BIT 10 (1970), 237–242

A METHOD TO CALCULATE THE DISTRIBUTION FUNCTION WHEN THE CHARACTERISTIC FUNCTION IS KNOWN

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Abstract.

A formula for numerical inversion of characteristic functions based on the Poisson formula is presented, and a numerical example is also given.

Key words: Numerical, inversion, characteristic functions.

Method.

In this paper we will discuss the numerical inversion of characteristic functions. We restrict ourselves to the case when the distribution function $F(x)$ has increments for non-negative x only. The characteristic function is equal to

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx}\,dF(x) = \int_{0}^{\infty} e^{itx}\,dF(x)\,.$$

The corresponding inversion formula reads

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\varphi(t)\,dt$$

from which follows

$$F(x)-F(-x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin tx}{t}\,\varphi(t)\,dt\,.$$

If we now make use of the Poisson summation formula we get the following relation

$$\sum_{\nu=-\infty}^{+\infty} \left(F(x+2\pi\nu/\lambda)-F(-x+2\pi\nu/\lambda)\right) = \frac{\lambda}{\pi}\sum_{\nu=-\infty}^{+\infty} \frac{\sin \lambda\nu x}{\lambda\nu}\,\varphi(\lambda\nu)\,.$$

This means that if λ is sufficiently small we have the following approximate relation

Received March 26, 1970.

$$F(x) \sim \sum_{\nu=-\infty}^{+\infty} \frac{\sin \lambda \nu x}{\pi \nu} \varphi(\lambda \nu) .$$

For $\nu = 0$ we define $\sin \lambda \nu x/\nu$ as its limit when $\nu \to 0$, that is λx. When we approach this formula from a numerical point of view we must consider the error introduced by replacing the infinite number of terms by a finite one. We will do this in the following way.

The function

$$C(t) = \begin{cases} (1-t) \cos \pi t + \dfrac{1}{\pi} |\sin \pi t| & \text{when } |t| < 1 \\ 0 & \text{when } |t| \geqq 1 \end{cases}$$

is a characteristic function. The corresponding random variable has mean value zero and variance π^2. If we replace $\varphi(t)$ by

$$\varphi(t) \cdot C(t/\lambda N)$$

the result will be a slight deformation of $F(x)$ resulting from the fact that $F(x)$ is replaced by a weighted mean of F-values in the neighbourhood of $F(x)$. We then have the following approximate formula

$$F(x) \sim \sum_{\nu=-N}^{N} \frac{\sin \lambda \nu x}{\pi \nu} C(\nu/N) \cdot \varphi(\lambda \nu) .$$

The choice of $C(t)$ for this purpose is related to the following fact. It can be shown that among characteristic functions which are identically zero for $|t| > 1$ the function $-C(t)$ has the smallest possible second derivative at $t = 0$, namely π^2. The result is that the use of $C(t)$ will cause a deformation of $F(x)$ which is as "local" as possible.

In order to discuss the value of λ we write the approximation formula as follows

$$(1) \qquad F(x) \sim \frac{\lambda x}{\pi} + \frac{2}{\pi} \sum_{\nu=1}^{N} \frac{\sin \lambda \nu x}{\nu} C(\nu/N) \operatorname{Re} \varphi(\lambda \nu)$$

where $\operatorname{Re} \varphi$ denotes the real part of φ. This version of the approximation formula is also the one best fitted for numerical calculations.

The formula (1) depends on λ and N. The choice of these two quantities should be made along the following lines.

The more computer time one can spend on the caulculations, the larger one can choose N, and the less will the deformation of $F(x)$ be.

The longer the "tail" of the distribution function $F(x)$ is, the smaller λ must be. A long tail means that $F(x)$ tends slowly to 1 as x tends to infinity. Since (1) is equal to one for $x = \pi/\lambda$ and equal to zero for $x = 0$

we can not hope for (1) to be precise outside the interval $(0, \pi/\lambda)$. From the Poisson formula one can draw the conclusion that in order for (1) to be accurate for large x it is necessary that λ tends to zero as x tends to infinity. This can evidently take place in different ways. However, as a result of some practical experiments I would suggest the following value of λ

$$\lambda = \pi/(m + 3\sigma + 2x) \tag{2}$$

where m and σ are the mean value and standard deviation of $F(x)$.

Examples.

