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# Polynomial boundedness of eigensolutions and the spectrum of Schrödinger operators

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## 1 Introduction and results

We consider the Schrödinger differential expression

$$T = -(\nabla - ib)^2 + V$$

over $\mathbf{R}^n$, where the functions $V: \mathbf{R}^n \to \mathbf{R}$ and $b = (b_1, \ldots, b_n)$: $\mathbf{R}^n \to \mathbf{R}^n$ are assumed to satisfy

$$\forall j \in \{1, \ldots, n\}: b_j \in L_{4,\mathrm{loc}}(\mathbf{R}^n),\ \nabla \cdot b \in L_{2,\mathrm{loc}}(\mathbf{R}^n)\ ,$$

$$V \in L_{2,\mathrm{loc}}(\mathbf{R}^n),\ V_- \in K(\mathbf{R}^n) + O(|x|^2)\ . \qquad \text{(A)}$$

Here $\nabla \cdot b$ is meant in the distributional sense, $V_- := \max\{0, -V\}$ denotes the negative part of $V$ and is decomposed as $V_1 + V_2$ with $V_1$ in the Kato class $K(\mathbf{R}^n)$ (see e.g. Cycon et al. [CFKS, Definition 1.10]) and $(1 + |\cdot|)^{-2} V_2$ bounded, where we may assume $V_1, V_2 \geqq 0$.

Under these conditions, $T | C_0^\infty(\mathbf{R}^n)$ defines an essentially self-adjoint operator in $L_2(\mathbf{R}^n)$. For $V_- = V_1$ this is claimed by Simon in [Si3, Theorem B.13.4], where Leinfelder and Simader [LS] is given as a reference for a proof. However, their Theorem 4 does not cover the situation given by (A) completely, since they have to assume $V_1$ to be $\Delta$-bounded. Therefore we will sketch a proof of essential self-adjointness on their line in Sect. 2.4.

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The self-adjoint operator defined as the closure of $T | C_0^\infty(\mathbf{R}^n)$ will be called $T$ again. The goal of this note is to provide a complete characterization of the spectrum of $T$ in terms of weak eigensolutions, i.e. functions $u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ satisfying

$$\forall \varphi \in C_0^\infty(\mathbf{R}^n)\colon \int \bar{u} T\varphi = \lambda \int \bar{u}\varphi$$

for some $\lambda \in \mathbf{R}$. (We then write $Tu = \lambda u$.) We will show that the spectrum is the closure of the set of those real numbers for which there is a polynomially bounded eigensolution. More precisely, our principal result reads as follows.

**Main Theorem.** *Let $b$ and $V$ satisfy* (A). *Assume in addition that* $|x|^{-2}V_2(x) \to 0$, *as* $|x| \to \infty$. *Then*

$$\sigma(T) = \overline{\{\lambda \in \mathbf{R}; \exists s > 0\ \exists u \neq 0, (1 + |\cdot|)^{-s} u \in L_\infty(\mathbf{R}^n)\colon Tu = \lambda u\}}\ .$$

The Main Theorem is an immediate consequence of the following two assertions which will be proved in Chap. 3.

**Proposition 1.** *Let $b$ and $V$ satisfy* (A). *Then*

$$\sigma(T) \subset \overline{\{\lambda \in \mathbf{R}; \exists s > 0\ \exists u \neq 0, (1 + |\cdot|)^{-s} u \in L_\infty(\mathbf{R}^n)\colon Tu = \lambda u\}}\ .$$

**Proposition 2.** *Let $b$ and $V$ satisfy the assumptions of the Main Theorem. Then*

$$\{\lambda \in \mathbf{R}; \exists s > 0\ \exists u \neq 0, (1 + |\cdot|)^{-s} u \in L_\infty(\mathbf{R}^n)\colon Tu = \lambda u\} \subset \sigma(T)\ .$$

Our proof of these two propositions will actually only cover the case $n \geqq 3$, because of citations from other work, where this restriction is made for convenience. However, all what is needed here also holds for $n = 1$ and $n = 2$.

The spectrum of Schrödinger operators $T$ consists of those energy levels, e.g. for systems of atomic or molecular particles, which are physically permissible, while bounded eigensolutions represent states where the particles live somewhere, though not necessarily localizable. The truth of the statement of our Main Theorem with "polynomial boundedness" replaced by "boundedness", i.e. $s = 0$, is thus a familiar assumption in Quantum Mechanics. To our knowledge, it is an open mathematical problem, whether Proposition 1 and therefore the Main Theorem should hold with $s = 0$ (see the discussion on p. 509 of [Si3]).

We want to point out that our result is optimal in several respects:

(i) The assumption of essential self-adjointness on $C_0^\infty(\mathbf{R}^n)$ is basic for the treated question. It shows that weak solutions as defined above make sense and it implies the desirable fact that a weak solution of $Tu = \lambda u$ is an eigenfunction of $T$ if and only if $u \in L_2(\mathbf{R}^n)$.

(ii) Once essential self-adjointness on $C_0^\infty(\mathbf{R}^n)$ is required for $T = -(\nabla - ib)^2 + V$, assumption (A) will be difficult to beat (despite for pathological situations): $b_j \in L_{4,\mathrm{loc}}(\mathbf{R}^n)$, $\nabla \cdot b \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ and $V \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ are necessary for $C_0^\infty(\mathbf{R}^n) \subset D(T)$. For the global behavior of $V_-$ like $O(|x|^\alpha)$, $\alpha = 2$ is known to be the borderline for essential self-adjointness. Finally the Kato class has turned out to be the biggest "nice" class describing local singularities. Of course, it contains physical N-body potentials.

(iii) The additional assumption $|x|^{-2}V_2(x) \to 0$ in Proposition 2 can not be dropped, i.e. weakened to boundedness of $|x|^{-2}V_2(x)$ as in (A). At least in the case $n = 1$ this follows from an example of Halvorsen [Ha]. His example actually treats the analogous question for operators on the half-line, but is easily seen to imply a similar example on the whole line. The discussion of this example in Hinz [Hi2, Chap. 5] shows that for $n \geqq 2$ the question is open if the condition $|x|^{-2} V_2(x) \to 0$ can be dropped in Proposition 2, and consequently in the Main Theorem.

The mathematical history of both problems contained in Propositions 1 and 2, respectively, started in the early 50s with the same person, Eh. Eh. Shnol', but followed quite different tracks thereafter.

Proposition 2, with $b = 0$ and $V$ continuous with $|x|^{-2} V_-(x) \to 0$, was proved by Shnol' in 1953 [Sh1, Theorem 1]. The later development concentrated on weakening the local assumptions on $V$. Hinz, based on Shnol''s method, showed that continuity of $V$ can be replaced by a local Stummel condition [Hi1, Corollary 3], while Simon [Si2, Theorem 1.2], employing semigroup techniques, needs only a local Kato condition, but has to assume $V_-$ to belong to the global Kato class. In [Si3] he reconsiders Shnol''s method (second proof of Theorem C.4.1; cf. [CFKS, Theorem 2.9]), which is in fact strong enough, as pointed out in [Hi2, Corollary 4.7], to recover Shnol''s result assuming only $V_- \in K_{\text{loc}}$ and $|x|^{-2} V_-(x) \to 0$, but with no restrictions on $V_+ := V + V_-$ at all.

Its basic idea, namely to construct a kind of singular sequence by cutting off the given polynomially bounded eigensolution outside balls with increasing diameters, will also be employed in our proof of Proposition 2 in Sect. 3.2. We are able to allow for non-vanishing $b$ by the observation that relative form boundedness with respect to $-\Delta$ implies relative form boundedness with respect to $-(\nabla - ib)^2$ and with the aid of an interpolation lemma which guarantees square-integrable first derivatives for locally bounded weak solutions. These tools are provided in Chap. 2 (Lemmas 2.3 and 2.2, respectively).

