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B-CONVERGENCE RESULTS FOR LINEARLY IMPLICIT ONE STEP METHODS

K. STREHMEL and R. WEINER

Sektion Mathematik, Martin-Luther-Universität Halle-Wittenberg, Weinbergweg 17, DDR-4020 Halle, German Democratic Republic

Abstract.

B-consistency and B-convergence of linearly implicit one step methods with respect to a class of arbitrarily stiff semi-linear problems are considered. Order conditions are derived. An algorithm for constructing methods of order > 1 is shown and examples are given. By suitable modifications of the methods the occurring order reduction is decreased.

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1. Introduction.

We consider the initial value problem

$$y' = f(t, y)$$
$$y(t_0) = y_0, \qquad f: [t_0, t_e] \times R^n \to R^n.$$

In classical concepts of consistency and convergence the Lipschitz constant of f is used to derive bounds for the local and global error. For stiff systems this Lipschitz constant is very large and the bounds become unrealistic in the smooth phase. On the other hand the accuracy of a numerical method is often worse than expected when the order of consistency is taken into account, and the method suffers from order reduction (see Verwer [17]). To derive realistic bounds for the error in the smooth phase Frank, Schneid and Ueberhuber (see e.g. [6]) developed the concept of B-consistency and B-convergence. Here the bounds do not depend on the classical Lipschitz constant but only on the one-sided Lipschitz constant of the system, which may be of moderate size for arbitrarily stiff systems.

In several papers (see e.g. [5], [4]) order results for implicit Runge-Kutta methods are proved.

This paper deals with the B-consistency and B-convergence of linearly implicit one step methods. In the last years these methods have been frequently used

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for the solution of stiff systems (see [11], [12], [15]). Because of the lack of B-stability we cannot expect positive B-convergence results for general nonlinear problems. We therefore will consider semi-linear problems

$$y' = f(t, y) := Ty + g(t, y) + r(t) \tag{1.1}$$

with

$$\begin{aligned} &1.\ \langle Tw, w\rangle \leq \mu\|w\|^2, \quad \mu \leq 0 \quad \text{for all } w \in R^n \\ &2.\ \|g(t,u) - g(t,v)\| \leq L\|u-v\| \quad \text{for all } t \in [t_0, t_e], \quad u, v \in R^n \\ &3.\ \|d^i g(t, y(t))/dt^i\| \leq M \quad \text{for all } t \in [t_0, t_e], \quad i = 1, \dots, p_0, \end{aligned} \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ is some inner product and $\|\cdot\|$ the corresponding norm. We will denote this class of problems by F.

The problems may be arbitrarily stiff, with no bound for the Lipschitz constant of f. On the other hand they satisfy a one-sided Lipschitz condition with constant $\mu + L$.

Problems of class (1.1) arise e.g. by discretization in space of an initial-boundary value problem for a parabolic differential equation of semi-linear type

$$\begin{aligned} &\frac{\partial u}{\partial t} = \sum_{i,k=1}^{\alpha} \frac{\partial}{\partial x_i}\left(b_{ik}(x)\frac{\partial u}{\partial x_k}\right) + q(t, x, u), \qquad x \in G, \quad 0 \leq t \leq t_e < \infty \\ &u(x, t) = \varphi(x, t), \quad x \in \partial G, \quad 0 \leq t \leq t_e \\ &u(x, 0) = u_0(x), \quad x \in G, \end{aligned} \tag{1.3}$$

where the matrix b_{ik} is uniformly positive definite and G is a spatial domain in R^α ($\alpha = 1, 2$ or 3) with boundary ∂G. The vector $r(t)$ arises from the time-dependent boundary conditions. Note that $\|r(t)\|$ tends to infinity if the parameter of space discretization tends to zero. The model problem of Prothero and Robinson [13] also belongs to class (1.1).

We will derive conditions for linearly implicit one-step methods to be B-consistent and B-convergent of order q on F. These conditions enable us to construct B-convergent methods of order $q > 1$ in a simple way.

It will be shown that in general an order reduction occurs. By a modification of the linearly implicit methods we will decrease this order reduction. The modification essentially depends on the class (1.1), (1.2), especially on the function $r(t)$. Finally we give some B-convergence results for a slightly more general class of problems.

In the proofs of our theorems we will use the following result for rational matrix functions of Hairer, Bader and Lubich [7]:

THEOREM 1.1. *Let $R(z)$ be a rational function and let the matrix A satisfy*

$\langle Aw, w\rangle \le \mu\|w\|^2$ for all $w \in R^n$ for some inner product. Then we have in the corresponding operator norm

$$\|R(A)\| \le \sup_{\mathrm{Re}\, z \le \mu} |R(z)|.$$

REMARK 1.1. 1. The class (1.1) without condition 3 of (1.2) was also considered by Hundsdorfer [10]. He shows, without consideration of B-consistency, that the linearly implicit methods of Van der Houwen [8, p. 44] under some natural assumptions are B-convergent of order 1. An order $q > 1$ of B-convergence is proved only for the special case $g(t, y) = 0$ (see (1.1)).

