x)_x$ in Q, $u(\cdot,-1) = u_0$ *in* $(-\zeta(-1), \zeta(-1))$, *and*

$$\pm\tilde{u}\overline{\psi} \leqq 0 \textit{ and } \tilde{u}(|\overline{\psi}| - \psi_\infty) = 0 \textit{ if } x = \pm\zeta(t) \textit{ for a.e. } t \in (-1,0), \tag{2.1}$$

where $\tilde{u}$ denotes the trace of the function u at the lateral boundaries $x = \pm\zeta(t)$ of Q.

Since u is a function of bounded variation, the trace of u is well defined. We observe that (2.1) is trivially satisfied if u satisfies, in the sense of traces, the Dirichlet boundary condition $u = 0$. In Sect. 3 we shall prove that this is not always the case, and condition (2.1) implies that if, for example, $\tilde{u} > 0$ at the boundary $x = \zeta(t)$, then $\overline{\psi}(\zeta(t),t) = -\psi_\infty$, i.e. $u_x(x,t) \to -\infty$ as $x \to \zeta(t)^-$.

In this section we shall prove the following result.

Theorem 2.2. *Let hypothesis* H *be satisfied. Then Problem I possesses a unique solution.*

The existence proof is based on the viscosity method, i.e., we consider the approximate problem

$$(\mathrm{I}_\varepsilon) \quad \begin{cases} u_t = \psi_\varepsilon(u_x)_x & \text{in } Q \\ u(\pm\zeta(t),t) = 0 & \text{if } -1 < t < 0 \\ u(x,-1) = u_0(x) & \text{if } |x| < \zeta(-1), \end{cases}$$

where $\varepsilon > 0$ and

$$\psi_\varepsilon(p) = \psi(p) + \varepsilon p \qquad \text{for } p \in \mathbf{R}.$$

Problem I_ε is uniformly parabolic and we denote its unique smooth solution [19] in the set $\{(x,t) : |x| \leqq \zeta(t), -1 \leqq t < 0\}$ by $u_\varepsilon(x,t)$.

In the following lemma's we give some estimates for u_ε.

Lemma 2.3. *Let u_ε denote the solution of Problem I_ε. Then:*

(i) $0 \leqq u_\varepsilon \leqq \max u_0$ *in* Q;

(ii) *for any compact subset* $K = [-a, a] \times [\tau, T]$ *of* Q *there exists a constant* C *such that*

$$\int_{-a}^{a} \psi_\varepsilon(u_{\varepsilon x}(s,t))_x^2 \, ds \leqq C \quad \text{for } \tau \leqq t \leqq T, \ 0 < \varepsilon \leqq 1 . \tag{2.2}$$

Proof. The first part follows at once from the maximum principle.

To prove (2.2) we choose a constant $b \in (a, \zeta(T))$, and a cut-off function $\chi \in C_0^\infty((-b, b))$ such that, for some $L > 0$,

$$0 \leqq \chi \leqq 1 \text{ and } |\chi'| \leqq L \text{ in } (-b, b), \qquad \chi \equiv 1 \text{ in } (-a, a) .$$

First we show that for some $C_0 > 0$ and for all ε

$$\iint_{K_0 = [-b,b] \times [-1,T]} \psi_\varepsilon(u_{\varepsilon x})_x^2 \chi^2 \, dx \, dt \leqq C_0 . \tag{2.3}$$

We multiply the equation for u_ε by $\psi_\varepsilon(u_{\varepsilon x})_x \chi^2$ and integrate by parts:

$$\iint_{K_0} \psi_\varepsilon(u_{\varepsilon x})_x^2 \chi^2 \, dx \, dt = - \iint_{K_0} \psi_\varepsilon(u_{\varepsilon x}) u_{\varepsilon x t} \chi^2 \, dx \, dt$$
$$-2 \iint_{K_0} \psi_\varepsilon(u_{\varepsilon x}) \psi_\varepsilon(u_{\varepsilon x})_x \chi \chi_x \, dx \, dt \equiv I_1 + I_2 . \tag{2.4}$$

Defining $\Psi_\varepsilon(p) = \int_0^p \psi_\varepsilon(s) \, ds$, we have

$$I_1 = - \int_{-b}^{b} \Psi_\varepsilon(u_{\varepsilon x}(x, T)) \chi^2 \, dx + \int_{-b}^{b} \Psi_\varepsilon(u_0'(x)) \chi^2 dx$$
$$\leqq - \int_{-b}^{b} \Psi_\varepsilon(u_{\varepsilon x}(x, T)) \chi^2 \, dx + C_1 , \tag{2.5}$$

for some C_1 which does not depend on ε. In addition we obtain from the inequalities of Cauchy-Schwartz and Young that

$$|I_2| = 2 \Big(\iint_{K_0} \psi_\varepsilon(u_{\varepsilon x})_x^2 \chi^2 \Big)^{1/2} \Big(\iint_{K_0} \psi_\varepsilon(u_{\varepsilon x})^2 \chi_x^2 \Big)^{1/2}$$
$$\leqq \frac{1}{2} \iint_{K_0} \psi_\varepsilon(u_{\varepsilon x})_x^2 \chi^2 + 2 \iint_{K_0} \psi_\varepsilon(u_{\varepsilon x})^2 \chi_x^2 . \tag{2.6}$$

It follows from [1, formula (4.1)] that, for all $0 < \varepsilon \leqq 1$,

$$\iint_{K_0} u_{\varepsilon x} \psi_\varepsilon(u_{\varepsilon x}) \leqq C_2$$

for some C_2. Since, for some $C_3 \geqq 1$,

$$\psi_\varepsilon(p)^2 = \psi(p)^2 + 2\varepsilon p\psi(p) + \varepsilon^2 p^2 \leqq C_3(1 + p\psi(p)) + \varepsilon p^2 \leqq C_3(1 + p\psi_\varepsilon(p)),$$

this implies that

$$\iint_{K_0} \psi_\varepsilon(u_{\varepsilon x})^2 \chi_x^2 \leqq L^2 C_3(2b + C_2). \tag{2.7}$$

Substituting (2.5), (2.6) and (2.7) into (2.4), we obtain (2.3).

Finally we prove (2.2). In view of (2.3) there exists for any $\varepsilon \in (0, 1]$ a time $\tau_\varepsilon \in [-1, \tau]$ such that

$$\int_{-b}^{b} \psi_\varepsilon(u_{\varepsilon x})_x^2(x, \tau_\varepsilon)\chi^2(x)\,dx \leqq \frac{C_0}{\tau + 1}. \tag{2.8}$$

We multiply the equation for $u_{\varepsilon x}$ by $\psi_\varepsilon(u_{\varepsilon x})_t \chi^2$ and integrate by parts over $K_\varepsilon = [-b, b] \times [\tau_\varepsilon, t]$, where $t \in [\tau, T]$:

$$0 \leqq \iint_{K_\varepsilon} \psi'_\varepsilon(u_{\varepsilon x})^2_{\varepsilon x t}\chi^2 = -\iint_{K_\varepsilon} \psi_\varepsilon(u_{\varepsilon x})_{xt}\psi_\varepsilon(u_{\varepsilon x})_x\chi^2$$
$$- 2\iint_{K_\varepsilon} \psi_\varepsilon(u_{\varepsilon x})_t\psi_\varepsilon(u_{\varepsilon x})_x\chi\chi_x \equiv I_3 + I_4. \tag{2.9}$$

It follows from (2.8) that

$$I_3 \leqq -\frac{1}{2}\int_{b}^{b} \psi_\varepsilon(u_{\varepsilon x})_x^2(x, t)\chi^2(x)\,dx + \frac{C_0}{2(\tau + 1)}. \tag{2.10}$$

From the Cauchy-Schwartz and Young inequalities we have that

$$|I_4| \leqq \frac{1}{2}\iint_{K_\varepsilon} \psi'_\varepsilon(u_{\varepsilon x})u^2_{\varepsilon x t}\chi^2\,dx\,dt + 2\iint_{K_\varepsilon} \psi'_\varepsilon(u_{\varepsilon x})\psi_\varepsilon(u_{\varepsilon x})_x^2\chi_x^2\,dx\,dt, \tag{2.11}$$

and, using (2.3) and the boundedness of ψ'_ε, we find that the latter term in (2.11) is uniformly bounded. Substituting (2.10) and (2.11) into (2.9), we obtain (2.2), and we have completed the proof of Lemma 2.3.

Lemma 2.4. *Let u_ε be the solution of Problem I_ε. Then $\{\psi_\varepsilon(u_{\varepsilon x})\}_{0<\varepsilon\leqq 1}$ is bounded in $C^{1/2,1/4}_{\text{loc}}(Q)$ and*

$$\limsup_{\varepsilon\to 0} \|\psi_\varepsilon(u_{\varepsilon x})\|_{L^\infty(K)} \leqq \psi_\infty \tag{2.12}$$

for all compact sets $K \subseteq Q$.

Proof. From (2.2) and the imbedding $H^1((-a,a)) \subseteq C^{1/2}([-a,a])$ we obtain the local uniform Hölder continuity of $\psi_\varepsilon(u_{\varepsilon x})$ with respect to x. Since $v_\varepsilon(x,t) = \psi_\varepsilon(u_{\varepsilon x}(x,t))$ satisfies the parabolic equation

$$v_t = \psi'_\varepsilon(u_{\varepsilon x})v_{xx}\,, \tag{2.13}$$

the coefficient of which is uniformly bounded, the local Hölder continuity with respect to t follows from [14].