In our proof of Proposition 1 in Sect. 3.1 we will use results on the existence of expansions in generalized eigenfunctions for Schrödinger operators. A history of this subject, which actually started in 1954 with a result on the half-line of Shnol' [Sh2, Theorem 1], can be found in Simon's account on eigenfunction expansions [Si3, C.5]. The latter also contains a proof of Proposition 1 for $b = 0$, $V \in K_{\text{loc}}$ and $V_- \in K + O(|x|)$. Results for non-zero $b \in C^1$ and $V_- = O(|x|^2)$ under stronger regularity assumptions on $V$ (Stummel condition) were given by Stolz [St1, Satz 1.4], see also [St2].

Whereas former proofs of eigenfunction expansion theorems showed that generalized eigenfunctions belong to polynomially weighted $L_2$-spaces, a different method of proof given in Poerschke et al. [PSW] in addition yields a priori information on the gradients of generalized eigenfunctions. Theorem 3.1 and Corollary 3.2 show how to apply this to Schrödinger operators with $b$ and $V$ satisfying (A). Since we are interested in singular magnetic potentials $b$, we will need these gradient properties in our proof of Proposition 1 in order to apply Kato's inequality. We provide an appropriate version of the latter in Sect. 2.1. From this we find local boundedness of weak solutions and finally are able to transform $L_2$-bounds on eigensolutions into pointwise bounds with the help of a mean value inequality [Hi2, Corollary 2.14].

## 2 Tools

In this chapter we provide some regularity properties of the operator $T$ and its eigensolutions, which are of some interest on their own and will be the main ingredients for the proofs of Propositions 1 and 2 in Chap. 3.

### 2.1 Kato's inequality

Kato introduced his famous inequality in [K2] to put aside the positive part $V_+$ of the potential in the question of self-adjointness of Schrödinger operators $T$. Compare [Hi2], where it has been made the base for local bounds on eigenfunctions $u$. This is also the way we use it here, but since our magnetic terms $b_j$ are not in $C^1$, as assumed by Kato, we have to strengthen the assumptions on $u$ in demanding some $L_p$-property of $\nabla u$.

If fact, Kato's inequality allows some singularity of $b$ without strenghtening the assumptions on $u$, compare Simon [Si1], but this does not cover $b$s as in (A). Actually, this was the reason that Leinfelder and Simader [LS] had to use new methods in their proof of essential self-adjointness for operators including such magnetic potentials.

In our applications, however, we will have sufficient a priori information about $u$ to use the following version of Kato's inequality.

**Lemma 2.1.** *Let $b$ satisfy* (A). *Then for any $u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ with $\nabla u \in L_{\frac{4}{3},\mathrm{loc}}(\mathbf{R}^n)$ and $(\nabla - ib)^2 u \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$:*

$$\Delta |u| \geqq \mathrm{re}[(\mathrm{sgn}\, u)(\nabla - ib)^2 u]$$

*holds in the sense of distributions. Here*

$$(\mathrm{sgn}\, u)(x) = \begin{cases} 0 \quad , & \text{if } u(x) = 0 \ ; \\ \frac{\overline{u(x)}}{|u(x)|}, & \text{if } u(x) \neq 0 \ . \end{cases}$$

*Proof.* We refer to Reed and Simon [RS, p. 189f]. Since they assume $b_j \in C^1$, we have to replace their lemma by the remark that our assumptions imply

$$(\nabla - ib)^2 u = \Delta u - i(\nabla \cdot b)u - 2ib \cdot \nabla u - |b|^2 u$$

in the distributional sense, thus showing that $\Delta u \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$ and hence

$$(\nabla - ib)^2 u_\delta \to (\nabla - ib)^2 u \text{ in } L_{1,\mathrm{loc}}(\mathbf{R}^n) \ ,$$

$u_\delta$ being the mollified $u$. The rest of the proof is identical to the one given in [RS]. □

### 2.2 $L_2$-property of gradients

Once local boundedness of eigensolutions is established, it will be necessary to have some information about their first derivatives. Lemma 3 of [Hi1] implies that $\nabla u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$, if $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$ and $\Delta u \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$. This holds true if we replace $\Delta$ by $(\nabla - ib)^2$ under the weakest possible assumptions on $b$.

**Lemma 2.2.** *Let $b\colon \mathbf{R}^n \to \mathbf{R}^n$ with $b_j \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ for any $j \in \{1, \ldots, n\}$ and $\nabla \cdot b \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$. If $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$ and $(\nabla - ib)^2 u \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$, then $\nabla u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$.*

*Proof.* In view of Weidmann [W, Theorem 4.25], it suffices to prove (cf. [Hi1, p. 175])

$$\forall R > 0 \ \exists c_R > 0 \ \forall \varepsilon \in ]0, 1]\colon \int_{B(0;R)} |\nabla u_\varepsilon(x)|^2 \, dx \leqq c_R \ ;$$

here $u_\varepsilon$ denotes the mollified $u$. We have for any $x$:

$$((\nabla - ib)^2 u)_\varepsilon(x) = \Delta(u_\varepsilon)(x) - i((\nabla\cdot b)u)_\varepsilon(x) + 2i \sum_{j=1}^{n} \partial_j((b_j u)_\varepsilon)(x) - (|b|^2 u)_\varepsilon(x)\,.$$

With $\zeta$ a smooth function with support in $B(0; 2R)$ and $\zeta = 1$ in $B(0; R)$, we may write

$$-\int \overline{u_\varepsilon}((\nabla - ib)^2 u)_\varepsilon \zeta = \int |\nabla(u_\varepsilon)|^2 \zeta + \int \overline{u_\varepsilon}\, \nabla u_\varepsilon \cdot \nabla\zeta + i \int \overline{u_\varepsilon}((\nabla\cdot b)u)_\varepsilon \zeta$$
$$+ \int \overline{u_\varepsilon}(|b|^2 u)_\varepsilon \zeta - 2i \sum_{j=1}^{n} \int \overline{u_\varepsilon}\partial_j((b_j u)_\varepsilon)\zeta\,.$$

Considering only the real part of this equation and making use of

$$\operatorname{re} \int \overline{u_\varepsilon}\, \nabla u_\varepsilon \cdot \nabla\zeta = -\tfrac{1}{2}\int |u_\varepsilon|^2 \,\Delta\zeta$$

and

$$2\left|\int \partial_j(\overline{u_\varepsilon})(b_j u)_\varepsilon \zeta\right| \leqq \tfrac{1}{2}\int |\partial_j u_\varepsilon|^2 \zeta + 2\int |(b_j u)_\varepsilon|^2 \zeta\,,$$

we arrive at

$$\int_{B(0;R)} |\nabla u_\varepsilon|^2 \leqq \int |\nabla u_\varepsilon|^2 \zeta$$
$$\leqq \| u \|_\infty \|(\nabla - ib)^2 u\|_1 + \frac{1}{2}\frac{\sigma_n}{n}(2R)^n \| u \|_\infty^2 \|\Delta\zeta\|_\infty$$
$$+ \| u \|_\infty^2 \|\nabla\cdot b\|_1 + \| u \|_\infty^2 \||b|\|_2^2 + 2n \| u \|_\infty^2 \||b|\|_2^2$$
$$+ 2\sum_{j=1}^{n} \| u \|_\infty \| u \|_2 \| b_j \|_2 \|\partial_j\zeta\|_\infty + \frac{1}{2}\int |\nabla u_\varepsilon|^2 \zeta\,,$$

where $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ denote the norms in $L_1$, $L_2$, and $L_\infty$ of $B(0; 2R+1)$, respectively. □

### 2.3. *Relative form boundedness*

Let $V_+ \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$ be non-negative and $b_j \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ be real-valued for any $j \in \{1, \ldots, n\}$. We define $D_j := \partial_j - ib_j$. Then the quadratic form $q_0$ with domain

$$D(q_0) = \{ f \in L_2(\mathbf{R}^n);\ V_+^{1/2} f \in L_2(\mathbf{R}^n), \forall j \in \{1, \ldots, n\}: D_j f \in L_2(\mathbf{R}^n)\}$$

and given by

$$q_0(f, g) = \sum_{j=1}^{n} (D_j f, D_j g) + (V_+^{1/2} f, V_+^{1/2} g)$$

is closed and positive. Let $T_0$ be the self-adjoint operator associated with $q_0$. Then form boundedness with respect to $-\Delta$ implies form boundedness with respect to $T_0$.