2. The assumption $\mu \le 0$ allows to simplify the proofs. Analogous results, however, will hold for $\mu > 0$ too.

2. Linearly implicit one-step methods.

Numerical results have shown that linearly implicit one-step methods are very efficient for the solution of stiff systems, if the Jacobian does not change very rapidly. These methods insert the Jacobian, or an approximation, directly into the integration formulae. Thus, while retaining favourable stability properties, in contrast to implicit Runge-Kutta methods only linear systems of algebraic equations have to be solved. In their general form linearly implicit one-step methods (also called linearly implicit Runge-Kutta methods) are given by

$$(2.1)\qquad u_{m+1}^{(1)} = u_m$$

$$u_{m+1}^{(i)} = R_0^{(i)}(c_i hT)u_m + h\sum_{j=1}^{i-1} A_{ij}(hT)[f_j - Tu_{m+1}^{(j)}], \qquad i = 2, \ldots, s$$

$$u_{m+1} = R_0^{(s+1)}(hT)u_m + h\sum_{j=1}^{s} B_j(hT)[f_j - Tu_{m+1}^{(j)}]$$

with $f_j = f(t_m + c_j h, u_{m+1}^{(j)})$.

$R_0^{(i)}(z)$ is a rational approximation to $\exp(z)$, $z \to 0$, while A_{ij} and B_j are rational functions (coefficients of the method), c_i real parameters ($c_1 = 0$) and T is an arbitrary (n, n)-matrix (usually an approximation to $f_y(t_m, u_m)$).

REMARK 2.1: There are two important classes of (2.1) which have been studied extensively in the last time

a) *ROW*- or *W*-methods (see [11], [12], [14]). With

$$A := \begin{bmatrix} 0 & & & \\ A_{21} & 0 & & \\ \vdots & & \ddots & \\ A_{s1} & \cdots & A_{s,s-1} & 0 \end{bmatrix}, \qquad B := (B_1, \ldots, B_s)^T, \ \tilde{T} := \mathrm{diag}(T, \ldots, T)$$
$$f := (f_1, \ldots, f_s)^T$$

they are given by

$$A = a(\Gamma - h\tilde{T}a)^{-1} \tag{2.2}$$
$$B = b(\Gamma - h\tilde{T}a)^{-1}$$
$$R_0^{(i)}(c_i hT) = I + hT \sum_{j=1}^{i-1} A_{ij}$$
$$R_0^{(s+1)}(hT) = I + hT \sum_{j=1}^{s} B_j$$

with

$$a := \begin{bmatrix} 0 & & & \\ a_{21}I & 0 & & \\ \vdots & & \ddots & \\ a_{s1}I & \dots & a_{s,s-1}I & 0 \end{bmatrix}, \quad \Gamma := \begin{bmatrix} I - h\gamma T & & & \\ -h\gamma_{21}T & I - h\gamma T & & \\ \vdots & & \ddots & \\ -h\gamma_{s1}T & \dots & -h\gamma_{s,s-1}T & I - h\gamma T \end{bmatrix}$$

$b := (b_1, \dots, b_s)^T$. Further $a_{ij}, b_j, \gamma_{ij}, \gamma$ are real parameters, which determine the method.

b) Adaptive Runge-Kutta methods (see [15], [16]).

Here $R_0^{(i)}(z)$ are arbitrary rational approximations to $\exp(z)$ and

$$A_{ij}(z) = \sum_{l=0}^{\varrho_i} R_{l+1}^{(i)}(c_i z) c_i^{l+1} \lambda_{lj}^{(i)}$$

$$B_j(z) = \sum_{l=0}^{\varrho_{s+1}} R_{l+1}^{(s+1)}(z) \lambda_{lj}^{(s+1)},$$

with $\lambda_{lj}^{(i)} \in R$ and $\quad R_1^{(i)}(z) = \dfrac{R_0^{(i)}(z) - 1}{z}$,

$$R_{l+1}^{(i)}(z) = \frac{lR_l(z) - 1}{z}, \qquad l = 1, 2, \dots, \varrho_i. \tag{2.3}$$

Because ROW-methods require $T = f_y(t_m, u_m)$ we will in the following consider especially W- and adaptive RK-methods.

REMARK 2.2. We will characterize the method (2.1) by the following parameter scheme

$$\begin{array}{c|cccc} c_2 & A_{21} & & & \\ \vdots & \vdots & \ddots & & \\ c_s & A_{s1} & \dots & A_{s,s-1} & \\ \hline & B_1 & \dots & B_{s-1} & B_s \end{array}$$

Furthermore we will use the following abbreviations and notations:

$$g_j := g(t_m + c_j h, u_{m+1}^{(j)}), \quad y^{(l)} := \left.\frac{d^l}{dt^l} y(t)\right|_{t = t_m}$$

$$r^{(l)} := \left.\frac{d^l}{dt^l} r(t)\right|_{t = t_m}, \quad g^{(l)} := \left.\frac{d^l g(t, y(t))}{dt^l}\right|_{t = t_m},$$

p : conventional order of consistency,
p_i: conventional stage order.

For our investigations we assume

(A1) The approximation order r_i of $R_0^{(i)}(z)$, $i = 2, \ldots, s+1$, to exp(z) is sufficiently high ($r_i \geq p_i$).

(A2) $R_0^{(i)}(z)$ has no pole for $\operatorname{Re} z \leq 0$ and $|R_0^{(i)}(\infty)| < \infty$.

(A3) $|zA_{ij}(z)|, |zB_j(z)|$ are uniformly bounded for $\operatorname{Re} z \leq 0$.

REMARK 2.3. For adaptive RK-methods (A3) holds if (A2) holds (see [15]). For W-methods $\gamma > 0$ is sufficient for (A2) and (A3).

3. *B*-consistency and *B*-convergence for the class *F*.

