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PRODUCT INTEGRATION RULES FOR VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract.

The numerical solution of Volterra integral equations of the first kind can be achieved via product integration. This paper establishes the asymptotic error expansions of certain product integration rules. The rectangular rules are found to produce expansions containing all powers of h, and the midpoint product method is found to produce even powers of h. Extrapolation to the limit is then applied.

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1. Introduction.

In this paper we consider some product integration methods for the solution of the Volterra integral equation of the first kind

$$\int_0^x k(x,t)y(t)dt = f(x), \qquad 0 \le x \le a, \tag{1.1}$$

where $f(x)$ is defined on $[0,a]$ and $k(x,t)$ on the domain $0 \le t \le x \le a$.

We assume the following conditions are satisfied:

C1. $k(x,x) \neq 0$ for $x \in [0,a]$,

C2. $k(x,t)$ and $\partial k(x,t)/\partial x$ are continuous on $0 \le t \le x \le a$,

C3. $f(0) = 0$ and $f'(x)$ is continuously differentiable on $0 \le x \le a$.

These conditions ensure the existence of a unique continuous solution to (1.1) (see Bôcher [4]). Further if $f(x) \in C^\infty[0,a]$ then $y(t) \in C^\infty[0,a]$ (McAlevey [22]).

Direct methods are obtained by replacing the integral by a numerical quadrature formula. Several authors have investigated the problem and proposed various numerical schemes. Low order methods have been examined by Jones [16], Kobayasi [17] and Linz [18], [19]. High order block by block methods

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have been investigated by De Hoog and Weiss [8], [9]. Brunner [5], [6] has studied the use of piecewise polynomial approximation. More recently linear multistep methods have been investigated by several authors: Gladwin and Jeltsch [13], Holyhead, McKee and Taylor [15], Holyhead and McKee [14], Taylor [24], Baker and Keech [3], Gladwin [11], [12], Andrade and McKee [2] and Wolkenfelt [25]. Product integration techniques have been examined by Anderseen and White [1] and Linz [20], [21]. Brunner [7] and Marchuk and Shaidurov [23] have used extrapolation techniques.

In the present paper we establish asymptotic error expansions for certain product integration methods. Extrapolation to the limit is then applied. In section 2 we present the product quadrature rules. We establish in section 3 some basic results and convergence results for these product methods. Asymptotic error expansions are derived for these methods in section 4. We consider in section 5 inexact moment integrals. Finally in sections 6 and 7 we discuss extrapolations and consider some numerical examples and their extrapolation tables.

2. Product integration rules.

The basic interval $[0, a]$ is divided into n subintervals of equal step length h, where $x_i = ih$, $i = 0, 1, \ldots, n$ and $nh = a$. We replace $y(t)$, $x_i \leq t \leq x_{i+1}$, by the approximation $Y_{i+\alpha}$, $0 \leq \alpha \leq 1$. Thus (1.1) gives

$$\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} k(x_n, t)dt\right) Y_{i+\alpha} = f(x_n) \quad \text{or} \tag{2.1}$$

$$\sum_{i=0}^{n-1} m_i(x_n) Y_{i+\alpha} = f(x_n) \tag{2.2}$$

where $m_i(x_n)$ are the moment integrals.

Linz [20] notes that using separate approximations for $k(x,t)$ and $y(t)$ is particularly useful when $k(x,t)$ varies rapidly, as seems to be the case in many practical applications. Setting $\alpha = 0$ defines the left rectangular product method, $\alpha = 1$ the right rectangular product method, while $\alpha = 1/2$ gives the midpoint product method.

3. Basic results and convergence.

The establishment of convergence and the derivation of the asymptotic error expansions for the rules are easily established in subsequent sections of the paper with the aid of the following theory.

Consider a class of quadrature formulae of the form:

$$I = \int_{x_0}^{x_n} f(t)dt = Q(h) + E_Q(h) + R_{Q,p} \quad \text{where} \tag{3.1}$$

$Q(h)$ is a linear quadrature rule of step length h,
$E_Q(h)$ is a series of correcting terms, and
$R_{Q,\,p}$ is the remainder associated with the truncation of $E_Q(h)$ after the pth term.
In this section we consider the rule

$$(3.2) \qquad Q(h) = h \sum_{i=0}^{n-1} f_{i+\alpha}, \quad \text{where } f_{i+\alpha} = f(x_0+(i+\alpha)h), \qquad 0 \le \alpha \le 1.$$

Using the shift operator E defined by $Ef(x) = f(x+h)$, we have

$$(3.3) \qquad Q(h) = h\left(\frac{E^\alpha}{E-1}\right)(E^n-1)f_0\,;$$

in particular we will consider the left and right rectangular rules, which occur when $\alpha = 0$ and 1 respectively, and the midpoint rule when $\alpha = 1/2$.