It remains to prove (2.12). Arguing by contradiction we suppose that there exist a $\delta > 0$, a sequence $\{\varepsilon_n\}$ converging to 0 and points $(x_n, t_n) \to (x_0, t_0)$ as $n \to \infty$, such that, for any n, $|\psi_{\varepsilon n}(u_{\varepsilon_n x}(x_n, t_n))| > \psi_\infty + 2\delta$. We restrict ourselves to the case in which

$$\psi_{\varepsilon n}(u_{\varepsilon_n x}(x_n, t_n)) > \psi_\infty + 2\delta\,.$$

In view of the local equicontinuity of $\psi_\varepsilon(u_{\varepsilon x})$ this means that there exist $N > 0$ and a neighbourhood Ω of (x_0, t_0) in Q such that

$$\psi_{\varepsilon n}(u_{\varepsilon_n x}) > \psi_\infty + \delta \quad \text{in } \Omega \text{ for } n > N\,,$$

and hence

$$u_{\varepsilon_n x} > \psi_{\varepsilon n}^{-1}(\psi_\infty + \delta) \quad \text{in } \Omega \text{ for } n > N\,. \tag{2.14}$$

Since $\psi_\varepsilon^{-1}(\psi_\infty+\delta) \to \infty$ as $\varepsilon \to 0$, we obtain from (2.14) that $\sup\limits_\Omega u_{\varepsilon_n} - \inf\limits_\Omega u_{\varepsilon_n} \to \infty$ as $n \to \infty$, which is a contradiction with Lemma 2.3(i).

It turns out that the inequality in (2.12) is strict and that it holds in compact subsets of Q^*.

Lemma 2.5. *Let u_ε be the solution of Problem I_ε. Then*

$$\limsup_{\varepsilon\to 0} \|\psi_\varepsilon(u_{\varepsilon x})\|_{L^\infty(K)} < \psi_\infty \tag{2.15}$$

for all compact subsets $K \subseteq Q^*$.

Proof. Without loss of generality we may suppose that K is a rectangle of the form $K = [-a,a] \times [-1,T]$. Let $b \in (a, \zeta(T))$ and $K_0 = [-b,b] \times [-1,T]$. Since $u_0 \in C^1([-\zeta(-1), \zeta(-1)])$, there exist constants $\tau_0 \in (-1,T)$ and $\delta > 0$ which do not depend on ε such that

$$|\psi_\varepsilon(u_{\varepsilon x})| < \psi_\infty - \delta \quad \text{in } [-b,b] \times [-1,\tau_0]\,.$$

Let $A > \psi_\infty$ be a constant to be chosen. By Lemma 2.4 there exists a constant $\varepsilon_A > 0$ such that

$$|\psi_\varepsilon(u_{\varepsilon x}(\pm b, t))| < A \quad \text{if } \tau_0 \leqq t \leqq T,\ 0 < \varepsilon \leqq \varepsilon_A\,,$$

and since the coefficient in (2.13) is uniformly bounded it follows from the maximum principle that there exists a constant B which does not depend on A such that

$$|\psi_\varepsilon(u_{\varepsilon x})| < A - \delta e^{-B(t-\tau_0)} \cos\left(\frac{\pi x}{2b}\right) \quad \text{if } |x| \leqq b,\ \tau_0 \leqq t \leqq T,\ 0 < \varepsilon \leqq \varepsilon_A . \tag{2.16}$$

Choosing $A > \psi_\infty$ so small that the right-hand side of (2.16) is strictly smaller than ψ_∞ in the set $[-a, a] \times [\tau_0, T]$, we have completed the proof of (2.15).

Lemma 2.5 implies that, locally in Q^*, u_ε satisfies an equation which is uniformly parabolic with respect to ε, and, from standard results on quasilinear uniformly parabolic equations, we obtain the following result.

Lemma 2.6. *Let hypothesis* H *be satisfied and let u_ε denote the solution of Problem I_ε. Then there exist a sequence $\{\varepsilon_n\}$ and a function $u \in C(Q^*) \cap C^{2,1}(Q)$ such that*

$$u_\varepsilon \to u \quad \text{in } C_{\text{loc}}(Q^*) \cap C^{2,1}_{\text{loc}}(Q) \text{ as } \varepsilon_n \to 0 ,$$

and u satisfies $u_t = \psi(u_x)_x$ in Q and $u(x, -1) = u_0(x)$ for $|x| < \zeta(-1)$.

To prove that u is a solution of Problem I, it remains to show that it satisfies the required properties at the lateral boundaries of Q. The following result will enable us to prove the uniform continuity of $\psi(u_x)$ in Q_T for $-1 < T < 0$.

Lemma 2.7. *Let $T \in (-1, 0)$. Let $\hat{\psi}_\varepsilon \in C(\mathbf{R})$ be defined by*

$$\hat{\psi}_\varepsilon(p) = \begin{cases} -\psi_\infty & \text{if } \psi_\varepsilon(p) \leqq -\psi_\infty \\ \psi_\varepsilon(p) & \text{if } -\psi_\infty < \psi_\varepsilon(p) < \psi_\infty \\ \psi_\infty & \text{if } \psi_\varepsilon(p) \geqq \psi_\infty . \end{cases}$$

Then the functions $\hat{\psi}_\varepsilon(u_{\varepsilon x})$ are equicontinuous in $\overline{Q}_T$. In addition, for any $\tau_0 \in (-1, T)$, there exist constants $c > 0$ and $\beta > 0$ which do not depend on ε such that

$$\pm u_{\varepsilon x}(x, t) \geqq \beta \quad \text{if } |x \pm \zeta(t)| < c,\ \tau_0 \leqq t \leqq T . \tag{2.17}$$

Proof. We only consider the boundary $x = \zeta(t)$.

Let $0 < c_0 < \zeta(t)$. Defining

$$\xi = x - \zeta(t) \quad \text{for } -c_0 \leqq x - \zeta(t) \leqq 0,\ -1 \leqq t \leqq T ,$$

and denoting $\bar{u}_\varepsilon(\xi, t) \equiv u_\varepsilon(x, t)$ by $u_\varepsilon(\xi, t)$ again, we find that u_ε satisfies the equation

$$u_t = \psi_\varepsilon(u_\xi)_\xi + \zeta' u_\xi \quad \text{in } (-c_0, 0) \times (-1, T] . \tag{2.18}$$

First we prove (2.17). Since $u_0 \in C^1([-\zeta(-1), \zeta(-1)])$, there exists a time $\tau \in (-1, T)$ such that $u_{\varepsilon\xi}$ in uniformly bounded in $(-c_0, 0) \times (-1, \tau]$. Without loss of generality we may assume that $\tau_0 = \tau$. By classical theory (the boundary point lemma), $u_{\varepsilon\xi}(0, \tau)$ is uniformly bounded away from zero, and, if we choose c_0 small enough, there exists a $C_0 > 0$ which does not depend on ε, such that

$$u_{\varepsilon\xi}(\xi, \tau) \leqq -C_0 \quad \text{for } -c_0 \leqq \xi \leqq 0 . \tag{2.19}$$

In particular, $u_\varepsilon(\xi,\tau) \geqq -C_0\xi$ for $-c_0 \leqq \xi \leqq 0$, and since, for some $C_1 > 0$, $u_\varepsilon(-c_0, t) \geqq C_1$ if $t \leqq \tau \leqq T$, it follows from (2.19) and the maximum principle applied to (2.18) that

$$u_\varepsilon \geqq -C_2\xi \quad \text{in } [-c_0, 0] \times [\tau, T],$$

where we have set $C_2 = \min\{C_0, C_1/c_0\}$. This implies that

$$u_{\varepsilon\xi}(0,t) \leqq -C_2 \quad \text{for } \tau \leqq t \leqq T.$$

The function $u_{\varepsilon\xi}$ satisfies the equation

$$w_t = \psi_\varepsilon(w)_{\xi\xi} + \zeta' w_\xi .$$

There exists a constant C_3 which does not depend on ε such that

$$u_{\varepsilon\xi}(-c_0, t) \leqq C_3 \quad \text{for } \tau \leqq t \leqq T,$$

and hence it follows from the maximum principle that

$$u_{\varepsilon\xi}(\xi, t) \leqq \overline{w}(\xi, t) \quad \text{for } -c_0 \leqq \xi \leqq 0,\ \ \tau \leqq t \leqq T,$$

where $\overline{w}$ is the uniformly bounded (and hence classical!) solution of the problem

$$\begin{cases} w_t = \psi_\varepsilon(w)_{\xi\xi} + \zeta' w_\xi & \text{if } -c_0 < \xi < 0,\ \ \tau < t \leqq T \\ w(-c_0, t) = C_3 \text{ and } w(0,t) = -C_2 & \text{if } \tau < t \leqq T \\ w(\xi, t) = u_{\varepsilon\xi}(\xi, \tau) & \text{if } -c_0 < \xi < 0 . \end{cases}$$

We obtain (2.17) if we choose $0 < c < c_0$ such that $\overline{w} \leqq 0$ in $[-c, 0] \times [\tau, T]$.

Since u_ε is strictly monotone near the boundary $\xi = 0$, we may introduce a new variable u, defined by

$$u = u_\varepsilon(\xi, t).$$

We choose $r > 0$ such that $u_\varepsilon(-c, 0) \geqq r$ for $\tau \leqq t \leqq T$ and for all ε, and we set

$$K = \{(u,t) : 0 < u < r, \tau < t \leqq T\}. \tag{2.20}$$

We define the functions $v_\varepsilon \in C^{2,1}(K) \cap C^{1,0}(\overline{K})$, $c_\varepsilon \in C^2(\mathbf{R}^-)$. $f_\varepsilon \in C^1([\tau, T])$ and $g_\varepsilon \in C^2((0, r]) \cap C^1(0, r])$ by

$$\begin{aligned} v_\varepsilon(u,t) &\equiv \psi_\varepsilon(u_{\varepsilon\xi}(\xi,t)) && \text{for } (u,t) \in \overline{K} \\ c_\varepsilon(s) &= -\frac{1}{\psi_\varepsilon^{-1}(s)} && \text{for } s < 0 \\ f_\varepsilon(t) &= v_\varepsilon(r,t) && \text{for } \tau \leqq t \leqq T \\ g_\varepsilon(u) &= c_\varepsilon(v_\varepsilon(u,\tau)) && \text{for } 0 \leqq u \leqq r . \end{aligned}$$