For a linearly implicit one-step method we will derive B-consistency and B-convergence results on class F. We assume that the matrix T of (2.1) is equal to the matrix T of (1.1).

In the following definitions the constants γ_i, γ, β, h_0, C_0 have to be independent of $\|T\|, r(t)$ and derivatives of $r(t)$, but they may depend on μ, L, M, t_0, t_e and on derivatives of the exact solution.

DEFINITION 3.1. A linearly implicit one-step method has the B-stage order q_i on F at the ith stage, if

$$\|y(t_m + c_i h) - u_{m+1}^{(i)}\| \leq \gamma_i h^{q_i + 1} \quad \text{for } u_m = y(t_m),\ h \leq h_0.$$

It is B-consistent of order q if

$$\|y(t_m + h) - u_{m+1}\| \leq \gamma h^{q+1} \quad \text{for } u_m = y(t_m),\ h \leq h_0.$$

DEFINITION 3.2. A linearly implicit one-step method is B-convergent of order q on F if

$$\|y(t_m) - u_m\| \leq \beta h^q \quad \text{for } u_0 = y_0, \quad h \leq h_0, \quad t_0 \leq t_m \leq t_e.$$

DEFINITION 3.3. (see [3]).

A linearly implicit one-step method is C-stable on F if for any two numerical solutions of (1.1)

$$\|u_{m+1}-v_{m+1}\| \leq (1+C_0h)\|u_m-v_m\| \text{ holds for all } 0 < h \leq h_0.$$

The order of B-consistency and B-convergence is therefore independent of the stiffness of the problem. In [3] the following theorem has been proved:

THEOREM 3.1. *Let a method be B-consistent of order q and C-stable on a given problem class. Then it is B-convergent of order q on this class.*

REMARK 3.1. Theorem 3.1. does not yield the best possible B-convergence result. The order of B-convergence may well be $q+1$ rather than q (see [1], [17] and also Theorem 3.5. of this paper).

For linearly implicit one-step methods with (A2) and (A3) one immediately proves the

THEOREM 3.2. *Let $R_0^{(s+1)}(z)$ be A-acceptable. Then the method* (2.1) *is C-stable on F.*

In the following we will derive conditions for B-consistency. The B-convergence follows from Theorem 3.1 and 3.2. Further, we present a simple algorithm for constructing B-convergent methods of order $q > 1$ on F.

Because of $u_{m+1}^{(1)} := u_m$ we set $q_1 = \infty$ and introduce the following index-sets

$$K_i := \{j | 1 \leq j \leq i-1, A_{ij}(z) \neq 0\}, \quad i = 2,\dots,s \tag{3.1}$$
$$K_{s+1} := \{j | 1 \leq j \leq s, \quad B_j(z) \neq 0\}.$$

THEOREM 3.3. *Let $q_i^{(1)} = \min_{j \in K_i} q_j$ (q_j: B-stage order at the j-stage) and let*

$$\sum_{j=1}^{i-1} A_{ij}c_j^l = c_i^{l+1}R_{l+1}^{(i)}(c_iz) \tag{3.2}$$

for $l = 0, 1, \dots, q_i^{(2)}$, where the $R_{l+1}^{(i)}$ are defined by (2.3). *Then the method* (2.1) *has at the ith stage the B-stage order $q_i = \min(q_i^{(1)}+1, q_i^{(2)})$.*

PROOF. From (1.1), (1.2) it follows

$$r(t_m+c_jh) = y'(t_m+c_jh)-Ty(t_m+c_jh)-g(t_m+c_jh, y(t_m+c_jh))$$

and with the assumptions of the theorem for $j \in K_i$

$$\|g(t_m+c_jh, u_{m+1}^{(j)})-g(t_m+c_jh, y(t_m+c_jh))\| \le L\|u_{m+1}^{(j)}-y(t_m+c_jh)\|$$

$$\le L\gamma_j h^{q_i^{(1)}+1} = O(h^{q_i}) \quad \text{for} \quad h \le h_0.$$

We get

$$(3.3) \quad u_{m+1}^{(i)} = R_0 y + h \sum_{j=1}^{i-1} A_{ij}[g_j + r(t_m+c_jh)]$$

$$= R_0 y + h \sum_{j=1}^{i-1} A_{ij}[y'(t_m+c_jh) - Ty(t_m+c_jh)] + O(h^{q_i+1})$$

$$= \sum_{l=0}^{q_i} \frac{1}{l!}(c_ih)^l y^{(l)} - \sum_{l=0}^{q_i} \frac{1}{l!}(c_ih)^l y^{(l)} + R_0 y + \sum_{l=0}^{q_i} \frac{1}{l!} h^{l+1} \left(\sum_{j=1}^{i-1} A_{ij} c_j^l \right) \cdot$$

$$\cdot (y^{(l+1)} - Ty^{(l)}) + O(h^{q_i+1})$$

$$= y(t_m+c_ih) + (R_0 - I - c_ihTR_1)y + \sum_{l=1}^{q_i} \frac{h^l}{l!} c_i^l(-I + lR_l - c_ihTR_{l+1})y^{(l)}$$

$$+ O(h^{q_i+1}).$$

With (2.3) we get finally

$$u_{m+1}^{(i)} = y(t_m+c_ih) + O(h^{q_i+1}). \qquad ■$$

COROLLARY 3.1. *Let* $q^{(1)} = \min_{j \in K_{s+1}} q_j$ *and let*

$$(3.4) \qquad \sum_{j=1}^{s} B_j c_j^l = R_{l+1}^{(s+1)}(z)$$

for $l = 0, \ldots, q^{(2)}$. *Then method* (2.1) *is B-consistent of order* $q = \min(q^{(1)}+1, q^{(2)})$ *on F.*

REMARK 3.2. For Theorem 3.3. and Corollary 3.1. assumption (3) of (1.2) is not necessary.