**Lemma 2.3.** *Let $V_1 \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$ be relatively form bounded with respect to $-\Delta$ with relative bound 0. Then $V_1$ is also relatively form bounded with respect to $T_0$ with relative bound 0.*

*Proof.* Since $C_0^\infty(\mathbf{R}^n)$ is a form core of $T_0$[LS, Theorem 1], it suffices to show that

$$\forall \iota > 0\ \exists c > 0\ \forall \varphi \in C_0^\infty(\mathbf{R}^n): |(\phi, V_1\varphi)| \leqq \iota \| |(\nabla - ib)\varphi| \|^2 + c\|\varphi\|^2 . \quad (1)$$

By assumption we have for the mollified $|\varphi|$:

$$|(|\varphi|_\varepsilon, V_1|\varphi|_\varepsilon)| \leqq \iota \| |(\nabla(|\varphi|_\varepsilon)| \|^2 + c\| |\varphi|_\varepsilon \|^2 ,$$

such that, since $|\varphi| \in W_2^1(\mathbf{R}^n)$,

$$|(\varphi, V_1\varphi)| \leqq \iota \| |\nabla(|\varphi|)| \|^2 + c\|\varphi\|^2 . \quad (2)$$

With

$$2|\varphi|\partial_j|\varphi| = \partial_j(|\varphi|^2) = \bar\varphi\partial_j\varphi + \varphi\partial_j\bar\varphi = \bar\varphi D_j\varphi + \varphi\overline{D_j\varphi} = 2\,\mathrm{re}(\bar\varphi D_j\varphi) ,$$

we get

$$|\varphi| |\nabla|\varphi\| \leqq |\varphi||(\nabla - ib)\varphi| .$$

From this it is immediate that (cf. Eastham and Kalf [EK, p. 239])

$$|\nabla|\varphi\| \leqq |(\nabla - ib)\varphi| ; \quad (3)$$

in fact, if $\varphi(x) = 0$, then the right hand side equals $|\nabla\varphi|$, and (3) follows from Gilbarg and Trudinger [GT, Lemma 7.6] in that case.

Inserting (3) into (2) yields (1). □

Lemma 2.3 allows to associate a self-adjoint operator $T_1$ with form domain $Q(T_1) = Q(T_0) = D(q_0)$ to the form $q_1(f, g) = q_0(f, g) - \int \bar f V_1 g$. The following property of the domain of $T_1$ will be useful later.

**Lemma 2.4.** *Let $g \in D(T_1)$ and $\psi \in C^2(\mathbf{R}^n) \cap W_\infty^2(\mathbf{R}^n)$. Then $\psi g \in D(T_1)$ and*

$$T_1(\psi g) = \psi T_1 g - (\Delta\psi)g - 2(\nabla\psi)\cdot(\nabla - ib)g . \quad (4)$$

*Proof.* The first representation theorem for quadratic forms (see Kato [K1, p. 322]) says that

$$D(T_1) = \{f \in Q(T_1); \exists h \in L_2(\mathbf{R}^n)\,\forall\varphi \in Q(T_1): (h, \varphi) = q_1(f, \varphi)\} , \quad (5)$$

and in this case $T_1 f = h$. Since $\psi g \in Q(T_1) = D(\nabla - ib) \cap D(V_+^{1/2})$, we deduce for $\varphi \in Q(T_1)$:

$$\begin{aligned} q_1(\psi g, \varphi) &= \int \overline{(\nabla - ib)(\psi g)}\cdot(\nabla - ib)\varphi + \int \overline{\psi g}(V_+ - V_1)\varphi \\ &= \int \overline{(\nabla - ib)g}\cdot(\nabla - ib)(\bar\psi\varphi) + \int \bar g(V_+ - V_1)\bar\psi\varphi \\ &\quad + \int \overline{(\nabla\psi)g}\cdot(\nabla - ib)\varphi - \int \overline{(\nabla - ib)g}\cdot(\nabla\bar\psi)\varphi \\ &= (T_1 g, \bar\psi\varphi) - \int \overline{(\Delta\psi)g}\varphi - 2\int \overline{(\nabla - ib)g}\cdot(\nabla\bar\psi)\varphi \\ &= (\psi T_1 g - (\Delta\psi)g - 2(\nabla\psi)\cdot(\nabla - ib)g, \varphi) , \end{aligned}$$

and (4) follows from (5). □

*2.4 Essential self-adjointness*

The goal of this section is the following.

**Theorem 2.5.** *Let $b$ and $V$ satisfy* (A). *Then* $-(\nabla - ib)^2 + V$ *is essentially self-adjoint on* $C_0^\infty(\mathbf{R}^n)$.

For $b = 0$, this result has been obtained in [Hi2, Theorem 3.4] for the even larger class of potentials $V \in L_{2,\text{loc}}(R^n)$ with $V_- \in K_{(1+|\cdot|)^{-1}}(\mathbf{R}^n)$. The proof there consists in a reduction to the case $V_- = V_1 \in K(\mathbf{R}^n)$ by cutting off $V_-$ outside compact sets. This reduction carries over to the case of non-vanishing $b$ with minor modifications, using the fact that (local) form boundedness with respect to $-\Delta$ implies (local) form boundedness with respect to $-(\nabla - ib)^2$ as in the proof of Lemma 2.3.

So we are left with proving Theorem 2.5 for $V_- = V_1 \in K(\mathbf{R}^n)$. This in turn can be done as with Corollary 3.3 in [Hi2], provided that $b \in C^1$, since then Kato's inequality holds for $u$, $\Delta u \in L_{1,\text{loc}}(\mathbf{R}^n)$, and local boundedness of eigensolutions can be established as in [Hi2, Chap. 2]. (Boundedness from below of $T$ is again a consequence of Lemma 2.3.) For general $b$, however, we have to adopt the method of Leinfelder and Simader, that is we prove, with $T_1$ as in Sect. 2.3:

**Lemma 2.6.** $C_0^\infty(\mathbf{R}^n)$ *is a core for* $T_1$.

The *proof* of this lemma may be taken from the proof of Theorem 2 in [LS] (which shows Lemma 2.6 in the $V_1 = 0$ case), after the following two facts have been established:

(i) Let $m \in \mathbf{R}$ such that $T_1 + m \geqq 1$. Then

$$\mathscr{C} := \{\varphi u;\ \varphi \in C_0^\infty(\mathbf{R}^n),\ u \in (T_1 + m)^{-1}(L_2(\mathbf{R}^n) \cap L_\infty(\mathbf{R}^n))\}$$

is a core for $T_1$.