From a straightforward calculation (see [1]) we obtain that v_ε satisfies

$$\begin{cases} c_\varepsilon(v)_t = v_{uu} & \text{in } K \\ v_u(0,t) = -\zeta'(t) & \text{for } \tau \leqq t \leqq T \\ v(r,t) = f_\varepsilon(t) & \text{for } \tau \leqq t \leqq T \\ c_\varepsilon(v(u,\tau)) = g_\varepsilon(u,\tau)) & \text{for } 0 \leqq u \leqq r\,. \end{cases}$$

It follows from the equation and boundary conditions for $v_{\varepsilon u}$ and the maximum principle applied in K that $v_{\varepsilon u}$ is uniformly bounded in K. Using the equation for v_ε, this implies that the functions $c_\varepsilon(v_\varepsilon)$ are uniformly continuous with respect to t (see also [1]), and thus the functions $c_\varepsilon(v_\varepsilon)$ are equicontinuous in $\overline{K}$. Hence there exist a subsequence of the sequence $\{\varepsilon_n\}$ of Lemma 2.6, which we shall denote by $\{\varepsilon_n\}$ again, and a function $\bar{c} \in C(\overline{K})$ such that

$$c_{\varepsilon_n}(v_{\varepsilon_n}) \to \bar{c} \quad \text{in } C(\overline{K}) \text{ as } \varepsilon_n \to 0\,. \tag{2.21}$$

We observe that, as $\varepsilon \to 0$,

$$c_\varepsilon(s) \to c(s) = \begin{cases} -\dfrac{1}{\psi^{-1}(s)} & \text{for } -\psi_\infty < s < 0 \\ 0 & \text{for } s \leqq -\psi_\infty \end{cases} \tag{2.22}$$

and it is natural to ask whether v_{ε_n} converges to a function v which satisfies the equation

$$c(v)_t = v_{uu} \quad \text{in } K\,. \tag{2.23}$$

By (2.22), equation (2.23) is of elliptic-parabolic type, i.e., formally it is a parabolic equation in the set Ω in which $-\psi_\infty < v < 0$, while (2.23) reduces to the elliptic equation $v_{uu} = 0$ in $K \setminus \overline{\Omega}$. These formal considerations lead to the following definitions of $\Omega \subseteq \overline{K}$, the free boundary $x = \alpha(t)$ which separates, at least if $\alpha(t) > 0$, the sets Ω and $K \setminus \overline{\Omega}$, and the function $v : \overline{K} \to \mathbf{R}$:

$$\begin{aligned} \Omega &= \{(u,t) \in \overline{K} : \bar{c}(u,t) > 0\} \\ \alpha(t) &= \inf\{u > 0 : \bar{c}(s,t) > 0 \text{ for } u < s < r\}, \quad \tau \leqq t \leqq T \end{aligned} \tag{2.24}$$

$$v(u,t) = \begin{cases} c^{-1}(\bar{c}(u,t)) & \text{if } (u,t) \in \Omega \\ -\psi_\infty - \zeta'(t)(u - \alpha(t)) & \text{for } 0 \leqq u \leqq \alpha(t) \text{ if } \alpha(t) > 0\,. \end{cases} \tag{2.25}$$

We observe that $0 \leqq \alpha(t) < r$, $\bar{c}(\alpha(t),t) = 0$ if $\alpha(t) > 0$, and, by (2.21) and the parabolicity of the equation (2.23) in Ω,

$$v_{\varepsilon_n} \to v \quad \text{in } C^{2,1}_{\text{loc}}(\Omega) \text{ as } n \to \infty\,. \tag{2.26}$$

In particular $c(v) = \bar{c}$ in $\overline{K}$ and, by (2.21),

$$c_{\varepsilon_n}(v_{\varepsilon_n}) \to c(v) \in C(\overline{K}) \quad \text{as } n \to \infty\,. \tag{2.27}$$

It follows from (2.25) that v is uniformly Lipschitz continuous with respect to u, and it is straightforward to show that v is a solution in the sense of distributions of the problem

$$\text{(II)}\begin{cases} c(v)_t = v_{uu} & \text{in } K \\ v_u(0,t) = -\zeta'(t) & \text{for } \tau \leqq t \leqq T \\ v(r,t) = f(t) & \text{for } \tau \leqq t \leqq T \\ c(v(u,t)) = g(u) & \text{for } 0 \leqq u \leqq r\,, \end{cases}$$

where the functions f and g are determined by the relations

$$f_{\varepsilon n} \to f \quad \text{in } C^1([\tau,T]) \text{ as } n \to \infty$$
$$g_{\varepsilon n} \to g \quad \text{in } C^2_{\text{loc}}((0,r]) \text{ as } n \to \infty\,.$$

Using the equicontinuity of $c_\varepsilon(v_\varepsilon)$ and arguing as in [1, Lemma 4.3], we find that the functions $\hat{\psi}_\varepsilon(u_\varepsilon x(x,t))$ are equicontinuous near the lateral boundary $x = \zeta(t)$, and the proof of Lemma 2.7 is complete.

Remarks. (i) In general the function v defined by (2.25), does not satisfy the inequality $v \geqq -\psi_\infty$, from which it easily follows that the functions $\psi_\varepsilon(u_{\varepsilon x})$ are not equicontinuous up to the lateral boundaries.

(ii) In Sect. 3 we shall give an interpretation of the following result, which we shall prove in the appendix:

Lemma 2.8. *Let α be defined by* (2.24). *Then*

$$\alpha \in C(\{t \in [\tau,T] : \zeta'(t) < 0\})\,,$$

and α is not necessarily continuous in $t \in [\tau,T]$ if $\zeta'(t) = 0$.

The next step is to prove that u has bounded variation up to the lateral boundaries.

Lemma 2.9. $u \in BV(Q_T) \cap L^\infty(0,T; BV((-\zeta(t),\zeta(t))$ *for any* $T \in (-1,0)$.

It is sufficient to prove the result near the lateral boundaries. The proof is quite similar to the one of Lemma 4.1 in [1], and we omit it. We observe that it follows immediately from (2.17) that $u \in L^\infty(0,T; BV((-\zeta(t),\zeta(t)))$.

The existence proof is completed by the following result.

Lemma 2.10. *Let u be defined by Lemma* 2.6. *Then u is a solution of Problem I.*

Proof. Lemma's 2.5, 2.6 and 2.7 imply that there exists a function $\overline{\psi}$ which is continuous in $\overline{Q}_T$ for any $T \in (-1,0)$ such that $\overline{\psi} = \psi(u_x)$ in Q_T. In view of Lemma's 2.6 and 2.9 it remains to show that the trace $\tilde{u}$ of u satisfies condition (2.1).

We consider only the boundary $x = \zeta(t)$. Let $t_0 \in (-1,0)$. If $-\psi_\infty < \overline{\psi}(\zeta(t_0),t_0) < \psi_\infty$, there exist $\varepsilon_0 > 0$ and $\alpha > 0$ such that $|\psi_\varepsilon(u_{\varepsilon x})| < \psi_\infty - \alpha$ in a neighbourhood of t_0. Hence $u_{\varepsilon x}$ is uniformly bounded in this neighbourhood and u_{ε_n} converges uniformly to u; in particular $\tilde{u}(\zeta(t),t) = 0$ for a.e. t for which $|\overline{\psi}(\zeta(t_0),t_0)| < \psi_\infty$.

To complete the proof we have to show that $\tilde{u}\overline{\psi}(\zeta(t),t) \leqq 0$ for a.e. t for which $|\overline{\psi}(\zeta(t),t)| = \psi_\infty$. By (2.17), $\overline{\psi}(\zeta(t),t) \leqq 0$ for all t, and the result follows from the fact that $\tilde{u}(\zeta(t),t) \geqq 0$ for a.e. t.

It remains to prove that the solution of Problem I is unique. For later purposes we shall prove a more general comparison principle for the following class of sub and supersolutions:

Definition 2.11. *A function $u : Q^* \to \mathbf{R}$ is a subsolution of Problem I if, for any $T \in (-1,0)$,*

(i) *$u \in W^{1,1}_{\mathrm{loc}}(Q) \cap C(Q^*) \cap BV(Q_T)$;*

(ii) *there exists a function $\overline{\psi} : \overline{Q} \to \mathbf{R}$ which, for some $\delta_T > 0$, is continuous in the set*

$$\{(x,t) \in \overline{Q}_T : x < -\zeta(t) + \delta_T \ \text{or} \ x > \zeta(t) - \delta_T\},$$

such that

$$\overline{\psi}(x,t) = \psi(u_x(x,t)) \quad \text{for a.e. } (x,t) \in Q;$$

(iii) *for any nonnegative Lipschitz continuous function $\chi : Q^* \to \mathbf{R}$ with compact support in Q^**

$$\int_{-\zeta(T)}^{\zeta(T)} u(x,T)\chi(x,T)dx \leqq \int_{-\zeta(-1))}^{\zeta(-1)} u_0(x)\chi(x,-1)dx + \iint_{Q_T} (u\chi_t - \psi(u_x)\chi_x)\, dx\, dt\,, \tag{2.28}$$

and

$$\pm \tilde{u}_+ \overline{\psi} \leqq 0 \ \text{and} \ \tilde{u}_+(|\overline{\psi}| - \psi_\infty) = 0 \quad \text{if } x = \pm\zeta(t) \quad \text{for a.e } t \in (-1,0)\,, \tag{2.29}$$

where $\tilde{u}$ denotes the trace of the function u at the lateral boundaries $x = \pm\zeta(t)$ of Q.

A supersolution of Problem I is defined similarly, with the reversed inequality in (2.28) *and with* (2.29) *replaced by*

$$\pm \tilde{u}_- \overline{\psi} \geqq 0 \ \textit{and} \ \tilde{u}_-(|\overline{\psi}| - \psi_\infty) = 0 \quad \textit{if } x = \pm\zeta(t) \quad \textit{for a.e } t \in (-1,0) \tag{2.30}$$

(*we have used the notations $a_+ = \max\{a,0\}$ and $a_- = -\min\{a,0\}$ for $a \in \mathbf{R}$*).