Theorem 3.3. and Corollary 3.1. allow to construct B-consistent linearly implicit one-step methods.

REMARK 3.3. Method (2.1) is translation invariant, iff (3.2) and (3.4) hold for $l = 0$. (For translation invariance see [2], [9]). Thus, our constructed B-consistent methods are translation invariant.

1-stage formula: Because of $c_1 = 0$ (3.4) can only hold for $l = 0$. The method

$$(3.5) \qquad \begin{array}{c|c} \; & \\ \hline & R_1 \end{array}$$

of order $p = 1$ is B-consistent ($q = 0$). However, it is B-convergent of order 1 (see Hundsdorfer [10]).

2-stage formulas: a) From (3.4) we have the conditions

$$B_1 + B_2 = R_1^{(3)}$$
$$c_2 B_2 = R_2^{(3)}.$$

The methods of order $p = 2$

$$\text{(3.6)} \qquad \begin{array}{c|cc} c_2 & c_2 R_1^{(2)} & \\ \hline & R_1^{(3)} - \dfrac{1}{c_2} R_2^{(3)} & \dfrac{1}{c_2} R_2^{(3)} \end{array}$$

have the B-consistency order $q = 1$.

b) For a 2-stage W-method of order $p = 2$ (i.e. $a_{21} = c_2$, $b_2 c_2 = \frac{1}{2}$, $b_1 + b_2 = 1$, $b_2 \gamma_{21} = -\gamma$, see [7]) we obtain from (2.2)

$$A_{21} = a_{21}(1-\gamma z)^{-1}, \quad R_0^{(2)}(c_2 z) = (1+(c_2-\gamma)z)/(1-\gamma z)$$

$$R_0^{(3)}(z) = \frac{1+(1-2\gamma)z+(\frac{1}{2}-2\gamma+\gamma^2)z^2}{(1-\gamma z)^2}$$

$$B_1 = \frac{b_1}{1-\gamma z} + (\tfrac{1}{2}-\gamma)\frac{z}{(1-\gamma z)^2}, \quad B_2 = \frac{b_2}{1-\gamma z}.$$

From (3.2), (3.4) we have

$$A_{21} = \frac{c_2}{1-\gamma z}, \quad B_2 = \frac{1}{c_2}\frac{\frac{1}{2}-\gamma^2 z}{(1-\gamma z)^2}.$$

One immediately sees that only $\gamma = 0$ or $\gamma = \frac{1}{2}$ is possible. For $\gamma = 0$ the method is not A-stable. With $\gamma = \frac{1}{2}$ we get a W-method of B-consistency order $q = 1$:

$$\text{(3.7)} \qquad \gamma_{21} = -c_2, \quad b_2 = \frac{1}{2c_2}, \quad b_1 = 1 - b_2.$$

This method is A-stable but not strongly A-stable.

3-stage formalas: Let us require second order B-consistency. Conditions (3.2) and (3.4) yield

$$\text{(3.8a)} \qquad A_{21} = c_2 R_1^{(2)}, \quad A_{31} + A_{32} = c_3 R_1^{(3)}, \quad A_{32} c_2 = c_3^2 R_2^{(3)}$$

$$B_1 + B_2 + B_3 = R_1^{(4)}, \quad B_2 c_2 + B_3 c_3 = R_2^{(4)}, \quad B_2 c_2^2 + B_3 c_3^2 = R_3^{(4)}.$$

Because $u_{m+1}^{(2)}$ is only of B-stage order $q_2 = 0$ it is also required that

(3.8b) $$B_2 = 0.$$

Under consideration of (3.2) we obtain from (3.8)

$$R_0^{(4)}(z) = \frac{1+(1-c_3/2)z+\frac{1}{2}(1-c_3)z^2}{1-c_3z/2}$$

and because of assumption (A2) we get $c_3 = 1$.

Therefore the 3-stage linearly implicit method

(3.9)
$$\begin{array}{c|ccc} c_2 & c_2R_1^{(2)} & & \\ 1 & R_1^{(3)}-c_2^{-1}R_2^{(3)} & c_2^{-1}R_2^{(3)} & \\ \hline & R_1^{(4)}-R_2^{(4)} & 0 & R_2^{(4)} \end{array}$$

with the stability function $R_0^{(4)}(z) = (1+z/2)/(1-z/2)$ has the B-consistency order $q = 2$.