(ii) $\mathscr{C} \subset L_\infty(\mathbf{R}^n) \cap W_2^2(\mathbf{R}^n) \cap W_4^1(\mathbf{R}^n)$.

So it remains to prove (i) and (ii).

*Proof of* (*i*). This is shown as in the proof of (3.16) of [LS], taking into account Lemma 2.4. □

The more complicated proof of (ii) will be prepared by two lemmas.

**Lemma 2.7.** *Let $b_n \to b$ in* $L_{2,\text{loc}}(\mathbf{R}^n)$, $V_+ \geqq 0$, $V_+ \in L_{1,\text{loc}}(\mathbf{R}^n)$, $V_1 \geqq 0$ *relatively formbounded with respect to* $-\Delta$ *with relative bound* 0. *Let $T_1$ (resp. $T_{n,1}$) be defined by using $b$, $V_+$ and $V_1$ (resp. $b_n$, $V_+$ and $V_1$) as in Sect.* 2.3. *Then $T_{n,1} \to T_1$ in the sense of strong resolvent convergence.*

*Proof.* Here the proof of [LS, (3.17)] can be mimicked, noting that the additional term arising from $V_1$ can be absorbed in the argument there by using its relative form boundedness. □

**Lemma 2.8.** *Let $m \in \mathbf{R}$, $u \in D(T_1)$ and $f := (T_1 + m)u \in L_{\infty,\text{loc}}(\mathbf{R}^n)$, with $T_1$ as in Lemma 2.6. Then $u \in L_{\infty,\text{loc}}(\mathbf{R}^n)$, and local $L_\infty$-bounds for $u$ are independent of $b$ and* $(V_+ - V_1 + m)_+$.

*Proof.* From $u \in D(T_1) \subset Q(T_1)$ it follows that $(\nabla - ib)u \in L_2(\mathbf{R}^n)$. Using also $bu \in L_{\frac{4}{3},\mathrm{loc}}(\mathbf{R}^n)$, one gets $\nabla u \in L_{\frac{4}{3},\mathrm{loc}}(\mathbf{R}^n)$. Lemma 2.1 applies, giving

$$\begin{aligned} \Delta|u| &\geqq \mathrm{re}[(\mathrm{sgn}\, u)(\nabla - ib)^2 u] \\ &= \mathrm{re}[(\mathrm{sgn}\, u)(f - (V_+ - V_1 + m)u)] \\ &\geqq -|f| - (V_+ - V_1 + m)_- |u|, \end{aligned}$$

i.e. $\Delta|u| + Q|u| + |f| \geqq 0$ with $Q = (V_+ - V_1 + m)_- \in K(\mathbf{R}^n)$. Now Lemma 2.4 of [Hi2] in connection with the method of the proof of Theorem 2.1 of [Hi2] yields $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$ with bounds independent of $b$ and $(V_+ - V_1 + m)_+$. □

The *proof of* (*ii*) is now very similar to that of [LS, Lemma 9]. Nevertheless, we prefer to give some details.

By mollifying $b$ we find $b_n \in C^\infty$ such that $b_n \to b$ in $L_{4,\mathrm{loc}}(\mathbf{R}^n)$ and $\nabla \cdot b_n \to \nabla \cdot b$ in $L_{2,\mathrm{loc}}(\mathbf{R}^n)$. Let $T_{n,1}$ be defined as in Lemma 2.7. Then the proof of Lemma 2.3 shows that there is an $m \geqq 0$ such that $T_1 + m \geqq 1$ and $T_{n,1} + m \geqq 1$ for every $n$.

Now let $\varphi \in C_0^\infty(\mathbf{R}^n)$ and $u \in (T_1 + m)^{-1}(L_2(\mathbf{R}^n) \cap L_\infty(\mathbf{R}^n))$. Then $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$ by Lemma 2.8, i.e. $\varphi u \in L_\infty(\mathbf{R}^n)$ proving the first of the three assertions in (ii).

By Lemma 2.7 (cf. [W, Theorem 9.15], which extends to $z = -m$)

$$u_n := (T_{n,1} + m)^{-1}(T_1 + m)u \to u \quad \text{in} \quad L_2(R^n)\,. \tag{6}$$

Furthermore

$$\|u_n\| \leqq \|(T_1 + m)u\|\,, \tag{7}$$

and Lemma 2.8 yields $u_n \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$ and

$$\|\varphi u_n\|_\infty \leqq C_1 < \infty \tag{8}$$

with $C_1$ independent of $n$. We also have

$$\begin{aligned} \|(\nabla - ib_n)u_n\|^2 &= ((T_{n,1} + m)u_n, u_n) - \int (V_+ - V_1 + m)|u_n|^2 \\ &\leqq ((T_1 + m)u, u_n) + \int V_1 |u_n|^2 \\ &\leqq ((T_1 + m)u, u_n) + \tfrac{1}{2}\|(\nabla - ib_n)u_n\|^2 + C_2\|u_n\|^2\,, \end{aligned}$$

with $C_2$ independent of $n$. Using (7) one gets

$$\|(\nabla - ib_n)u_n\| \leqq C_3\|(T_1 + m)u\|\,.$$

Let $v_n := \varphi u_n$, then by Lemma 2.4 $v_n \in D(T_{n,1})$ and

$$T_{n,1}v_n = \varphi T_{n,1}u_n + 2\nabla\varphi \cdot (\nabla - ib_n)u_n + (\Delta\varphi)u_n$$

and therefore

$$\begin{aligned} \|T_{n,1}v_n\| &\leqq \|\varphi\|_\infty(\|(T_{n,1} + m)u_n\| + m\|u_n\|) \\ &\quad + 2\|\nabla\varphi\|_\infty\|(\nabla - ib_n)u_n\| + \|\Delta\varphi\|_\infty\|u_n\| \\ &\leqq \|(T_1 + m)u\|((m + 1)\|\varphi\|_\infty + 2C_3\|\nabla\varphi\|_\infty + \|\Delta\varphi\|_\infty) =: a\,. \end{aligned}$$

From $v_n \in D(T_{n,1})$ we have $(\nabla - ib_n)v_n \in L_2(\mathbf{R}^n)$, so $\nabla v_n \in L_2(\mathbf{R}^n)$ by $b_n v_n \in L_2(R^n)$. This means that $T_{n,1}v_n$ can be computed as a distribution (compare with (3.13) of [LS]) to give

$$T_{n,1}v_n = -\Delta v_n + 2ib_n \cdot \nabla v_n + (i\nabla \cdot b_n + |b_n|^2 + V_+ - V_1)v_n ,$$

proving $v_n \in W_2^2(\mathbf{R}^n)$ (use (8)). As in the proof of [LS, Lemma 8] one gets

$$\| \Delta v_n \| \leqq 2 \| T_{n,1} v_n \| + d \| v_n \|_\infty \leqq 2a + dC_1 < \infty .$$

Now the proof of $\varphi u \in W_2^2(\mathbf{R}^n)$ is completed as the proof of [LS, Lemma 9] by using (6).

The remaining assertion $\varphi u \in W_4^1(\mathbf{R}^n)$ follows from $\varphi u \in L_\infty(\mathbf{R}^n) \cap W_2^2(\mathbf{R}^n)$ by [LS, Lemma 7]. □

## 3 Proof of the Main Theorem

As obvious from the introduction, the proof of our Main Theorem is decomposed in a natural way into showing Propositions 1 and 2 separately. This will be achieved in the following two sections.