Observe that a solution of Problem I (according to Definition 2.1) is both a subsolution and a supersolution of Problem I, and the uniqueness of the solution of Problem I is a consequence of the following comparison principle:

Theorem 2.12. *Let hypothesis* H *be satisfied, and let u and v be, respectively, a subsolution and a supersolution of Problem I. Then*

$$u \leqq v \quad \text{a.e. in } Q\,.$$

Proof. Let $T \in (-1,0)$, and let $\delta \in \left(0, \frac{1}{2}\delta_T\right)$. We define the function $\chi_\delta \in W^{1,\infty}(Q_T)$ by

$$\chi_\delta(x,t) = \begin{cases} \frac{1}{\delta}(x + \zeta(t) - \delta) & \text{if } \ -\zeta(t) + \delta \leqq x \leqq -\zeta(t) + 2\delta \\ 1 & \text{if } \ -\zeta(t) + 2\delta < x < \zeta(t) - 2\delta \\ \frac{1}{\delta}(\zeta(t) - \delta - x) & \text{if } \ \zeta(t) - 2\delta \leqq x \leqq \zeta(t) - \delta \\ 0 & \text{if } \ |x \pm \zeta(t)| \leqq \delta \end{cases}$$

for $-1 \leqq t \leqq T$. Let $\tau \in (-1, T)$, let $\varepsilon > 0$ be small enough, and let $g_{\tau\varepsilon} \in C^1([-1, T])$ satisfy $g_{\tau\varepsilon} \equiv 1$ in $[\tau + \varepsilon, T]$, $g_{\tau\varepsilon} \equiv 0$ in $[-1, \tau]$, $0 < g_{\tau\varepsilon} < 1$ in $(\tau, \tau + \varepsilon)$ and $0 \leqq g'_{\tau\varepsilon} \leqq 2/\varepsilon$ in $(-1, T]$. Substituting the function $\chi = (u - v)_+ \chi_\delta g_{\tau\varepsilon}$ into the integral inequalities (2.28) for u and v respectively, subtracting the two inequalities, and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\int_{-\zeta(T)}^{\zeta(T)} (u - v)_+^2(x, T)\chi_\delta(x, T)\, dx \\
&\leqq \int_{-\zeta(\tau)}^{\zeta(\tau)} (u - v)_+^2(x, \tau)\chi_\delta(x, \tau)\, dx \\
&\quad + \iint_{\{u>v\}} \Bigg(\frac{1}{2}((u - v)^2\chi_\delta)_t + \frac{1}{2}(u - v)^2\chi_{\delta t} \\
&\quad - (u - v)_x(\psi(u_x) - \psi(v_x))\chi_\delta - (\psi(u_x) - \psi(v_x))(u - v)\chi_{\delta x} \Bigg),
\end{aligned}$$

where we have set $\{u > v\} = \{(x, t) \in Q_T : t > \tau,\ u(x, t) > v(x, t)\}$, and where we have used the convergence

$$\iint_{Q_T} (u - v)_+ \chi_\delta g'_{\tau\varepsilon}\, dx\, dt \to \int_{-\zeta(\tau)}^{\zeta(\tau)} (u - v)_+ \chi_\delta(x, \tau) dx$$

as $\varepsilon \to 0$, since u and v are continuous and bounded functions in Q^*. Since $\chi_{\delta t} \leqq 0$ and $(p - q)(\psi(p) - \psi(q)) \geqq 0$ for $p, q \in \mathbf{R}$, we find that

$$\begin{aligned}
&\frac{1}{2} \int_{-\zeta(T)}^{\zeta(T)} (u - v)_+^2(x, T)\chi_\delta(x, T)\, dx \\
&\leqq \frac{1}{2} \int_{-\zeta(\tau)}^{\zeta(\tau)} (u - v)_+^2(x, \tau)\chi_\delta(x, \tau)\, dx \\
&\quad - \frac{1}{\delta} \int_\tau^T \left(\int_{-\zeta(t)+\delta}^{-\zeta(t)+2\delta} - \int_{\zeta(t)-2\delta}^{\zeta(t)-\delta} \right) (u - v)_+(\overline{\psi}_u - \overline{\psi}_v),
\end{aligned}$$

where $\overline{\psi}_u$ and $\overline{\psi}_v$ indicate the function $\overline{\psi}$ in Definition 2.1 corresponding to, respectively, u and v. Letting first $\tau \to -1$ and then $\delta \to 0$, this leads to

$$\frac{1}{2} \int_{-\zeta(T)}^{\zeta(T)} (u - v)_+^2(x, T)\, dx \leqq \int_0^T [(\tilde{u} - \tilde{v})_+(\overline{\psi}_u - \overline{\psi}_v)]_{x=-\zeta(t)}^{x=\zeta(t)}\, dt\,. \tag{2.31}$$

It remains to show that the right-hand side of (2.31) is nonpositive, i.e., that for a.e. $t \in (-1, T)$,

$$(\overline{\psi}_u - \overline{\psi}_v)(\zeta(t), t) \leqq 0 \quad \text{if } (\tilde{u} - \tilde{v})(\zeta(t), t) > 0 \tag{2.32}$$

and

$$(\overline{\psi}_u - \overline{\psi}_v)(-\zeta(t), t) \geqq 0 \quad \text{if } (\tilde{u} - \tilde{v})(-\zeta(t), t) > 0 .$$

We only prove (2.32): if $\tilde{u}(\zeta(t)) > 0$, it follows from (2.29) that $\overline{\psi}_u(\zeta(t), t) = -\psi_\infty$ and thus $(\overline{\psi}_u - \overline{\psi}_v)(\zeta(t), t) \leqq 0$; if $\tilde{u}(\zeta(t)) \leqq 0$, we may assume that $\tilde{v}(\zeta(t)) < 0$ and hence, by (2.30), $\overline{\psi}_v(\zeta(t), t) = \psi_\infty$, which implies that $(\overline{\psi}_u - \overline{\psi}_v)(\zeta(t), t) \leqq 0$.

3 Discontinuities at the lateral boundaries

We introduce a family of travelling wave solutions of (1.3), which we shall use to prove that the solution of Problem I does not necessarily satisfy the boundary condition at $x = \pm\zeta(t)$. In particular we are interested in travelling waves with unbounded gradient.

Choosing $c > 0$ and setting $\eta = x - ct$, we look for the solution $v(\eta; c) \in C^2(\mathbf{R}^+)$ of the problem

$$(\text{TW}_c) \begin{cases} \psi(v')' + cv' = 0 & \text{in } \mathbf{R}+ \\ v(0^+) = 0, \quad v'(0^+) = +\infty . \end{cases}$$

We observe that if $v(\eta)$ is a solution of Problem TW_1, then $v(\eta; c)$, defined by

$$v(\eta; c) = \frac{1}{c} v(c\eta), \qquad \eta > 0 ,$$

is a solution of Problem TW_c.

In order to solve Problem TW_1, we integrate twice:

$$\psi(v') + v = \psi_\infty \Rightarrow v' = \psi^{-1}(\psi_\infty - v), \qquad \eta > 0 ,$$

and thus the function v defined by

$$\int_0^{v(\eta)} \frac{1}{\psi^{-1}(\psi_\infty - s)} ds = \eta, \qquad \eta > 0 \tag{3.1}$$

is the unique solution of Problem TW_1. We notice that $v(+\infty) = \psi_\infty$.

For any $c > 0$ and $A \geqq 0$ we define

$$v(\eta; c, A) = a + \frac{1}{c} v(c\eta), \qquad \eta > 0 . \tag{3.2}$$

Hence $v(\eta; c, A)$ satisfies

$$\begin{cases} \psi(v')' + cv' = 0 & \text{in } \mathbf{R}^+ \\ v(0^+) = A, \quad v'(0^+) = +\infty, \quad v(+\infty) = A + \dfrac{\psi_\infty}{c} . \end{cases}$$

We use the travelling waves to prove the main result of this section.

Theorem 3.1. *Let ψ and ζ satisfy hypotheses* H1 *and* H2. *Then there exist initial functions u_0 satisfying hypothesis* H3 *such that the solution u of Problem I satisfies for some $-1 < t_0 < t_1 < 0$*

$$\liminf_{x\to\pm\zeta(t)} u(x,t) > 0 \quad \text{if } t_0 < t < t_1 .$$

Proof. Let $C > 0$ be a constant to be determined, and let u_0 satisfy hypothesis H3 such that

$$u_0(x) \geqq C\cos\left(\frac{\pi x}{2\zeta(-\frac{1}{2})}\right) \quad \text{if } |x| \leqq \zeta\left(-\frac{1}{2}\right).$$

Since ψ' is uniformly bounded, it follows from the comparison principle (Theorem 2.12), applied in the set $K = \left[-\zeta\left(-\frac{1}{2}\right), \zeta\left(-\frac{1}{2}\right)\right] \times \left[-1, -\frac{1}{2}\right]$, that for some $B > 0$, which does not depend on C,

$$u(x,t) \geqq Ce^{-B(t+1)}\cos\left(\frac{\pi x}{2\zeta(-\frac{1}{2})}\right) \quad \text{for } (x,t) \in K,$$

whence, in particular,

$$u(0,t) \geqq Ce^{-B/2} \quad \text{for } -1 \leqq t \leqq -\tfrac{1}{2}.$$

We set

$$c = \zeta(-1) - \zeta\left(-\frac{1}{2}\right), \qquad A = Ce^{-B/2} - \frac{\psi_\infty}{c},$$

and we choose C so large that $A > 0$. Let $v(\eta; c, A)$ be defined by (3.2) and let

$$\begin{aligned}
x_0 &= -\frac{\zeta(-1) + \zeta(-\frac{1}{2})}{2} \\
w(x,t) &= v((x - x_0) - c(t+1); c, A) \\
\tau &= \sup\{-1 < t \leqq 0 : -\zeta(s) < x_0 + c(s+1) \text{ for } -1 \leqq s < t\}.
\end{aligned}$$

We observe that $-\zeta(-1) < x_0 < -\zeta\left(-\frac{1}{2}\right)$, and, since $-\zeta\left(-\frac{1}{2}\right) = x_0 + c\left(1 - \frac{1}{2}\right)$, we have $-1 < \tau \leqq -\frac{1}{2}$.