4-stage formulas: From (3.2) and (3.4) we find that with our construction principle only $q = 2$ is possible. This implies that the coefficients A_{21}, A_{31}, A_{32} and B_1, B_2, B_3 and B_4 are uniquely determined by c_2, c_3 and c_4 ($c_3 \neq c_4$). The coefficients A_{41}, A_{42} and A_{43} are not uniquely determined; they only have to fulfil

$$A_{41}+A_{42}+A_{43} = c_4R_1^{(4)}$$
$$A_{42}c_2+A_{43}c_3 = c_4^2R_2^{(4)}.$$

One family of methods of B-consistency order $q = 2$ is given by

(3.10)
$$\begin{array}{c|cccc} c_2 & c_2R_1^{(2)} & & & \\ c_3 & c_3R_1^{(3)}-\dfrac{c_3^2}{c_2}R_2^{(3)} & \dfrac{c_3^2}{c_2}R_2^{(3)} & & \\ c_4 & c_4R_1^{(4)}+\left(\dfrac{c_4^2}{c_2}-\dfrac{2c_4^2}{c_3}\right)R_2^{(4)} & -\dfrac{c_4^2}{c_2}R_2^{(4)} & 2\dfrac{c_4^2}{c_3}R_2^{(4)} & \\ \hline & B_1 & B_2 & B_3 & B_4 \end{array}$$

with

$$B_2 = 0, \quad B_3 = \frac{c_4R_2^{(5)}-R_3^{(5)}}{c_3(c_4-c_3)}, \quad B_4 = \frac{R_3^{(5)}-c_3R_2^{(5)}}{c_4(c_4-c_3)}, \quad B_1 = R_1^{(5)}-B_3-B_4.$$

These linearly implicit methods have the B-consistency order $q = 2$. Note that for adaptive RK-methods the stability functions $R_0^{(i)}$ can be chosen arbitrarily (under consideration of the assumptions (A1) and (A2)). From the order conditions in [15] it follows that the conventional order p is at least 3. For the special choice

$$c_2 = c_3 = \tfrac{1}{2} \qquad \text{and } c_4 = 1,$$

we have a method of order $p = 4$ which for $T = 0$ is reduced to the explicit England method.

In the same manner one can construct methods of higher order (e.g. $q = 3$, $s = 7$). To have a B-consistency order q the number of stages has to be greater than for the same conventional order p. For special problems of class F we can derive better results.

3.1. *The case* $r(t) = 0$.

We consider

$$y' = Ty + g(t, y). \tag{3.11}$$

THEOREM 3.4. *Let* $q_i^{(1)} = \min_{j \in K_i} q_j$ *and let* (3.2) *hold for* $l = 0, \ldots, q_i^{(2)}$. *Then the linearly implicit method* (2.1) *has at the ith stage the B-stage order*

$$q_i = \min(q_i^{(1)}, q_i^{(2)}) + 1.$$

PROOF. The beginning of the proof is analogous to that of Theorem 3.3. From (3.3) with (2.3) we then have

$$\begin{aligned} u_{m+1}^{(i)} &= y(t_m + c_i h) + \frac{1}{q_i!} h^{q_i+1} \left(c_i^{q_i+1} R_{q_i+1}^{(i)} - \sum_{j=1}^{i-1} A_{ij} c_j^{q_i} \right) T y^{(q_i)} + O(h^{q_i+1}) \\ &= y(t_m + c_i h) + \frac{1}{q_i!} h^{q_i+1} \left(c_i^{q_i+1} R_{q_i+1}^{(i)} - \sum_{j=1}^{i-1} A_{ij} c_j^{q_i} \right) (y^{(q_i+1)} - g^{(q_i)}) + O(h^{q_i+1}) \end{aligned}$$

and with (1.2) we get

$$u_{m+1}^{(i)} = y(t_m + c_i h) + O(h^{q_i+1}). \qquad \blacksquare$$

COROLLARY 3.2. *Let* $q^{(1)} = \min_{j \in K_{s+1}} q_j$ *and let* (3.4) *hold for* $l = 0, \ldots, q^{(2)}$. *Then method* (2.1) *is B-consistent of order* $q = \min(q^{(1)}, q^{(2)}) + 1$.

EXAMPLE: For method (3.5) condition (3.4) must only hold for $l = 0$. For (3.11) the method has B-consistency order 1.

Analogously we find that the methods (3.6), (3.7), (3.10) have B-consistency order $q = 2;\ 2;\ 3$, respectively.

3.2. *The case* $g(t, y) = 0$.
We consider

$$y' = Ty + r(t). \tag{3.12}$$

The S-stability model problem of Prothero and Robinson [13] belongs to this class. Such problems also arise by discretization in space of problems (1.3) with $q(t, x, u) \equiv 0$.

THEOREM 3.5. *Let* $R_0^{(s+1)}(z)$ *be A-acceptable and let* (3.4) *hold for* $l = 0, \ldots, q$. *Then it follows*:

a) *The method is B-consistent of order* q.

b) *If in addition*

$$D(z) = (R_1^{(s+1)}(z))^{-1}\left(R_{q+2}^{(s+1)} - \sum_{j=1}^{s} B_j c_j^{q+1}\right) \tag{3.13}$$

is uniformly bounded for $z \in \mathbb{C}^-$ *then the method is B-convergent of order* $q+1$.

PROOF. The property a) follows from Corollary 3.1., because the last stage is independent of the internal values $u_{m+1}^{(i)}$, $i = 2, \ldots, s$, To see that b) holds we consider

$$\varepsilon_{m+1} := u_{m+1} - y(t_{m+1}) = u_{m+1} - v_{m+1} + v_{m+1} - y(t_{m+1}), \quad \varepsilon_0 = 0,$$

where v_{m+1} is a numerical solution with $v_m = y(t_m)$. We have

$$u_{m+1} - v_{m+1} = R_0^{(s+1)}(hT)\varepsilon_m$$

$$v_{m+1} - y(t_{m+1}) = \frac{1}{(q+1)!} h^{q+1} hT \left[R_{q+2}^{(s+1)} - \sum_{j=1}^{s} B_j c_j^{q+1} \right] y^{(q+1)} + O(h^{q+2}).$$