### *3.1 Proof of Proposition 1*

The key in the proof of Proposition 1 is to use results of [PSW] on the existence of expansions in generalized eigenfunctions. To this end we prove the following theorem, where $T_1$ is defined as in Sect. 2.3. (Note that $V_1 \in K(\mathbf{R}^n)$ is relatively form bounded with respect to $-\Delta$ with relative bound 0.) From Theorem 2.5 we know that, under the assumptions of Proposition 1, $T$ and $T_1$ are essentially self-adjoint on $C_0^\infty(\mathbf{R}^n)$. For any real $s$ let $k_s := (1 + |\cdot|^2)^{s/2}$ on $\mathbf{R}^n$.

**Theorem 3.1.** *Let $z \in \rho(T)$. Then $k_{-s}(T-z)^{-\ell}$ and $T_1 k_{-s}(T-z)^{-\ell}$ are Hilbert-Schmidt operators if $s > 0$ and $\ell \in \mathbf{N}$ are sufficiently large.*

Before we present the proof of this theorem, we will show how it can be used to prove Proposition 1 readily.

**Corollary 3.2.** *Let $b$ and $V$ satisfy* (A). *Then*

$$\sigma(T) \subset \overline{\{\lambda \in \mathbf{R}; \exists s > 0\ \exists u \in k_s L_2(\mathbf{R}^n) \setminus \{0\},\ \nabla u \in L_{\frac{4}{3},\mathrm{loc}}(\mathbf{R}^n)\colon Tu = \lambda u\}} .$$

*Proof.* An application of Theorems 1 and 2 of [PSW] using Theorem 3.1 shows the existence of an expansion in generalized eigenfunctions $u \in k_s D(T_1)$ corresponding to $T$. In particular this means that a non-trivial solution $u \in k_s D(T_1)$ of $Tu = \lambda u$ exists for almost every $\lambda$ with respect to a spectral measure for $T$. The set of these $\lambda$s being relatively dense in $\sigma(T)$, we get

$$\sigma(T) \subset \overline{\{\lambda \in \mathbf{R}; \exists s > 0\ \exists u \in k_s D(T_1) \setminus \{0\}\colon Tu = \lambda u\}} .$$

Since the form domain of $T_1$ is given by

$$Q(T_1) = \{ f \in L_2(\mathbf{R}^n); V_+^{1/2} f \in L_2(\mathbf{R}^n), \forall j \in \{1, \ldots, n\}\colon D_j f \in L_2(\mathbf{R}^n)\} ,$$

we have in particular

$$D(T_1) \subset Q(T_1) \subset D(\nabla - ib) \subset W^1_{\frac{4}{3},\mathrm{loc}}(\mathbf{R}^n)\,,$$

the last inclusion following from $b_j \in L_{4,\mathrm{loc}}(\mathbf{R}^n)$. □

*Proof of Proposition* 1. By Corollary 3.2 it remains to show that $(1 + |\cdot|)^{-s}u \in L_2(\mathbf{R}^n)$ with $\nabla u \in L_{\frac{4}{3},\mathrm{loc}}(\mathbf{R}^n)$ and $Tu = \lambda u$ implies $(1 + |\cdot|)^{-s'}u \in L_\infty(\mathbf{R}^n)$ for some $s' > 0$. As mentioned before, this can be done directly by the methods of [Hi2], provided $b \in C^1$. In our general case, however, we are obliged to use Kato's inequality in the version of Lemma 2.1, which is possible since $(\nabla - ib)^2 u = (V - \lambda)u \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$. We get

$$\begin{aligned} \Delta|u| &\geqq \mathrm{re}[(\mathrm{sgn}\, u)(\nabla - ib)^2 u] = \mathrm{re}[(\mathrm{sgn}\, u)(V - \lambda)u] \\ &= (V - \lambda)|u| \geqq -(V - \lambda)_-|u|\,. \end{aligned}$$

With $p := (V - \lambda)_- \in K_{\mathrm{loc}}(\mathbf{R}^n)$ and $v := |u|$ we are in the situation of [Hi2, Theorem 2.1], whence $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$. Now Lemma 2.2 may be employed, which guarantees $\nabla u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ and consequently $|u| \in W^1_{2,\mathrm{loc}}(\mathbf{R}^n)$ [GT, L.7.6]. Since $V_- \in K(\mathbf{R}^n) + O(|x|^2) \subset K_{(1+|\cdot|)^{-1}}(\mathbf{R}^n)$ by [Hi2, Proposition 1.5], the mean value inequality [Hi2, Corollary 2.14] applies, and we arrive at

$$\exists c > 0\ \forall x \in \mathbf{R}^n\colon |u(x)|^2 \leqq c(1 + |x|)^n \int_{B(x;1)} |u(y)|^2\, dy\,.$$

From this $(1 + |\cdot|)^{-s'}u \in L_\infty(\mathbf{R}^n)$ follows with $s' = s + \frac{n}{2}$. □

The rest of this section is devoted to the proof of Theorem 3.1, which will be achieved in a series of lemmas. At first it will be sufficient to assume $b_j \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$, $j \in \{1, \ldots, n\}$, and $V_+ \in L_{1,\mathrm{loc}}(\mathbf{R}^n)$. Let the self-adjoint operator $T_0$ be defined as in Sect. 2.3.

**Lemma 3.3.** *$(T_0 + 1)^{-\ell}$ is bounded as an operator from $L_2(\mathbf{R}^n)$ into $L_\infty(\mathbf{R}^n)$ for real $\ell \geqq \ell_0$, where $\ell_0$ is the smallest integer which is bigger than $\frac{n}{4}$.*

*Proof.* From Lemma 6 of [LS] we have for every $f \in L_2(\mathbf{R}^n)$

$$|(T_0 + 1)^{-1}f| \leqq (-\Delta + 1)^{-1}|f|$$

pointwise a.e. with respect to Lebesgue-measure. Iterating this we get

$$|(T_0 + 1)^{-\ell}f| \leqq (-\Delta + 1)^{-\ell}|f| \quad \text{for every} \quad \ell \in \mathbf{N}\,.$$

The lemma follows from this and the boundedness of $(-\Delta + 1)^{-\ell}$ from $L_2(\mathbf{R}^n)$ to $L_\infty(R^n)$ for $\ell > \frac{n}{4}$. The latter is a consequence of the fact that $(-\Delta + 1)^{-\ell}$ is a convolution with an $L_2$-function. □

We note that the assertion of Lemma 3.3 could be shown for every real $\ell > \frac{n}{4}$ using results on $L_p$-boundedness of $(-\Delta + 1)^{-\ell}$ and $L_p$-interpolation. This would be necessary for finding optimal values for $\ell$ in Theorem 3.1. We are not interested in this here.

In the sequel $B_p$ denotes the $p$-th Schatten class of operators in $L_2(\mathbf{R}^n)$, i.e. the set of those compact operators on $L_2(\mathbf{R}^n)$, whose singular values form an $\ell_p$-sequence. $B_2$ is the class of Hilbert-Schmidt operators; by $B_\infty$ we denote the set of compact operators.

**Lemma 3.4.** *For any real* $s > \frac{n}{2}$ *and real* $\ell \geqq \ell_0$ *we have* $k_{-s}(T_0 + 1)^{-\ell} \in B_2$. *If* $p \in (2, \infty]$ *then* $k_{-s}(T_0 + 1)^{-\ell} \in B_p$ *for any* $s > \frac{n}{p}$ *and* $\ell > \frac{2}{p}\ell_0$.

*Proof.* For $\ell \geqq \ell_0$ Lemma 3.3 and the Dunford-Pettis theorem [CFKS, p. 24] say that $(T_0 + 1)^{-\ell}$ is an integral operator with kernel $K(x, y)$ such that $\sup_x \int |K(x, y)|^2 dy < \infty$. So $k_{-s}(x)K(x, y) \in L^2(dx \times dy)$ for $s > \frac{n}{2}$, yielding the first statement of Lemma 3.4.