Since

$$w(0,t) < v(+\infty; c, A) = A + \frac{\psi_\infty}{c} = Ce^{-B/2} \leqq u(0,t) \quad \text{for } -1 \leqq t \leqq -\frac{1}{2},$$

it follows from the comparison principle (Theorem 2.12) applied in the set $\{(x,t) : x_0 + c(t+1) \leqq x \leqq 0, -1 \leqq t \leqq \tau\}$, that if u_0 satisfies

$$u_0(x) \geqq w(x,0) \quad \text{for } x_0 < x \leqq 0,$$

then

$$u(x,\tau) \geqq w(x,\tau) \quad \text{for} \quad -\zeta(t) < x \leqq 0\,.$$

Hence $\liminf_{x\to-\zeta(\tau)} u(x,\tau) \geqq A > 0$.

Choosing x_0 slightly smaller, one proves in a similar way that u_0 can be chosen such that $\liminf_{x\to-\zeta(t)} u(x,t) > 0$ for $t \in [t_0,\tau]$, with $t_0 < \tau$.

Finally we consider the regularity of u near the lateral boundaries.

Theorem 3.2. *Let hypothesis* H *be satisfied and let u be a solution of Problem I. Then the functions*

$$u(\zeta(t)^-,t) \quad \textit{and} \quad u(-\zeta(t)^+,t)$$

are continuous at $t_0 \in (-1,0)$ if $\zeta'(t_0) < 0$. If $\zeta'(t_0) = 0$, these functions are not necessarily continuous at t_0.

Proof. We restrict ourselves to the function $u(\zeta(t)^-,t)$. Then it follows from the proof of Lemma 2.7 that $u(\zeta(t)^-,t) = \alpha(t)$, where $\alpha(t)$ is defined by (2.24), and Theorem 3.2 is a consequence of Lemma 2.8.

4 Theorem A

In this section we consider the case in which ζ satisfies

$$\int_{-1}^{0} \frac{1}{\zeta(t)}\,dt = +\infty\,. \tag{4.1}$$

In order to prove Theorem A, we introduce the new variables (see also [18])

$$\begin{cases} y = \dfrac{x}{\zeta(t)} & \text{for } |x| \leqq \zeta(t), \quad -1 \leqq t < 0 \\ \tau = \int_{-1}^{t} \dfrac{1}{\zeta(s)}\,ds & \text{for } -1 \leqq t < 0\,, \end{cases} \tag{4.2}$$

i.e. $-1 \leqq y \leqq 1$ and $0 \leqq \tau < +\infty$. Thus $t = t(\tau)$ is a function of τ, and we shall denote the functions $\overline{u}(y,\tau) \equiv u(x,t)$ and $\overline{u}_0(y) \equiv u_0(x)$ by, respectively, $u(y,\tau)$ and $u_0(y)$. Hence u satisfies the equation

$$\mathscr{L}(u) = 0 \quad \text{in } D = (-1,1) \times \mathbf{R}^+\,,$$

where we have set

$$\mathscr{L}(u) = u_\tau - \psi\left(\frac{u_y}{\zeta(t(\tau))}\right)_y - y\zeta'(t(\tau))u_y\,. \tag{4.3}$$

We shall construct a supersolution of the form

$$\overline{u}(y,\tau) = \zeta(t(\tau))g(y) + f(\tau)\,,$$

where $g \in C^2([-1,1])$ and $f \in W^{1,\infty}(\mathbf{R}^+)$ are functions to be determined; in particular we require that $\bar{u}$ satisfies

$$\begin{cases} \mathcal{L}(\bar{u}) \geqq 0 & \text{a.e. in } D \\ \bar{u}(y,0) \geqq u_0(y) & \text{for } |y|<1 \\ \bar{u}(\pm 1,\tau) \geqq 0 & \text{for } \tau > 0 . \end{cases}$$

Hence, by the comparison principle,

$$u \leqq \zeta(t(\tau))g(y) + f(\tau) \quad \text{in } D . \tag{4.4}$$

To determine g and f, we calculate

$$\mathcal{L}(\bar{u}) = \zeta\zeta'(g - yg') + f' - \psi(g')' \quad \text{a.e. in } D .$$

Let $\alpha \in (0, \psi_\infty)$ and let g be defined by

$$\begin{cases} -\psi(g')' = \alpha & \text{for } |y| < 1 \\ g(\pm 1) = 0 , \end{cases}$$

i.e., $g(y) = -\int_y^1 \psi^{-1}(-\alpha s)ds$. Substituting g into $\mathcal{L}(\bar{u})$ we obtain that for some constant $C > 0$

$$\mathcal{L}(\bar{u}) \geqq C\zeta\zeta' + f' + \alpha \quad \text{in } D .$$

It remains to determine $f(\tau)$. In order to satisfy the inequalities at the parabolic boundary of D, we require that

$$f(0) = \max_{-1 \leqq y \leqq 1} u_0(y) \quad \text{and} \quad f \geqq 0 \quad \text{in } \mathbf{R}^+ .$$

In view of the condition that $\mathcal{L}(\bar{u}) \geqq 0$ in D, this leads to a function f defined by:

$$\begin{cases} f'(\tau) = \begin{cases} 0 & \text{if } f(\tau) = 0 \text{ and} \\ & C\zeta(t(\tau))\zeta'(t(\tau)) + \alpha > 0 \\ -C\zeta(t(\tau))\zeta'(t(\tau)) - \alpha & \text{otherwise} \end{cases} \\ f(0) = \max_{-1 \leqq y \leqq 1} u_0(y) . \end{cases}$$

Since

$$\int_0^\infty |\zeta'(t(\tau))|\zeta(t(\tau))\, d\tau = -\int_{-1}^0 \zeta'(t)\, dt = \zeta(-1) < \infty ,$$

it follows immediately from the definition of f that

$$f(\tau) \to 0 \quad \text{as } \tau \to \infty , \tag{4.5}$$

and that, if $\zeta(t)\zeta'(t) = 0$ as $t \to 0$, there exists a $\tau_1 > 0$ such that

$$f(\tau) \to 0 \quad \text{for } \tau \geqq \tau_1 . \tag{4.6}$$

Clearly (1.5) follows from (4.4) and (4.5), while (1.6) is a consequence of (4.4) and (4.6), and so we have proved Theorem A.

5 Theorem B

In this section we consider the case in which

$$\int_{-1}^{0} \frac{1}{\zeta(t)} dt < \infty \tag{5.1}$$

and we construct solutions which do not vanish at the vertex of Q. Theorem B is an immediate consequence of the following lemma:

Lemma 5.1. *Let hypothesis* H *and condition* (5.1) *be satisfied, and let* c *and* a_0 *be constants satisfying*

$$c > \psi_\infty \quad \textit{and} \quad a_0 > c \int_{-1}^{0} \frac{1}{\zeta(t)} dt . \tag{5.2}$$

If

$$u_0(x) \geqq \left[\int_0^x \psi^{-1}\left(- \frac{cs}{\zeta(-1)} \right) ds + a_0 \right]_+ \quad \textit{for } |x| < \frac{\psi_\infty}{c}\zeta(-1) , \tag{5.3}$$

then the solution u *of Problem I satisfies*

$$u(0,t) \geqq a_0 - c \int_{-1}^{0} \frac{1}{\zeta(t)} dt > 0 \quad \textit{for all } t \in [-1, 0) . \tag{5.4}$$

Proof. We define for any $(x,t) \in Q^*$ such that $|x| < \dfrac{\psi_\infty}{c}\zeta(t)$

$$\underline{u}(x,t) \left[\int_0^x \psi^{-1}\left(- \frac{cs}{\zeta(t)} \right) ds + f(t) \right]_+ , \tag{5.5}$$

where $f \in C^1([-1,0))$ is a positive and nonincreasing function to be determined. Let Ω be the subset of the set of definition of $\underline{u}$ in which $\underline{u}$ is strictly positive. Since $\underline{u}$ is nonincreasing with respect to t, it follows that there exists a continuous nonincreasing function $\underline{\zeta}$, which satisfies hypothesis H2, such that

$$\Omega = \{(x,t) \in Q^* : |x| < \underline{\zeta}(t)\} .$$

Hence we obtain from the comparison principle (Theorem 2.12) in Ω that if $\underline{u}$ satisfies

$$\mathcal{L}(\underline{u}) \equiv \underline{u}_t - \psi(\underline{u}_x)_x \leqq 0 \quad \text{in } \Omega , \tag{5.6}$$

then

$$u \geqq \underline{u} \quad \text{in } \Omega \tag{5.7}$$

(we observe that $\underline{u}$ satisfies (2.29) if $x = \pm\underline{\zeta}(t)$).

From (5.5) we find that in Ω

$$\mathscr{L}(\underline{u}) = f'(t) + \frac{c\zeta'(t)}{\zeta^2(t)} \int_0^x s(\psi^{-1})'\left(-\frac{cs}{\zeta(t)}\right)ds + \frac{c}{\zeta(t)} \leqq f'(t) + \frac{c}{\zeta(t)},$$

and thus (5.6) is satisfied if we define $f(t)$ by

$$f(t) = a_0 - c\int_{-1}^{t} \frac{1}{\zeta(s)}ds.$$

Hence we obtain (5.7), which, in view of the definition of $\underline{u}$, yields (5.4).