Now we consider

$$\hat{\varepsilon}_{m+1} := \varepsilon_{m+1} + h^{q+1} A y^{(q+1)}(t_m + h)$$

with

$$A = \frac{1}{(q+1)!} (R_1^{(s+1)}(hT))^{-1} \left(R_{q+2}^{(s+1)}(hT) - \sum_{j=1}^{s} B_j c_j^{q+1} \right).$$

With our assumptions we get

$$\hat{\varepsilon}_{m+1} = \varepsilon_{m+1} + O(h^{q+1}).$$

On the other hand we obtain

$$\hat{\varepsilon}_{m+1} = R_0^{(s+1)}\hat{\varepsilon}_m + h^{q+1}(A - R_0^{(s+1)}A + hTR_1^{(s+1)}A)y^{(q+1)} + O(h^{q+2})$$

and from (2.3) we get

$$\hat{\varepsilon}_{m+1} = R_0^{(s+1)}\hat{\varepsilon}_m + O(h^{q+2}).$$

After simple manipulations we get

$$\hat{\varepsilon}_{m+1} = O(h^{q+1}) \text{ and finally } \varepsilon_{m+1} = O(h^{q+1}). \qquad \blacksquare$$

REMARK 3.4. Our proof uses the idea of considering $\hat{\varepsilon}_{m+1}$ of Hundsdorfer [10], where an analogous version of result b) is proved. Our formulation, however, allows a simple derivation of corresponding methods.

EXAMPLE: Method (3.6) is B-consistent of order 1. With

$$R_0^{(3)}(z) = \frac{1+(1-2\gamma)z+(\frac{1}{2}-2\gamma+\gamma^2)z^2}{(1-\gamma z)^2} \quad \text{and} \quad \gamma > \tfrac{1}{4},$$

we have

$$[R_1^{(3)}(z)]^{-1}[R_3^{(3)}(z) - c_2^2 B_2(z)] = \frac{2\gamma(1-\gamma)-c_2/2+\gamma^2(c_2-1)z}{1+(\frac{1}{2}-2\gamma)z}.$$

This expression is uniformly bounded and by Theorem 3.5. the method is B-convergent of order 2 for (3.12). For $\gamma = \frac{1}{4}$ we get:

$$[R_1^{(3)}(z)]^{-1}[R_3^{(3)}(z) - c_2^2 B_2(z)] = \tfrac{3}{8} - \frac{c_2}{2} + \tfrac{1}{16}(c_2-1)z.$$

Therefore the method is B-convergent of order $q = 2$ for $c_2 = 1$.

4. Modifications of linearly implicit one-step methods.

Here we will modify method (2.1) in order to get for (1.1) the same B-consistency results as for (3.11). With the assumptions of Theorem 3.4. we have

$$u_{m+1}^{(i)} = y(t_m + c_i h) + \frac{1}{q_i!}h^{q_i+1}\left(c_i^{q_i+1}R_{q_i+1}^{(i)} - \sum_{j=1}^{i-1} A_{ij}c_j^{q_i}\right) \times$$
$$\times\, (y^{(q_i+1)} - g^{(q_i)} - r^{(q_i)}) + O(h^{q_i+1}).$$

One immediately sees, if we add

$$\frac{1}{q_i!}h^{q_i+1}\left(c_i^{q_i+1}R_{q_i+1}^{(i)}-\sum_{j=1}^{i-1}A_{ij}c_j^{q_i}\right)r^{(q_i)}$$

then the method has the B-stage order q_i.

For the modified linearly implicit one-step method

$$\begin{aligned}(4.1)\qquad u_{m+1}^{(i)} &= R_0^{(i)}(c_ihT)u_m+h\sum_{j=1}^{i-1}A_{ij}[f_j-Tu_{m+1}^{(j)}]+\\ &+\frac{1}{q_i!}h^{q_i+1}\left(c_i^{q_i+1}R_{q_i+1}^{(i)}-\sum_{j=1}^{i-1}A_{ij}c_j^{q_i}\right)r^{(q_i)}\\ u_{m+1} &= R_0^{(s+1)}(hT)u_m+h\sum_{j=1}^{s}B_j(f_j-Tu_{m+1}^{(j)})+\\ &+\frac{1}{q!}h^{q+1}\left(R_{q+1}^{(s+1)}-\sum_{j=1}^{s}B_jc_j^q\right)r^{(q)},\end{aligned}$$

we have

THEOREM 4.1. *a) Let* $q_i^{(1)}=\min_{j\in K_i}q_j$ *and let* (3.2) *hold for* $l=0,\dots,q_i^{(2)}$. *Then at the ith stage the modified method* (4.1) *has the B-stage order* $q_i=\min(q_i^{(1)},q_i^{(2)})+1$ *on F.*

b) Let $q^{(1)}=\min_{j\in K_{s+1}}q_j$ *and let* (3.4) *hold for* $l=0,\dots,q^{(2)}$. *Then* (4.1) *is B-consistent of order* $q=\min(q^{(1)},q^{(2)})+1$ *on F.*

The same order results are obtained by another modification, where the function $r(t)$ is treated separately:

$$\begin{aligned}(4.2)\qquad u_{m+1}^{(i)} &= R_0^{(i)}u_m+h\sum_{j=1}^{i-1}A_{ij}g_j+\sum_{l=0}^{q_i}\frac{1}{l!}(c_ih)^{l+1}R_{l+1}^{(i)}r^{(l)}\\ u_{m+1} &= R_0^{(s+1)}u_m+h\sum_{j=1}^{s}B_jg_j+\sum_{l=0}^{q}\frac{1}{l!}h^{l+1}R_{l+1}^{(s+1)}r^{(l)}.\end{aligned}$$