Listed below are the results of the application of formula (1) with λ-value (2). The function φ is equal to

$$\varphi(t) = \frac{1-p}{1 - \dfrac{p}{1-p}it} + \frac{p}{1 - \dfrac{1-p}{p}it}.$$

For all values of p larger than 0 but less than 1 this is a characteristic function. The corresponding distribution function has mean value 1 and standard deviation

$$\sigma = \sqrt{2/(p-p^2)-7}\,.$$

The table below gives approximate values and exact values of $F(x)$ for x equal to $m, m+\sigma, \ldots m+6\sigma$. The mean value m is equal to 1 and σ is given above. The values of p are equal to 0.5, 0.7, 0.9 and 0.99. The larger the value of p is the longer is the "tail" of the distribution. It is seen from the table that it is easier to get a good approximation in the tail of the distribution than elsewhere. This has to do with the fact that the deformation caused by the use of the function C becomes larger if $F(x)$ is more pronounced convex or concave. In the tail of the distribution the second derivative of $F(x)$ is small and thus $F(x)$ is almost linear.

p	N	λ	x	F exact	F appr.
0.50000	10	0.52360	1.00000	0.63212	0.57042
0.50000	10	0.39270	2.00000	0.86466	0.81908
0.50000	10	0.31416	3.00000	0.95021	0.92050
0.50000	10	0.26180	4.00000	0.98168	0.96300
0.50000	10	0.22440	5.00000	0.99326	0 98146
0.50000	10	0.19635	6.00000	0.99752	0.98981
0.50000	10	0.17453	7.00000	0.99909	0.99375

p	N	λ	x	F exact	F appr.
0.50000	40	0.52360	1.00000	0.63212	0.62807
0.50000	40	0.39270	2.00000	0.86466	0.86194
0.50000	40	0.31416	3.00000	0.95021	0.94860
0.50000	40	0.26180	4.00000	0.98168	0.98079
0.50000	40	0.22440	5.00000	0.99326	0.99278
0.50000	40	0.19635	6.00000	0.99752	0.99727
0.50000	40	0.17453	7.00000	0.99909	0.99894
0.50000	70	0.52360	1.00000	0.63212	0.63080
0.50000	70	0.39270	2.00000	0.86466	0.86378
0.50000	70	0.31416	3.00000	0.95021	0.94969
0.50000	70	0.26180	4.00000	0.98168	0 98140
0.50000	70	0.22440	5.00000	0.99326	0.99311
0.50000	70	0.19635	6.00000	0.99752	0.99744
0.50000	70	0.17453	7.00000	0.99909	0.99904
0.50000	100	0.52360	1.00000	0.63212	0.63148
0.50000	100	0.39270	2.00000	0.86466	0.86423
0.50000	100	0.31416	3.00000	0.95021	0.94996
0.50000	100	0.26180	4.00000	0.98168	0.98155
0.50000	100	0.22440	5.00000	0.99326	0.99319
0.50000	100	0.19635	6.00000	0.99752	0.99748
0.50000	100	0.17453	7.00000	0.99909	0.99907
0.70000	10	0.40453	1.00000	0.73669	0.59775
0.70000	10	0.28708	2.58865	0.89941	0.86859
0.70000	10	0.22248	4.17730	0.94988	0.93541
0.70000	10	0.18162	5.76595	0.97465	0.96373
0.70000	10	0.15343	7.35460	0.98717	0.97858
0.70000	10	0.13282	8.94325	0.99351	0.98683
0.70000	10	0.11709	10.53190	0.99671	0.99154
0.70000	40	0.40453	1.00000	0.73669	0.72914
0.70000	40	0.28708	2.58865	0.89941	0.89830
0.70000	40	0.22248	4.17730	0.94988	0.94924
0.70000	40	0.18162	5.76595	0.97465	0.97417
0.70000	40	0.15343	7.35460	0.98717	0.98682
0.70000	40	0.13282	8.94325	0.99351	0.99325
0.70000	40	0.11709	10.53190	0.99671	0.99654
0.70000	70	0.40453	1.00000	0.73669	0.73449
0.70000	70	0.28708	2.58865	0.89941	0.89912
0.70000	70	0.22248	4.17730	0.94988	0.94969
0.70000	70	0.18162	5.76595	0.97465	0.97450
0.70000	70	0.15343	7.35460	0.98717	0.98706
0.70000	70	0.13282	8.94325	0.99351	0.99343
0.70000	70	0.11709	10.53190	0.99671	0.99666