6 Theorem C

In this section we shall prove Theorem C. By the comparison principle (Theorem 2.12), it is sufficient to consider the case in which

$$\zeta(t) = c\sqrt{-t} \qquad (c > 0). \tag{6.1}$$

Introducing the new variables

$$y = \frac{x}{\sqrt{-t}}, \qquad \tau = -\log(-t),$$

we obtain the following equation for $\tilde{u}(y, \tau) \equiv u(x, t)$:

$$\tilde{u}_\tau = e^{-1/2\tau}\psi(e^{1/2\tau}\tilde{u}_y)_y - \tfrac{1}{2}y\tilde{u}_y \quad \text{in } (-c, c) \times \mathbf{R}^+.$$

Hence the function

$$v(y, \tau) = e^{1/2\tau}\tilde{u}(y, \tau) \quad \text{in } (-c, c) \times \mathbf{R}^+ \tag{6.2}$$

satisfies the equation

$$v_\tau = \psi(v_y)_y - \tfrac{1}{2}yv_y + \tfrac{1}{2}v \quad \text{in } (-c, c) \times \mathbf{R}^+. \tag{6.3}$$

An important role will be played by the steady state problem corresponding to (6.3):

$$(\mathrm{III}_c) \begin{cases} \psi(\varphi')' - \frac{1}{2}y\varphi' + \frac{1}{2}\varphi = 0 & \text{in } (-c, c) \\ \varphi(\pm c) \geqq 0 \text{ and } -\varphi'(\pm c) = \pm\infty & \text{if } \varphi(\pm c) > 0. \end{cases}$$

We shall call $\varphi \in C^2((-c, c)) \cap C([-c, c])$ a *positive solution of* Problem III_c if $\varphi > 0$ in $(-c, c)$ and if φ satisfies the equation and boundary conditions (where $\varphi'(c)$ indicates the one-sided limit $\varphi'(c^-)$) of Problem I.

The proof of Theorem C consists of several lemma's. First we consider the linearized steady state problem.

Lemma 6.1. *For any $c > 0$ the eigenvalue problem*

$$(L_c)\begin{cases} \psi'(0)\varphi'' - \frac{1}{2}y\varphi' = -\lambda\varphi & \text{in } (-c,c) \\ \varphi(\pm c) = 0 \end{cases}$$

has a principal eigenvalue λ_c and a positive eigenfunction $\varphi_c \in C^2([-c,c])$, which is decreasing and concave in $(0,c)$. In addition λ_c satisfies

$$0 < c_1 < c_2 \;\Rightarrow\; \lambda_{c_1} > \lambda_{c_2} > 0\,, \tag{6.4}$$

and

$$\lambda_c \to \begin{cases} 0 & \text{as } c \to \infty \\ \infty & \text{as } c \to 0^+ . \end{cases} \tag{6.5}$$

In particular there exists a unique c_0 such that $\lambda_{c_0} = \frac{1}{2}$ and

$$\lambda_c \begin{cases} > \frac{1}{2} & \text{if } 0 < c < c_0 \\ < \frac{1}{2} & \text{if } c > c_0 . \end{cases} \tag{6.6}$$

Proof. Rewriting the equation in divergence form as

$$\psi'(0)(e^{-y^2/(4\psi'(0))}\varphi')' = -\lambda e^{-y^2/(4\psi'(0))}\varphi \quad \text{in } (-c,c)\,,$$

it follows from standard theory that λ_c exists and that

$$\lambda_c = \psi'(0) \min_{\varphi\in H_0^1((-c,c))}\left\{\int_{-c}^{c} e^{-y^2/(4\psi'(0))}(\varphi')^2 dy;\; \int_{-c}^{c} e^{-y^2/(4\psi'(0))}\varphi^2 dy = 1\right\}. \tag{6.7}$$

In particular the minimum in (6.7) is attained in a positive eigenfunction φ_c, and (6.4) and (6.5) are simple consequences of (6.7).

The existence of c_0 follows at once from (6.4), (6.5) and the continuous dependence of λ_c on c. The monotonicity and concavity of φ_c are an immediate consequence of the equation and the positivity of φ_c.

As a first consequence of Lemma 6.1 we obtain the following result about Problem I:

Lemma 6.2. *Let* H1 *be satisfied and let ζ and c_0 be given by* (6.1) *and* (6.6). *If*

$$c < c_0\,,$$

then there exist initial functions u_0 satisfying H3 *such that the corresponding solutions of Problem I vanish as $(x,t) \to (0,0)$.*

Proof. Let $\delta > -\psi'(0)$ be a constant to be determined, and let $\lambda_{c,\delta}$ and $\varphi_{c,\delta}$ denote, respectively, the principal eigenvalue and a positive eigenfunction of Problem L_c in which $\psi'(0)$ is replaced by $\psi'(0)+\delta$. By (6.6) $\lambda_c > \frac{1}{2}$, and hence we may choose $\delta > 0$ so small that

$$\lambda_{c,-\delta} \geqq \tfrac{1}{2}\,.$$

Choosing $a > 0$ so small that

$$\psi'(a\varphi'_{c,-\delta}(y)) \geqq \psi'(0) - \delta \quad \text{in } (-c,c)\,,$$

we find that $\varphi_{c,-\delta}$ satisfies

$$\psi'(a\varphi')a\varphi'' - \tfrac{1}{2}ya\varphi' + \tfrac{1}{2}a\varphi \leqq \left(\tfrac{1}{2} - \lambda_{c,-\delta}\right)a\varphi \leqq 0 \quad \text{in } (-c,c)\,.$$

Hence, in view of the transformation (6.2), the function $\overline{u}$, defined by

$$\overline{u}(x,t) = a\sqrt{-t}\,\varphi_{c,-\delta}\left(\frac{x}{\sqrt{-t}}\right) \quad \text{for } (x,t) \in Q^*\,,$$

is a supersolution of Problem I if u_0 satisfies

$$u_0(x) \leq a\varphi_{c,-\delta}(x) \quad \text{for } |x| < c\,.$$

Since $\overline{u}(x,t) \to 0$ as $(x,t) \to (0,0)$, it follows from the comparison principle that the solution with initial function u_0 vanishes at $(0,0)$.

Lemma 6.3. *Let* H1 *be satisfied and let* ζ *and* c_0 *be given by* (6.1) *and* (6.6). *If* $c > c_0$, *then there exists* u_0 *satisfying* H3 *such that the solution* u *of Problem I satisfies*

$$u(x,t) \to 0 \quad \text{as } (x,t) \to (0,0) \tag{6.8}$$

if and only if

$$\textit{Problem } III_c \textit{ has a positive solution.} \tag{6.9}$$

Proof. By Lemma 6.1 $\lambda_c < \frac{1}{2}$ and thus we have that $\lambda_{c,\delta} < \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \lambda_c\right)$ for $\delta > 0$ small enough, where $\lambda_{c,\delta}$ and the corresponding positive eigenfunction $\varphi_{c,\delta}$ are defined as in the proof of Lemma 6.2.

Let $\mu > 0$ be a constant to be determined below. Hence there exist arbitrarily small constants $a > 0$ and $\delta > 0$ such that $\lambda_{c,\delta} < \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \lambda_c\right)$ and

$$1 - \mu \leq \frac{\psi'(a\varphi'_{c,\delta})}{\psi'(0)+\delta} \leq 1 \quad \text{in } (-c,c)\,. \tag{6.10}$$

It follows from the second inequality in (6.10) that $\varphi_{c,\delta}$ satisfies

$$\psi'(a\varphi')a\varphi'' - \tfrac{1}{2}ya\varphi' + \tfrac{1}{2}a\varphi \geq \left(\tfrac{1}{2} - \lambda_{c,\delta}\right)a\varphi > 0 \quad \text{in } (-c,c)\,. \tag{6.11}$$

By the comparison principle we may restrict ourselves to solutions of Problem I with initial functions

$$u_0 = a\varphi_{c,\delta}$$

and to steady-state solutions (in the (y,τ) variables) φ which satisfy

$$\varphi \geq a\varphi_{c,\delta} \quad \text{in } (-c,c),$$

with a arbitrarily small. Because of (6.11), the function $v(y,\tau)$, corresponding to the solution $u(x,t)$ of Problem I, is nondecreasing with respect to τ, and we may distinguish two cases:

$$w(y) = \lim_{\tau\to\infty} v(y,\tau) < \infty \quad \text{for } y \in (-c,c) \tag{6.12}$$

and

$$\lim_{\tau\to\infty} v(y,\tau) = \infty \quad \text{for some } y \in (-c,c). \tag{6.13}$$

We claim that, for a and δ sufficiently small, (6.12) implies that

$$w \text{ is a positive solution of Problem III}_c, \tag{6.14}$$

and (6.13) implies that

$$\limsup_{(x,t)\to(0,0)} u(x,t) > 0. \tag{6.15}$$

Obviously (6.13) implies that Problem III_c does not have a positive solution larger than $a\varphi_{a,\lambda}$, while it follows from (6.12) that $u(x,t) \to 0$ as $(x,t) \to (0,0)$. Hence the proof of Lemma 6.3 is complete if we prove (6.14) and (6.15).

First we prove the following monotonicity property of v_y:

$$v_y(y,\tau_2) \leq v_y(y,\tau_1) \quad \text{for} \quad 0 \leq \tau_1 \leq \tau_2,\ 0 \leq y \leq c. \tag{6.16}$$

Setting $z = \psi(v_y)$, z satisfies

$$\begin{cases} z_\tau = \psi'(\psi^{-1}(z))z_{yy} - \frac{1}{2}yz_y & \text{in } (0,c)\times\mathbf{R}^+ \\ z(0,\tau) = 0 & \text{for } \tau > 0 \\ z_y(c,\tau) - \frac{1}{2}c\psi^{-1}(z(c,\tau)) = 0 \quad \text{if } z(c,\tau) > -\psi_\infty & \text{for } \tau > 0. \end{cases}$$

We claim that, for a and δ small enough, at $\tau = 0$

$$\psi'(\psi^{-1}(z))z_{yy} - \tfrac{1}{2}yz_y \leq 0 \quad \text{in } (0,c).$$

Indeed, setting $\varphi = \varphi_{c,\delta}$ and $\lambda = \lambda_{c,\delta}$, we have that

$$\begin{aligned} &\frac{\psi'(0)+\delta}{a\psi'(a\varphi')}\left(\psi'(a\varphi')\psi(a\varphi')'' - \frac{1}{2}y\psi(a\varphi')'\right) \\ &\quad = a\psi''(a\varphi')\frac{\left(\frac{1}{2}y\varphi' - \lambda\varphi\right)^2}{\psi'(0)+\delta} + \left(\frac{\psi'(a\varphi')}{\psi'(0)+\delta} - 1\right)\frac{1}{2}y\left(\frac{1}{2}y\varphi' - \lambda\varphi\right) \\ &\quad\quad + \psi'(a\varphi')\left(\frac{1}{2} - \lambda\right)\varphi' \\ &\quad \leq 0 \quad \text{in } (0,c) \end{aligned}$$

for a and δ small enough, where we have used (6.10) (with μ sufficiently small) and the inequalities $\lambda_{c,\delta} < \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \lambda_c\right)$, $|\psi''(a\varphi')| \leqq aC_1 y$ in $(0, c)$ for some $C_1 > 0$, and $\varphi' \leqq -C_2 y$ in $(0, c)$ for some $C_2 > 0$. Hence it follows from the comparison principle that $\psi(v_y)$ is nonincreasing with respect to τ in $(0, c)$, as long as $\psi(v_y(c, \tau)) > -\psi_\infty$. If there exists $T > 0$ such that $\psi(v_y(c, T)) = -\psi_\infty$, then the monotonicity of $\psi(v_y)$ in the interval (T, ∞) follows from the fact that then $\psi(v_y)$ satisfies the Dirichlet boundary condition $\psi(v_y) = -\psi_\infty$ on $\{c\} \times (T, \infty)$. Thus we have proved (6.16).