THEOREM 4.2. *For* (4.2) *the results of Theorem* 4.1. *hold.*

PROOF. $u_{m+1}=R_0u_m+h\sum_{j=1}^{s}B_jg_j+\sum_{l=0}^{q}\frac{1}{l!}h^{l+1}R_{l+1}r^{(l)}$

$$=R_0u_m+\sum_{l=0}^{q-1}\frac{h^{l+1}}{l!}R_{l+1}(g^{(l)}+r^{(l)})+\frac{h^{q+1}}{q!}\left(\sum_{j=1}^{s}B_jc_j^qg^{(q)}+R_{q+1}r^{(q)}\right)+O(h^{q+2})$$

$$= R_0 y + \sum_{l=0}^{q-1} \frac{h^{l+1}}{l!} R_{l+1}(y^{(l+1)} - Ty^{(l)}) + \frac{h^{q+1}}{q!} R_{q+1} r^{(q)} + O(h^{q+1})$$

$$= y(t_m+h) + \frac{h^{q+1}}{q!} R_{q+1}(Ty^{(q)} + r^{(q)}) + O(h^{q+1})$$

$$= y(t_m+h) + \frac{h^{q+1}}{q!} R_{q+1}(y^{(q+1)} - g^{(q)}) + O(h^{q+1})$$

$$= y(t_m+h) + O(h^{q+1}).$$

The proof for $u_{m+1}^{(i)}$ is analogous. ■

One immediately sees that the modification has no influence on the C-stability of the method. A linearly implicit method (4.1) or (4.2) with B-consistency order q and an A-acceptable stability function $R_0^{(s+1)}(z)$ is therefore B-convergent of order q.

The modifications require the evaluation of derivatives of $r(t)$.

On the other hand with fewer stages the same order q as for (2.1) is achieved which reduces the number of function evaluations and the number of back-substitutions. These modifications are particularly advantageous for semi-discretized problems (1.3) where almost all components of $r(t)$ are zero (for one dimensional problems only the first and the last components of $r(t)$ are not zero).

EXAMPLES: 1. The modified version of (3.5) with $q = 1$ is B-consistent and B-convergent of order 1. The modified versions of (3.6) with $q_2 = 1$, $q = 2$ and of (3.10) with $q_2 = 1$, $q_3 = q_4 = 2$, $q = 3$ are B-consistent and B-convergent of order 2 and 3, respectively.

2. The modified W-method

$$(I - \tfrac{1}{2}hT)k_1 = f_m + \frac{h}{2} r.$$

$$(I - \tfrac{1}{2}hT)k_2 = g(t_m + c_2 h, u_m + c_2 h k_1) + f_m - g_m + \tfrac{1}{2}hr'_m + \tfrac{1}{2}h^2 c_2 r''_m$$

$$u_{m+1} = u_m + h\left\{\left(1 - \frac{1}{2c_2}\right)k_1 + \frac{1}{2c_2}k_2\right\}$$

is of B-consistency and B-convergence order 2.

5. Some generalizations.

We consider the slightly more general class FT of problems

$$y' = A(t)y + g(t, y) + r(t)$$

and assume that (1.2) holds (with $T := A(t)$). We consider the modified method (3.5) with $T = A(t_m)$:

$$u_{m+1} = R_0(hT)u_m + hR_1(hT)(f_m - Tu_m) + h^2R_2(A'(t_m)u_m + r'(t_m)). \tag{5.2}$$

THEOREM 5.1. *Method* (5.2) *is B-consistent of order* 1 *on* FT.

PROOF. With (2.3) we have

$$\begin{aligned} u_{m+1} &= u_m + hR_1 f_m + h^2R_2(A'(t_m)u_m + r'(t_m)) \\ &= y + hy' + h^2R_2(y'' - g') = y(t_m + h) + O(h^2). \qquad \blacksquare \end{aligned}$$

To show C-stability and consequently B-convergence we need an additional assumption.

THEOREM 5.2. *Let* $R_0(z)$ *be A-acceptable and let* $\|A^{-1}(t)A'(t)\|$ *be uniformly bounded on* FT *for all* $t \in [t_0, t_e]$. *Then* (5.2) *is C-stable on* FT.

PROOF.

$$u_{m+1} - w_{m+1} = R_0[u_m - w_m] + hR_1[g(t_m, u_m) - g(t_m, w_m)] + h^2R_2A'(t_m)(u_m - w_m).$$

With (1.2) and (2.3) we obtain

$$\begin{aligned} \|u_{m+1} - w_{m+1}\| &\le [1 + hL\|R_1\| + h\|R_1 - I\|\|A^{-1}A'\|]\|u_m - w_m\| \\ &\le (1 + C_0h)\|u_m - w_m\|. \qquad \blacksquare \end{aligned}$$

REMARK 5.1. The modification (5.2) requires the evaluation of $A'(t)$. This additional effort may be justified if $A(t) = a(t)\cdot B$, where $a(t)\colon [t_0, t_e] \to R$ and B is a constant (n, n)-matrix. Such problems arise by semidiscretization of

$$\frac{\partial u}{\partial t} = a(t) \sum_{i,k=1}^{\alpha} \frac{\partial}{\partial x_i}\left(b_{ik}(x)\frac{\partial u}{\partial x_k}\right) + q(t, x, u), \quad a(t) > 0, \tag{5.3}$$

with time dependent Dirichlet boundary conditions. The condition for C-stability then reads $|a'(t)/a(t)|$ uniformly bounded.