p	N	λ	x	F exact	F appr.
0.70000	100	0.40453	1.00000	0.73669	0.73579
0.70000	100	0.28708	2.58865	0.89941	0.89930
0.70000	100	0.22248	4.17730	0.94988	0.94979
0.70000	100	0.18162	5.76595	0.97465	0.97458
0.70000	100	0.15343	7.35460	0.98717	0.98712
0.70000	100	0.13282	8.94325	0.99351	0.99346
0.70000	100	0.11709	10.53190	0.99671	0.99668
0.90000	10	0.21365	1.00000	0.91040	0.47078
0.90000	10	0.13958	4.90157	0.94199	0.92514
0.90000	10	0.10365	8.80313	0.96240	0.95609
0.90000	10	0.08243	12.70470	0.97563	0.96926
0.90000	10	0.06842	16.60627	0.98420	0.97860
0.90000	10	0.05848	20.50783	0.98976	0.98503
0.90000	10	0.05106	24.40940	0.99336	0.98938
0.90000	40	0.21365	1.00000	0.91040	0.90183
0.90000	40	0.13958	4.90157	0.94199	0.94248
0.90000	40	0.10365	8.80313	0.96240	0.96245
0.90000	40	0.08243	12.70470	0.97563	0.97552
0.90000	40	0.06842	16.60627	0.98420	0.98405
0.90000	40	0.05848	20.50783	0.98976	0.98961
0.90000	40	0.05106	24.40940	0.99336	0.99323
0.90000	70	0.21365	1.00000	0.91040	0.90958
0.90000	70	0.13958	4.90157	0.94199	0.94269
0.90000	70	0.10365	8.80313	0.96240	0.96261
0.90000	70	0.08243	12.70470	0.97563	0.97565
0.90000	70	0.06842	16.60627	0.98420	0.98417
0.90000	70	0.05848	20.50783	0.98976	0.98972
0.90000	70	0.05106	24.40940	0.99336	0.99332
0.90000	100	0.21365	1.00000	0.91040	0.91068
0.90000	100	0.13958	4.90157	0.94199	0.94273
0.90000	100	0.10365	8.80313	0.96240	0.96264
0.90000	100	0.08243	12.70470	0.97563	0.97568
0.90000	100	0.06842	16.60627	0.98420	0.98419
0.90000	100	0.05848	20.50783	0.98976	0.98974
0.90000	100	0.05106	24.40940	0.99336	0.99334
0.99000	10	0.06998	1.00000	0.99010	0.17761
0.99000	10	0.04314	14.96496	0.99140	0.96724
0.99000	10	0.03118	28.92993	0.99253	0.98838
0.99000	10	0.02441	42.89489	0.99352	0.98946
0.99000	10	0.02006	56.85985	0.99437	0.99100
0.99000	10	0.01702	70.82482	0.99511	0.99240
0.99000	10	0.01479	84.78978	0.99575	0.99350

p	N	λ	x	F exact	F appr.
0.99000	40	0.06998	1.00000	0.99010	0.63472
0.99000	40	0.04314	14.96496	0.99140	0.99207
0.99000	40	0.03118	28.92993	0.99253	0.99332
0.99000	40	0.02441	42.89489	0.99352	0.99417
0.99000	40	0.02006	56.85985	0.99437	0.99486
0.99000	40	0.01702	70.82482	0.99511	0.99545
0.99000	40	0.01479	84.78978	0.99575	0.99598
0.99000	70	0.06998	1.00000	0.99010	0.89248
0.99000	70	0.04314	14.96496	0.99140	0.99226
0.99000	70	0.03118	28.92993	0.99253	0.99340
0.99000	70	0.02441	42.89489	0.99352	0.99421
0.99000	70	0.02006	56.85985	0.99437	0.99489
0.99000	70	0.01702	70.82482	0.99511	0.99548
0.99000	70	0.01479	84.78978	0.99575	0.99600
0.99000	100	0.06998	1.00000	0.99010	0.97542
0.99000	100	0.04314	14.96496	0.99140	0.99228
0.99000	100	0.03118	28.92993	0.99253	0.99341
0.99000	100	0.02441	42.89489	0.99352	0.99422
0.99000	100	0.02006	56.85985	0.99437	0.99489
0.99000	100	0.01702	70.82482	0.99511	0.99548
0.99000	100	0.01479	84.78978	0.99575	0.99601

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