From this we get that $k_{-\varepsilon_1}(T_0 + 1)^{-\varepsilon_2}$ is compact for arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by writing it as a norm limit of compact operators (decompose by multiplying from the left with characteristic functions of large balls resp. their complements, and from the right with appropriate spectral projections corresponding to $T_0$). The remaining statement for $p \in (2, \infty)$ follows from the $p = 2$ and $p = \infty$ cases by interpolation in $B_p$-spaces: Let $s_0 > \frac{n}{2}$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Define

$$F(z) = k_{-\varepsilon_1 - z(s_0 - \varepsilon_1)}(T_0 + 1)^{-\varepsilon_2 - z(\ell_0 - \varepsilon_2)} .$$

Then for $f, g \in L_2(\mathbf{R}^n)$ we see that $z \mapsto (f, F(z)g)$ is analytic in $0 < \operatorname{re} z < 1$ and continuous in $0 \leqq \operatorname{re} z \leqq 1$. Furthermore $F(iy) \in B_\infty$ and $F(1 + iy) \in B_2$ for any $y \in \mathbf{R}$. Now Theorem 2.9 of [Si4] together with Proposition 8 on p. 44 of [RS] shows that $F(z) \in B_{2/\operatorname{re} z}$ for any $z$ with $0 \leqq \operatorname{re} z \leqq 1$. Given $s > \frac{n}{p}$ and $\ell > \frac{2}{p}\ell_0$ we may arrange $s_0$, $\varepsilon_1$ and $\varepsilon_2$ in a way to conclude $k_{-s}(T_0 + 1)^{-\ell} \in B_p$. □

**Lemma 3.5.** *For every* $m > -\inf \sigma(T_1)$

$$k_{-s}(T_1 + m)^{-1/2} \in B_p \quad \text{for } p > 4\ell_0, s > \frac{n}{p}, \tag{9}$$

*and for every* $z \in \rho(T_1)$

$$k_{-s}(T_1 - z)^{-1} \in B_p \quad \text{for } p > 2\ell_0, s > \frac{n}{p}. \tag{10}$$

*Proof.* By Lemma 2.3, $T_0$ and $T_1$ have equal form domains, so $(T_0 + 1)^{1/2}(T_1 + m)^{-1/2}$ is bounded and (9) follows from Lemma 3.4.

By the Hölder-property of $B_p$-spaces we get from this

$$k_{-s}\ (T_1 + m)^{-1} k_{-s} \in B_p \quad \text{for } p > 2\ell_0, s > \frac{n}{2p}. \tag{11}$$

In the following lemma we will show that

$$[k_{-s}, (T_1 + m)^{-1}] = -(T_1 + m)^{-1}\{(\Delta k_{-s}) + 2(\nabla k_{-s})\cdot(\nabla - ib)\}(T_1 + m)^{-1} \tag{12}$$

for every $s > 0$, where $[\cdot,\cdot]$ denotes the commutator. From this we find

$$k_{-2s}(T_1+m)^{-1} = k_{-s}(T_1+m)^{-1}k_{-s} - k_{-s}(T_1+m)^{-1}(\Delta k_{-s})(T_1+m)^{-1}$$

$$- 2k_{-s}(T_1+m)^{-1}(\nabla k_{-s})\cdot(\nabla - ib)(T_1+m)^{-1}\,. \tag{13}$$

By $D(T_1) \subset Q(T_0) \subset D(\nabla - ib)$ we get boundedness of $(\nabla - ib)(T_1+m)^{-1}$. Using (11) to treat the remaining terms on the right hand side of (13) we arrive at (10) in the case $z = -m$; this extends to general $z$ by the boundedness of $(T_1+m)(T_1-z)^{-1}$. □

**Lemma 3.6.** (12) *holds for every* $s > 0$.

*Proof.* We have to show that for $g \in D(T_1)$ and $f \in D(T_1)$

$$((T_1+m)g, k_{-s}f) - (k_{-s}g, (T_1+m)f)$$

$$= -(g, (\Delta k_{-s})f + 2(\nabla k_{-s})\cdot(\nabla - ib)f)\,, \tag{14}$$

since from this we get the result by choosing $f = (T_1+m)^{-1}f_0$ and $g = (T_1+m)^{-1}g_0$ where $f_0 \in L_2$ and $g_0 \in L_2$ are arbitrary. The choice $\psi = k_{-s}$ in Lemma 2.4 shows that $k_{-s}f \in D(T_1)$ and

$$T_1 k_{-s} f = k_{-s}T_1 f - (\Delta k_{-s})f - 2(\nabla k_{-s})\cdot(\nabla - ib)f\,. \tag{15}$$

This implies (14). □

**Lemma 3.7.** *Assume that* $b$ *and* $V$ *satisfy* (A). *Then for every* $s \geqq 2$ *and* $z \in \rho(H)$ *the operators* $T_1 k_{-s}(T-z)^{-1}$ *and* $(\partial_j - ib_j)k_{-s}(T-z)^{-1}$, $j \in \{1, \ldots, n\}$ *are bounded (in particular,* $s \geqq 2$ *and* $f \in D(T)$ *implies* $k_{-s}f \in D(T_1)$).

*Proof.* Boundedness of $(\partial_j - ib_j)k_{-s}(T-z)^{-1}$ follows from boundedness of $(\partial_j - ib_j)(T_1+m)^{-1}$ and boundedness of $T_1 k_{-s}(T-z)^{-1}$.

To prove boundedness of $T_1 k_{-s}(T-z)^{-1}$ we will show that for some $C > 0$

$$\| T_1 k_{-s} f \| \leqq C(\| Tf \| + \| f \|) \quad \text{for every} \quad f \in C_0^\infty(\mathbf{R}^n)\,.$$

The result then follows from the essential self-adjointness of $T$ on $C_0^\infty(\mathbf{R}^n)$. Boundedness of $k_{-s}$, $k_{-s}V_2$ and $\Delta k_{-s}$ implies

$$\| k_{-s}T_1 f - (\Delta k_{-s})f \| = \| k_{-s}Tf + k_{-s}V_2 f - (\Delta k_{-s})f \|$$

$$\leqq C(\| Tf \| + \| f \|)\,.$$

Taking into account (15) it remains to prove a similar estimate for $(\nabla k_{-s})\cdot(\nabla - ib)f$. Let $r \geqq 1$, then

$$\| k_{-r}Tf \|^2 + \| k_{-r}f \|^2 \geqq 2\mathrm{re}(k_{-r}Tf, k_{-r}f)$$

$$= 2\,\mathrm{re}\{((\nabla - ib)f, (\nabla - ib)k_{-2r}f) + (Vk_{-r}f, k_{-r}f)\}$$

$$\geqq 2\mathrm{re}\{\| k_{-r}(\nabla - ib)f \|^2 + 2(k_{-r}(\nabla - ib)f, (\nabla k_{-r})f)$$

$$- (V_1 k_{-r}f, k_{-r}f) - (V_2 k_{-r}f, k_{-r}f)\}\,. \tag{16}$$

Lemma 2.3 shows that

$$
\begin{aligned}
|(V_1 k_{-r}f, k_{-r}f)| &\leqq \tfrac{1}{4}\,\|(\nabla - ib)k_{-r}f\|^2 + C_1\,\|k_{-r}f\|^2 \\
&\leqq \tfrac{1}{2}\,\|k_{-r}(\nabla - ib)f\|^2 + \tfrac{1}{2}\,\|(\nabla k_{-r})f\|^2 + C_1\,\|k_{-r}f\|^2 \\
&\leqq \tfrac{1}{2}\,\|k_{-r}(\nabla - ib)f\|^2 + C_2\,\|f\|^2 .
\end{aligned}
$$