The following example illustrates the advantage of the modified method (5.2)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = -\frac{t}{\varepsilon}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} y_1 - v_1(t) \\ y_2 - v_2(t) \end{bmatrix} + \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix}, \quad \begin{aligned} v_1(t) &= 1 + e^{-t} \\ v_2(t) &= 1 - e^{-t} \end{aligned}$$

$$t \in [0.5, 2].$$

The initial values correspond to the exact solution

$$y_1(t) = 1+e^{-t}+\exp(-t^2/2\varepsilon)+\exp(-t^2/\varepsilon)$$
$$y_2(t) = 1+e^{-t}+\exp(-t^2/2\varepsilon)+2\exp(-t^2/\varepsilon).$$

Table 5.1. shows the Euclidean norm of the absolute error at the endpoint err(1/20) obtained with constant stepsize $h = 1/20$ and err(1/40) with $h = 1/40$ for various ε. Further, it shows the numerically obtained order of B-convergence

$$q = \log_2 \frac{\text{err}(1/20)}{\text{err}(1/40)}.$$

Table 5.1. *Results for methods* (3.5) *and the modified method* (5.2)

	(3.5) with $R_0 = \frac{1}{1-z}$			(3.5), $R_0 = \frac{1+z/2}{1-z/2}$			(5.2), $R_0 = \frac{1}{1-z}$			(5.2), $R_0 = \frac{1+z/2}{1-z/2}$		
ε	err(1/20)	err(1/40)	q	err(1/20)	err(1/40)	q	err(1/20)	err(1/40)	q	err(1/20)	err(1/40)	q
10^{-2}	9.8 $E-3$	4.9 $E-3$	1.0	4.8 $E-3$	2.4 $E-3$	1.0	2.3 $E-4$	5.7 $E-5$	2.01	1.2 $E-4$	3.0 $E-5$	2.0
10^{-4}	9.8 $E-3$	4.9 $E-3$	1.0	1.6 $E-2$	8.7 $E-3$	0.88	2.4 $E-4$	6.0 $E-5$	2.0	1.8 $E-3$	4.7 $E-4$	1.94
10^{-6}	9.8 $E-3$	4.9 $E-3$	1.0	1.7 $E-2$	8.2 $E-3$	1.05	2.4 $E-4$	6.0 $E-5$	2.0	1.9 $E-3$	4.9 $E-4$	1.96
10^{-8}	9.8 $E-3$	4.9 $E-3$	1.0	1.7 $E-2$	8.3 $E-3$	1.03	2.4 $E-4$	6.0 $E-5$	2.0	1.9 $E-3$	4.9 $E-4$	1.96

For this example method (3.5) has the B-convergence order $q = 1$. For the modified method (5.2) one shows analogously to Theorem 3.5. that the order of B-convergence is $q = 2$ if

$$D(z) = (R_1^{(s+1)}(z))^{-1}R_3^{(s+1)}(z)$$

is uniformly bounded for $z \in \mathbb{C}^-$. For $R_0(z) = (1+z/2)/(1-z/2)$ we have $D(z) = \frac{1}{2}$ which implies B-convergence of order 2. For $R_0(z) = 1/(1-z)$ we obtain $D(z) = (1+z)/z$, which is unbounded for $z \to 0$; the order of B-convergence is 1. For the given stepsizes, however, $D(z)$ is bounded so that numerically we also obtain $q = 2$.

Note, that the accuracy of our modified methods is clearly superior.

Modifications of higher order require higher derivatives of $A(t)$. For problems of type (5.3), however, they may be useful.

THEOREM 5.3. *The method*

$$(5.4)\qquad \begin{aligned} u_{m+1}^{(2)} &= R_0^{(2)}u_m + c_2hR_1^{(2)}[g+r] + c_2^2h^2R_2^{(2)}[A'u_m+r'] \\ u_{m+1} &= R_0^{(3)}u_m + h[(R_1^{(3)} - c_2^{-1}R_2^{(3)})g + c_2^{-1}R_2^{(3)}g_2 + R_1^{(3)}r] + \\ &\quad + h^2[R_2^{(3)}(A'u_m+r') + c_2^{-1}R_3^{(3)}(A'u_{m+1}^{(2)} - A'u_m)] + \tfrac{1}{2}h^3R_3^{(3)}(A''u_m+r'') \end{aligned}$$

with $A' := A'(t_m)$ *is B-consistent of order* $q = 2$ *on* FT.

THEOREM 5.4. *Let* $R_0^{(3)}(z)$ *be A-acceptable and let* $\|A^{-1}(d^iA)/dt^i)\|$, $i = 1, 2$ *be uniformly bounded for all* $t \in [t_0, t_e]$. *Then* (5.4) *is C-stable on* FT.

The proofs of Theorems 5.3. and 5.4. are analogous to those of Theorems 5.1. and 5.2. ■

6. Conclusions.

Our investigations have shown that linearly implicit one step methods with respect to B-convergence properties are suitable for semi-linear problems (1.1), (1.2). The methods suffer from order reduction but by the modifications (4.1), (4.2) this order reduction can be decreased or completely avoided. The modification of linearly implicit one step methods also permits B-convergence results of order ≥ 1 for a slightly more general class of problems.

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