Next we claim that the function w, defined by (6.12), is concave. Arguing by contradiction, we suppose that there exist $-c < y_1 < y_2 < y_3 < c$ such that

$$w(y_2) < w(y_1) + \frac{w(y_3) - w(y_1)}{y_3 - y_1}(y_2 - y_1).$$

Let $\varepsilon > 0$ be so small that

$$w(y_2) < w(y_1) - \varepsilon + \frac{w(y_3) - w(y_1)}{y_3 - y_1}(y_2 - y_1). \tag{6.17}$$

Then there exists $\tau_0 > 0$ such that

$$v(y_1, \tau_0) \geqq w(y_1) - \varepsilon \quad \text{and} \quad v(y_3, \tau_0) \geqq w(y_3) - \varepsilon\,, \tag{6.18}$$

and, in the set $(y_1, y_3) \times (\tau_0, \infty)$, $v(y, \tau)$ is a supersolution of the Cauchy-Dirichlet problem

$$\begin{cases} q_\tau = \psi(q_y)_y & \text{in } (y_1, y_3) \times (\tau_0, \infty) \\ q(y_1, \tau) = v(y_1, \tau_0) & \text{for } \tau > \tau_0 \\ q(y_3, \tau) = v(y_3, \tau_0) & \text{for } \tau > \tau_0 \\ q(y, \tau_0) = v(y, \tau_0) & \text{for } y_1 < y < y_3\,, \end{cases}$$

i.e. $v(y, \tau)$ is larger than the corresponding solution $q(y, \tau)$ in $(y_1, y_3) \times (\tau_0, \infty)$. The derivative q_y is bounded, since it is bounded on the parabolic boundary of $(y_1, y_3) \times (\tau_0, \infty)$. This implies that the problem for q is uniformly parabolic and, by standard theory, $q(y, \tau)$ converges to the unique steady state

$$\bar{q}(y) = v(y_1, \tau_0) + \frac{v(y_3, \tau_0) - v(y_1, \tau_0)}{y_3 - y_1}(y - y_1)$$

as $\tau \to \infty$. By (6.17) and (6.18), $w(y_2) < \bar{q}(y_2)$, and hence there exists $\tau_1 > \tau_0$ such that

$$w(y_2) < q(y_2, \tau_1) \leqq v(y_2, \tau_1)\,,$$

and, since $w(y_2) \geqq v(y_2, \tau)$ for all τ, we have found a contradiction. Thus w is concave in $(-c, c)$.

From the concavity of w it follows that w' is locally bounded in $(-c, c)$, and since v_y is monotone with respect to τ in $(0, c)$ and, by symmetry, in $(-c, 0)$, it follows that v_y is uniformly bounded in sets of the form $(-c + \varepsilon, c - \varepsilon) \times \mathbf{R}^+$. Thus, by classical theory, w satisfies the equation $\psi(w')' - \frac{1}{2}yw' + \frac{1}{2}w = 0$ in $(-c, c)$, and it follows easily that w satisfies $w'(c^-) = -\infty$ if $w(c^-) > 0$. Hence w is a positive steady state and we have proved (6.14).

Finally we prove (6.15). The set in which $v(y, \tau)$ tends to infinity as $\tau \to \infty$ is a nonempty connected interval I and we may define $y_0 \in [0, c]$ by $\bar{I} = [-y_0, y_0]$. We claim that $y_0 = c$.

Arguing by contradiction, we define

$$w(y) = \lim_{\tau\to\infty} v(y, \tau) \quad \text{for } y_0 < y \leqq c\,.$$

Arguing as in the proof of (6.14), it follows that w is concave in (y_0, c) and $w(y_0^+) = \infty$. But such a function w does not exist and we have found a contradiction.

Hence $v(y, \tau) \to \infty$ as $\tau \to \infty$ for $|y| < c$, which implies that, for any $|y| < c$,

$$\frac{u(y/\sqrt{-t}, t)}{\sqrt{-t}} \to \infty \quad \text{as } t \to 0^-,$$

and it is not difficult to show that there exists $t_0 \in (-1, 0)$ such that condition (5.3) is satisfied by $u(x, t_0)$, with $t = -1$ replaced by $t = t_0$ (in particular condition (5.2) becomes $a_0 > C\sqrt{-t_0}$ for some $C > 0$). Finally (6.15) follows from Lemma 5.1.

Lemma 6.4. *Let ψ satisfy hypothesis* H1. *Then there exists $c^* \geqq c_0$ such that Problem III_c does not have positive solutions for $c > c^*$, and such that, if $c^* > c_0$, Problem III_c has positive solutions for $c_0 < c < c^*$.*

Proof. In view of the comparison principle and Lemma 6.3, it is sufficient to show that for c large enough Problem III_c does not possess positive solutions.

Let μ be a nonnegative constant such that

$$\psi'(p) < \psi'(0) + \mu \quad \text{if } p > 0\,,$$

and let $\lambda_{c,\mu}$ and $\varphi_{c,\mu}(y)$ be defined as in the proof of Lemma 6.2. By Lemma 6.1,

$$\lambda_{c,\mu} < \tfrac{1}{2}$$

for c large enough, and we claim that for such values of c Problem III_c does not possess positive solutions.

We argue by contradiction and suppose that φ is a positive solution. Let $A > 0$ be defined by

$$A = \max\{a > 0 : a\varphi_{c,\mu} \leqq \varphi \text{ in } (-c, c)\}\,.$$

Setting

$$\mathscr{L}(\varphi) = \psi(\varphi')' - \tfrac{1}{2}y\varphi' + \tfrac{1}{2}\varphi\,,$$

we have $\mathscr{L}(A\varphi_{c,\mu}) > 0$ in $(-c, c)$, and it follows from the maximum principle and the boundary point lemma that there exists $\varepsilon > 0$ such that

$$\varphi - A\varphi_{c,\mu} \geqq \varepsilon\varphi_{c,\mu} \quad \text{in } (-c, c)\,.$$

The positivity of ε is a contradiction with the definition of A.

Substituting $\mu = 0$ in the proof of Lemma 6.4 we find a class of functions ψ for which the constants c^* and c_0 coincide:

Corollary 6.5. *Let ψ satisfy hypothesis* H1 *and let c_0 and c^* be defined by Lemma's* 6.1 *and* 6.4. *If*

$$\psi'(p) < \psi'(0) \quad \text{for } p > 0\,,$$

then $c^* = c_0$.

Proof of Theorem C. Theorem C is a consequence of the Lemma's 6.2, 6.3 and 6.4.

To conclude this section we prove that c^* and c_0 do not coincide for all functions ψ.

Lemma 6.6. *There exist functions ψ which satisfy hypothesis* H1 *and for which $c^* > c_0$ (more precisely, for any constant c there exists a function ψ satisfying* H1 *for which $c^* > c$).*

Proof. Let $\psi'(0)$ be given and let $c > c_0$. We define the function $\bar{v} \in C([-c, c])$ by

$$\bar{v}(y) = \alpha(L^2 - (|y| - y_0)^2)\,,$$

where $\alpha, L > 0$ and $y_0 < 0$. Choosing $L = c - y_0$ we have that $\bar{v}(\pm c) = 0$, $\bar{v} > 0$ in $(-c, c)$, and, for $0 < y < c$,

$$\begin{aligned}\mathcal{L}(\bar{v}) &\equiv \psi(\bar{v}')' - \tfrac{1}{2}y\bar{v}' + \tfrac{1}{2}\bar{v} \\ &= -2\alpha\psi'(-2\alpha(y - y_0)) + \alpha y(y - y_0) + \tfrac{1}{2}\alpha(L^2 - (y - y_0)^2) \\ &\leqq \alpha(-2\psi'(-2\alpha(y - y_0)) + (y_0 + L)L + \tfrac{1}{2}(L^2 - y_0^2))\,.\end{aligned}$$

We observe that $|-2\alpha(y - y_0)|$ belongs to the interval

$$I_\alpha = [2\alpha|y_0|, 2\alpha L]\,,$$

and hence $\mathcal{L}(\bar{v}) \leqq 0$ in $(0, c)$ if ψ satisfies the condition

$$2\psi'(s) \geqq (y_0 + L)L + \tfrac{1}{2}(L^2 - y_0^2) \quad \text{for } |p| \in I_\alpha\,.$$

Since $\bar{v}'(0^+) = 2\alpha y_0 < 0$, the function

$$\sqrt{-t}\,\bar{v}(x/\sqrt{-t})$$

is a supersolution of Problem I if

$$u_0 \leqq \bar{v} \quad \text{in } (-c, c)\,.$$

Hence, by the comparison principle, $u(x, t) \to 0$ as $(x, t) \to (0, 0)$, and thus $c \leqq c^*$.