$|V_2^{1/2}k_{-r}| \leqq C_3$ yields $|(V_2 k_{-r}f, k_{-r}f)| \leqq C_3\,\|f\|^2$. Finally

$$|2(k_{-r}(\nabla - ib)f, (\nabla k_{-r})f)| \leqq \tfrac{1}{4}\,\|k_{-r}(\nabla - ib)f\|^2 + C_4\,\|f\|^2 .$$

Inserting everything into (16) we arrive at

$$\|k_{-r}Tf\|^2 + \|k_{-r}f\|^2 \geqq \tfrac{1}{2}\,\|k_{-r}(\nabla - ib)f\|^2 - C_5\,\|f\|^2 .$$

From this we find $\|k_{-r}(\nabla - ib)f\| \leqq C_6(\|Tf\| + \|f\|)$. Now $|\nabla k_{-s}| \leqq Ck_{-s-1}$ gives the desired estimate for $(\nabla k_{-s})\cdot(\nabla - ib)f$. □

**Lemma 3.8.** *Let $z \in \rho(T)$. Then*

(i) $k_{-s}(T - z)^{-1} \in B_p$ *for* $p > 2\ell_0$, $s > \dfrac{n}{p} + 2$,

(ii) $(\partial_j - ib_j)k_{-s}(T - z)^{-1} \in B_p$ *for* $p > 4\ell_0$, $s > \dfrac{n}{p} + 2, j \in \{1, \ldots, n\}$,

(iii) $\overline{k_{-s}(\partial_j - ib_j)(T - z)^{-1}} \in B_p$ *for* $p > 4\ell_0$, $s > \dfrac{n}{p} + 2, j \in \{1, \ldots, n\}$.

*Proof.* (i) follows from

$$k_{-s}(T - z)^{-1} = k_{-(s-2)}(T_1 + m)^{-1}(T_1 + m)k_{-2}(T - z)^{-1}$$

by using Lemmas 3.5 and 3.7.

$(\partial_j - ib_j)(T_1 + m)^{-1/2}$ is bounded, so for (ii) it is enough to consider

$$
\begin{aligned}
(T_1 + m)^{1/2}k_{-s}(T - z)^{-1} &= (T_1 + m)^{-1/2}(T_1 + m)k_{-(s-2)}k_{-2}(T - z)^{-1} \\
&= (T_1 + m)^{-1/2}\{k_{-(s-2)}(T_1 + m) - (\Delta k_{-(s-2)}) \\
&\quad - 2(\nabla k_{-(s-2)})\cdot(\nabla - ib)\}k_{-2}(T - z)^{-1}
\end{aligned}
$$

and (ii) follows from this by Lemmas 3.5 and 3.7.

Finally (iii) follows from

$$\overline{k_{-s}(\partial_j - ib_j)(T - z)^{-1}} = (\partial_j - ib_j)k_{-s}(T - z)^{-1} - (\partial_j k_{-s})(T - z)^{-1} ,$$

which is immediate on the dense set $(T - z)C_0^\infty(\mathbf{R}^n)$ and extends to the closure. □

To complete the *proof of Theorem* 3.1, we observe that for $s > 0$ and $z \in \rho(T)$

$$
\begin{aligned}
[k_{-s}, (T - z)^{-1}] = (T - z)^{-1}\{&-(\Delta k_{-s})(T - z)^{-1} \\
&- 2\overline{(\nabla k_{-s})\cdot(\nabla - ib)(T - z)^{-1}}\} . \qquad (17)
\end{aligned}
$$

This follows from $[k_{-s}, (T - z)^{-1}] = (T - z)^{-1}[T - z, k_{-s}](T - z)^{-1}$ and the explicit form of $[T - z, k_{-s}]$ on $C_0^\infty(\mathbf{R}^n)$.

Let $s = s_1 + s_2, s_1 > 0, s_2 > 0$. By (17) we have

$$k_{-s}(T-z)^{-2} = k_{-s_1}(T-z)^{-1}\{k_{-s_2}(T-z)^{-1} - (\Delta k_{-s_2})(T-z)^{-2} - \overline{2(\nabla k_{-s_2})\cdot(\nabla - ib)(T-z)^{-2}}\} .$$

Now Lemma 3.8 and the Hölder-property of $B_p$ show

$$k_{-s}(T-z)^{-2} \in B_p, p > \frac{4\ell_0}{3}, s > \frac{n}{p} + 4 .$$

In a similar way we get

$$(\partial_j - ib_j)k_{-s}(T-z)^{-2} \in B_p, p > 2\ell_0, s > \frac{n}{p} + 4$$

and as in Lemma 3.8

$$\overline{k_{-s}(\partial_j - ib_j)(T-z)^{-2}} \in B_p, p > 2\ell_0, s > \frac{n}{p} + 4 .$$

It is obvious how this procedure extends to successively higher powers of the resolvent. After a finite number of iterations we get that for $s$ and $\ell$ sufficiently large $k_{-s}(T-z)^{-\ell} \in B_2$, i.e. the first assertion of Theorem 3.1, and also

$$\overline{k_{-s}(\partial_j - ib_j)(T-z)^{-\ell}} \in B_2 \quad \text{for} \quad j \in \{1, \ldots, n\} . \tag{18}$$

From (17) we deduce

$$T_1 k_{-s}(T-z)^{-\ell} = T_1 k_{-s_1}(T-z)^{-1}\{k_{-s_2}(T-z)^{-\ell+1} - (\Delta k_{-s_2})(T-z)^{-\ell} - \overline{2(\nabla k_{-s_2})\cdot(\nabla - ib)(T-z)^{-\ell}}\} .$$

The second assertion of Theorem 3.1 now follows from the first assertion, (18) and Lemma 3.7. □

## *3.2 Proof of Proposition 2*

In preparation for the proof of Proposition 2 we need a little lemma.

**Lemma 3.9.** *Let $b$ and $V$ satisfy* (A). *Let $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n)$ be a solution of $Tu = \lambda u$. Then $\Delta(|u|^2) = 2(V-\lambda)|u|^2 + 2|(\nabla - ib)u|^2$.*

*Proof.* The assumptions on $u$ imply $\nabla u \in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ (Lemma 2.2). A straightforward calculation shows for every real-valued $\varphi \in C_0^\infty(\mathbf{R}^n)$:

$$\int \overline{(\nabla - ib)^2 u} u_\varepsilon \varphi = -\int \bar{u}(\nabla - ib)^2 u_\varepsilon \varphi - 2\int \overline{.(\nabla - ib)u}\cdot(\nabla - ib)u_\varepsilon \varphi + \int \bar{u} u_\varepsilon \Delta\varphi .$$

Since $(\nabla - ib)^2 u_\varepsilon \to (\nabla - ib)^2 u$ in $L_{1,\mathrm{loc}}(\mathbf{R}^n)$ (cf. the proof of Lemma 2.1), $u_\varepsilon \to u$, $\nabla u_\varepsilon \to \nabla u$, and $bu_\varepsilon \to bu$ in $L_{2,\mathrm{loc}}(\mathbf{R}^n)$, we arrive at

$$\begin{aligned}\int |u|^2 \Delta\varphi &= 2\mathrm{re}\int \overline{(\nabla - ib)^2 u}\, u\varphi + 2\int |(\nabla - ib)u|^2 \varphi \\ &= 2\int (V-\lambda)|u|^2 \varphi + 2\int |(\nabla - ib)u|^2 \varphi .\end{aligned}$$

□

*Proof of Proposition* 2. Let $u \neq 0$ with $(1+|\cdot|)^{-s}u \in L_\infty(\mathbf{R}^n)$ for some $s>0$ be a solution of $Tu=\lambda u$. We have to show that $\lambda \in \sigma(T)$. We may assume $u \notin L_2(\mathbf{R}^n)$, since otherwise $\lambda \in \sigma_p(T)$, and we are done.