Appendix: Proof of Lemma 2.8

The elliptic-parabolic Problem II has been extensively studied by Hulshof e.a. [3, 10, 15–17] in the case in which c is uniformly Lipschitz continuous. Most of the results carry over to the more general case in which $c'(-\psi_\infty^-)$ is not necessarily finite. In particular Problem II has a unique weak solution, which satisfies a comparison principle [17] (below we shall use the comparison principle several times and for its precise form we refer to [17]; important is the fact that at the initial time the value of $c(v)$ is important for the comparison principle, rather than the value of v itself).

The properties of the interface $x = \alpha(t)$ were studied in [15,16]. In particular it can be deduced from [16, Theorem 1.1(i)] that α is not necessarily continuous at points at which ζ' vanishes. It remains to prove that

$$-\zeta'(t_0) > 0 \quad \Rightarrow \quad \alpha(t) \text{ is continuous at } t_0. \tag{A.1}$$

Hulshof has proved (A.1) in the case in which c is Lipschitz continuous. His proof yields in addition a modulus of continuity of α. Below we shall indicate a simplification of his proof, which allows us to work with general functions c, but which does not provide a modulus of continuity.

Proof of (A.1). Following [15, Lemma 1], we have immediately from the continuity of $c(v)$ that

$$\limsup_{t\to t_0} \alpha(t) \leqq \alpha(t_0)\,.$$

In particular α is continuous at t_0 if $\alpha(t_0) = 0$.

So let $\alpha(t_0) > 0$. First we prove that

$$\liminf_{t\to t_0^+} \alpha(t) \geqq \alpha(t_0)\,. \tag{A.2}$$

Let $\varepsilon > 0$ be arbitrary and let $\delta_0 > 0$ be such that

$$-\zeta' \geqq \delta_0 > 0 \tag{A.3}$$

in a neighbourhood of t_0. Then the function

$$\bar{v}_\varepsilon(u) = -\psi_\infty + \varepsilon + \delta_0(u - \alpha(t_0))$$

is a supersolution of Problem II in $[0, \alpha(t_0)] \times [t_0, t_\varepsilon]$ for $t_\varepsilon - t_0$ small enough, and hence, by the comparison principle, $\bar{c}(v_\varepsilon(u)) \geqq c(v(u,t))$ in this set. In particular

$$\alpha(t) \geqq \alpha(t_0) - \varepsilon/\delta_0 \quad \text{for } t_0 < t < t_\varepsilon$$

and (A.2) follows.

It remains to show that

$$\liminf_{t\to t_o^-} \alpha(t) \geqq \alpha(t_0)\,. \tag{A.4}$$

First we shall prove that

$$\liminf_{t\to t_0^-} \alpha(t) = \limsup_{t\to t_0^-} \alpha(t)\,. \tag{A.5}$$

We set

$$b_0 = \limsup_{t \to t_0^-} \alpha(t) .$$

Let $t_n \to t_0^-$ as $n \to \infty$ such that $\alpha(t_n) \to b_0$. Since $c(v)$ is continuous, for any $\varepsilon > 0$ there exists a time $t_\varepsilon \in [\tau, t_0)$ such that

$$v(b_0, t) < -\psi_\infty + \varepsilon \quad \text{for } t_\varepsilon \leqq t \leqq t_0 .$$

If we define

$$\bar{v}(u) = -\psi_\infty + \varepsilon + \delta_0(u - b_0) \quad \text{for } 0 \leqq u \leqq b_0 ,$$

there exists $n_\varepsilon \in \mathbf{N}$ such that

$$c(v(u, t_{n_\varepsilon})) \leqq c(\bar{v}(u)) \quad \text{for } 0 \leq u \leq b_0.$$

Hence it follows from the comparison principle that

$$c(v(u(t))) \leq c(\bar{v}(u)) \quad \text{in } [0, b_0] \times [t_{n_\varepsilon}, t_0] ;$$

in particular $\alpha(t) \geqq b_0 - \varepsilon\delta_0$ in (t_ε, t_0), and, since ε is arbitrary, (A.5) follows.

From (A.5) it follows that $\lim_{t \to t_0^-} \alpha(t)$ exists, and to complete the proof of (A.4), we have to show that

$$\lim_{t \to t_0^-} \alpha(t) \geqq \alpha(t_0) . \tag{A.6}$$

Arguing by contradiction, we suppose that

$$a_0 = \lim_{t \to t_0^-} \alpha(t) < \alpha(t_0) .$$

Let $d_0 = \frac{1}{2}(a_0 + \alpha(t_0))$ and

$$\underline{v}(u) = -\psi_\infty < \delta(u - d_0) \quad \text{for } d_0 \leqq u \leqq r ,$$

where $\delta > 0$ and where r is defined by (2.20). By the definition of a_0 and d_0, there exists $t_1 \in [\tau, t_0)$ such that

$$\underline{v}(d_0) = -\psi_\infty < v(d_0, t) \quad \text{for } t_1 \leqq t \leqq t_0 .$$

Using the continuity of $c(v)$, we may choose $\delta > 0$ so small that

$$\underline{v}(r) \leq v(r, t) \quad \text{for } t_1 \leqq t \leqq t_0 ,$$

and

$$c(\underline{v}(u)) \leqq c(v(u, t_1)) \quad \text{for } d_0 \leqq u \leqq r .$$

Hence, by the comparison principle,

$$c(\underline{v}(u)) \leqq c(v(u, t)) \quad \text{in } [d_0, r] \times [t_1, t_0] ;$$

in particular $\alpha(t_0) \leqq d_0$, and we have found a contradiction.

References

1. Bertsch, M., Dal Passo, R.: Hyperbolic phenomena in a strongly degenerate parabolic equation. Arch. Ration. Mech. Anal. **117**, 349–387 (1992)
2. Bertsch, M., Dal Passo, R.: A parabolic equation with a mean-curvature type operator. In: Lloyd, N.G., Ni, W. M., Peletier, L. A., Serrin, J. (eds.) Nonlinear diffusion equations and their equilibrium states, 3. pp. 89–97. Basel, Boston, Stuttgart: Birkhäuser 1992
3. Bertsch, M., Hulshof, J.: Regularity results for an elliptic-parabolic free-boundary problem. Trans. Am. Math. Soc. **297**, 337–350 (1986)
4. Blanc, Ph.: Existence de solutions discontinues pour des équations paraboliques. C.R. Acad. Sci. Paris **310**, 53–56 (1990)
5. Blanc, Ph.: Sur une classe d'équations paraboliques dégénérées à une dimension d'espace possédant des solutions discontinues. Thesis EPFL Lausanne (1989)
6. Blanc, Ph.: A degenerate parabolic equation. In: Clément, Ph., Invernizzi, S., Mitidieri, E., Vrabie, I.I.: Trends in semigroup theory and applications. Lect. Notes Pure Appl. Math. **116**, 59–65 (1989)
7. Blanc, Ph.: On the regularity of the solutions of some degenerate parabolic equations. To appear
8. Dal Passo, R.: Uniqueness of the entropy solution of a strongly degenerate parabolic equation. To appear
9. Dal Passo, R., Ughi, M.: Problème de Dirichlet pour une classe d'équations paraboliques non linéaires dégénérées dans des ouverts non cylindriques. C.R. Acad. Sci. Paris **308**, 555–558 (1989)
10. Van Duyn, C. J., Peletier, L. A.: Nonstationary filtration in partially saturated porous media. Arch. Ration. Mech. Anal. **78**, 173–198 (1982)
11. Evans, L. C., Gariepy, R. F.: Wiener's criterion for the heat equation. Arch. Ration. Mech. Anal. **78**, 293–314 (1982)
12. Gariepy, R. F., Ziemer. W. P.: Thermal capacity and boundary regularity. J. Differ. Equations **45**, 374–388 (1982)
13. Garofalo, N., Lanconelli, E,: Wiener's criterion for parabolic equations with variable coefficients and its consequences. Trans. Am. Math. Soc. **308**, 811–836 (1988)
14. Gilding, B. H.: Hölder continuity of solutions of parabolic equations. J. Lond. Math. Soc. **13**, 103–106 (1976)
15. Hulshof, J.: An elliptic-parabolic free boundary problem: continuity of the interface. Proc. R. Soc. Edinb., Sect. A **106**, 327–339 (1987)
16. Hulshof, J.: The fluid flow in a partially saturated porous medium: behaviour of the free boundary. Thesis University of Leiden, Leiden (1986)
17. Hulshof, J., Peletier, L. A.: An elliptic-parabolic free boundary problem. Nonlinear Anal. Theory Methods Appl. **10**, 1327–1346 (1986)
18. Kondrat'ev, V. A.: Boundary problems for parabolic equations in closed domains. Tr. Mosk. Mat. O.-va **15**, 400–451 (1966) (Russian); Trans. Mosc. Math. Soc. **15**, 450–504 (1966)
19. Ladyzenskaya, O. A., Solonnikov, V. A., Ural'tzeva, N. N.: Linear and quasilinear equations of parabolic type. Translations Math. Monographs 23, American Math. Soc., Providence, RI (1968)
20. Lanconelli, E.: Sul problema di Dirichlet per l'equazione del calore. Ann. Mat. Pura Appl., IV. Ser. **97**, 83–114 (1973)
21. Landis, E. M.: Necessary and sufficient conditions for regularity of boundary points to the Dirichlet problem for the heat-conduction equation. Sov. Math. **10**, 380–384 (1969)
22. Petrovsky, J.: Zur ersten Randwertaufgabe der Wärmeleitungsgleichung. Compos. Math. **1**, 383–419 (1935)
23. Pini, B.: Sulle soluzione generalizzate di Wiener per il primo problema di valori al contorno nel caso parabolico. Rend. Semin. Mat. Univ. Padova **23**, 422–434 (1954)
24. Rosenau, Ph.: Free energy functionals at the high gradient limit. Phys. Rev. A. **41**, 2227–2230 (1990)

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