For every $R \geqq 1$ we will construct a $w_R \in D(T)$ with $\|w_R\|=1$, such that $w_R \xrightarrow{w} 0$, as $R \to \infty$, and $\liminf_{R\to\infty} \|(T-\lambda)w_R\| = 0$. From this our result follows; in fact we have $\lambda \in \sigma_e(T)$ by Theorem 7.24 in [W].

We choose $\Theta \in C^\infty(\mathbf{R})$ with $0 \leqq \Theta \leqq 1$, $\Theta(r)=1$ for $r \leqq 0$, and $\Theta(r)=0$ for $r \geqq 1$. We define for all $R \geqq 1$:

$$\forall x \in \mathbf{R}^n\colon \Theta_R(x) = \Theta\left(\frac{|x|-R}{R}\right) \quad \text{and} \quad u_R := \Theta_R u .$$

By Lemma 2.2, $u \in L_{\infty,\mathrm{loc}}(\mathbf{R}^n) \cap W^1_{2,\mathrm{loc}}(\mathbf{R}^n)$ and we may write for $\varphi \in C_0^\infty(\mathbf{R}^n)$:

$$\begin{aligned}\int \overline{u_R}\, T\varphi &= \int \overline{\lambda u_R}\,\varphi + \int (V-\lambda)\bar u \Theta_R \varphi - \int \bar u (\nabla - ib)^2(\Theta_R\varphi)\\ &\quad + \int \bar u \Delta\Theta_R \varphi + 2\int \bar u \nabla\Theta_R \cdot \nabla\varphi - 2i\int \bar u b\cdot \nabla\Theta_R\varphi\\ &= \int \overline{\lambda u_R}\,\varphi - \int \bar u \Delta \Theta_R\varphi - 2\int \overline{(\nabla - ib)u}\cdot \nabla\Theta_R\varphi ,\end{aligned}$$

whence $u_R \in D(T)$ and $(T-\lambda)u_R = -u\Delta\Theta_R - 2(\nabla - ib)u\cdot\nabla\Theta_R$. Therefore

$$\begin{aligned}\|(T-\lambda)u_R\|^2 &\leqq 2\|u\Delta\Theta_R\|^2 + 8\|(\nabla-ib)u\cdot\nabla\Theta_R\|^2\\ &\leqq 2\int |u|^2|\Delta\Theta_R|^2 + 8\int |(\nabla-ib)u|^2|\nabla\Theta_R|^2 . \qquad (19)\end{aligned}$$

By Lemma 3.9

$$\int |(\nabla-ib)u|^2|\nabla\Theta_R|^2 \leqq \tfrac12 \int |u|^2\Delta(|\nabla\Theta_R|^2) + \int (V_- + \lambda)|u|^2|\nabla\Theta_R|^2, \qquad (20)$$

where by assumption $V_- = V_1 + V_2$ with $0 \leqq V_1 \in K(\mathbf{R}^n)$ and $0 \leqq (1+|\cdot|)^{-2}V_2$ bounded and vanishing at infinity. From Lemma 2.3 we have relative form boundedness of $V_1$ with respect to $-(\nabla - ib)^2$, i.e.

$$\forall \iota > 0\ \exists c(\iota)>0\ \forall \varphi \in C_0^\infty(\mathbf{R}^n)\colon \int V_1|\varphi|^2 \leqq \iota \int |(\nabla - ib)\varphi|^2 + c(\iota)\int|\varphi|^2 .$$

By regularization we may apply this to $\varphi := u(\partial_j\Theta_R) \in W_2^1(\mathbf{R}^n) \cap L_4(\mathbf{R}^n)$, $j \in \{1,\ldots,n\}$, and get

$$\begin{aligned}\int V_1|u|^2|\nabla\Theta_R|^2 &\leqq \iota \sum_{j=1}^n \int |(\nabla-ib)(u(\partial_j\Theta_R))|^2 + c(\iota)\sum_{j=1}^n \int |u(\partial_j\Theta_R)|^2\\ &\leqq 2\iota \int |(\nabla - ib)u|^2|\nabla\Theta_R|^2 + 2\iota \sum_{j,k=1}^n \int |u|^2|\partial_k\partial_j\Theta_R|^2 + c(\iota)\int |u|^2|\nabla\Theta_R|^2 .\end{aligned}$$

Inserting this into (20), choosing $\iota = \frac14$ and rearranging terms, we obtain

$$\begin{aligned}&\int |(\nabla-ib)u|^2|\nabla\Theta_R|^2\\ &\quad \leqq \int |u|^2\left\{\Delta(|\nabla\Theta_R|^2) + \sum_{j,k=1}^n |\partial_k\partial_j\Theta_R|^2 + 2(c(\tfrac14) + V_2 + \lambda)|\nabla\Theta_R|^2\right\}.\end{aligned}$$

Observing the definition of $\Theta_R$, (19) yields

$\exists c > 0 \; \forall R \geqq 1$:

$$\|(T-\lambda)u_R\|^2 \leqq \frac{c}{R^2}\left(1 + \sup_{R \leqq |x| \leqq 2R} |V_2(x)|\right) \int_{R \leqq |y| \leqq 2R} |u(y)|^2 dy \,. \tag{21}$$

We define $w_R := \frac{u_R}{\|u_R\|}$. So $w_R \in D(T)$, $\|w_R\| = 1$, and $w_R \overset{w}{\to} 0$, because for every $\varphi \in C_0^\infty(\mathbf{R}^n)$ we have $(w_R, \varphi) = \frac{(u, \varphi)}{\|u_R\|}$ for $R$ large enough, and $\|u_R\| \to \infty$, as $R \to \infty$, since $u \notin L_2(\mathbf{R}^n)$. Finally, from (21), we get

$$\|(T-\lambda)w_R\|^2 \leqq \frac{c}{R^2}\left(1 + \sup_{R \leqq |x| \leqq 2R} |V_2(x)|\right) \frac{\int_{R \leqq |y| \leqq 2R} |u(y)|^2 dy}{\int_{|y| \leqq R} |u(y)|^2 dy} \,.$$

Since

$$\frac{1}{R^2} \sup_{R \leqq |x| \leqq 2R} |V_2(x)| \leqq 4 \sup_{R \leqq |x|} \frac{|V_2(x)|}{|x|^2} \to 0, \quad \text{as } R \to \infty \,,$$

we are left with showing that

$$\liminf_{R \to \infty} \frac{\int_{R \leqq |y| \leqq 2R} |u(y)|^2 dy}{\int_{|y| \leqq R} |u(y)|^2 dy} < \infty \,.$$

But this is done precisely as in [Hil, p. 181] by proving that for any $R \geqq 1$:

$$\inf_{r \geqq R} \frac{\int_{r \leqq |y| \leqq 2r} |u(y)|^2 dy}{\int_{|y| \leqq r} |u(y)|^2 dy} + 1 \leqq 2^{2\mu} \,,$$

as soon as $(1 + |\cdot|)^{-\mu} u \in L_2(\mathbf{R}^n)$, i.e., for $\mu > s + \frac{n}{2}$.

This completes the proof of Proposition 2. □

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