The Euler-Maclaurin formula has the remainder

$$(3.4) \qquad R = \frac{n}{(2m)!} B_{2m} h^{2m+1} f^{(2m)}(\xi)$$

where the highest derivative retained in the correction term is of order $2m-3$, $x_0 < \xi < x_n$, and B_{2m} is a Bernoulli number.

For fixed h the remainder term will in general not tend to zero as m increases. However, for fixed m, there is in general a sufficiently small value of h for which the remainder term may be considered negligible. This justifies the use of this class of quadrature formulae, which are generally asymptotic, but provided h is sufficiently small, have the property that truncation at a sufficiently small term yields an error with the same order of magnitude as this term. We assume this holds for the integrals approximated in this paper. In the asymptotic expansions that follow the remainder term is omitted for brevity.

Letting $E_Q(h) = I - Q(h)$ we have

$$(3.5) \qquad E_Q(h) = D^{-1}(E^n-1)f_0 - h\,\frac{E^\alpha}{E-1}\,(E^n-1)f_0$$

$$= h\left[(hD)^{-1} - \frac{E^\alpha}{E-1}\right](f_n - f_0),$$

where the derivative operator D is defined by $Df(x) = f'(x)$. With the aid of the expansions (see [10])

$$(3.6) \qquad \frac{1}{E-1} = (hD)^{-1} - \frac{1}{2} + \frac{1}{12}hD - \frac{1}{729}h^3D^3 + \frac{1}{30{,}240}h^5D^5 - \ldots$$

$$\operatorname{cosech}(hD) = (hD)^{-1} - \frac{1}{6}hD + \frac{7}{360}h^3D^3 - \frac{31}{15{,}120}h^5D^5 + \ldots$$

we obtain the asymptotic error expansions for the rules above. We assume $f(t)$ to be sufficiently smooth so that all derivatives under discussion exist.

For $\alpha = 0$, the left rectangular rule gives

$$E_L(h) = [L_1 h f(t)]_0^{x_n} + [L_2 h^2 df(t)/dt]_0^{x_n} + [L_3 h^4 d^3 f(t)/dt^3]_0^{x_n} + \dots, \tag{3.7}$$

with L_i constants ($L_1 = 1/2$, $L_2 = -1/12$ and $L_3 = 1/720$).

The midpoint rule is obtained by setting $\alpha = 1/2$, giving

$$E_M(h) = [M_1 h^2 df(t)/dt]_0^{x_n} + [M_2 h^4 d^3 f(t)/dt^3]_0^{x_n} + [M_3 h^6 d^5 f(t)/dt^5]_0^{x_n} + \dots, \tag{3.8}$$

with M_i constants.

Finally $\alpha = 1$, gives the right rectangular rule,

$$E_R(h) = [R_1 h f(t)]_0^{x_n} + [R_2 h^2 df(t)/dt]_0^{x_n} + [R_3 h^4 d^3 f(t)/dt^3]_0^{x_n} + \dots, \tag{3.9}$$

with R_i constants.

DEFINITION 1.

Let $Y_0, Y_1, \dots, Y_n$ denote the approximations obtained by a given method for some fixed step length h ($nh = a$). Then the method is said to be convergent if and only if

$$\max_{0 \le i \le n} |y(x_i) - Y_i| \to 0$$

as $h \to 0$, $n \to \infty$, such that $nh = a$.

DEFINITION 2.

A method is said to be of order p, if p is the largest real number for which there exists a finite constant C such that

$$|y(x_i) - Y_i| \le Ch^p \quad \text{for} \quad i = 0, 1, \dots, n$$

for all $h > 0$.

In the subsequent analysis of this section we assume:

(i) Conditions C1, C2 and C3 of section 1 to be valid,

(ii) $k(x, t)$ has a convergent Taylor series in the second variable, $0 \le t \le x \le a$.

To show convergence of the midpoint product method we set $\alpha = 1/2$ in (2.2) and obtain

$$\sum_{i=0}^{n-1} m_i(x_n) Y_{i+1/2} = f(x_n). \tag{3.10}$$

Expanding the moment integral about the point $t = x_{i+1/2}$ gives

$$h \sum_{i=0}^{n-1} \left(k(x_n, x_{i+1/2}) + \frac{(h/2)^2}{3!} k^{02}(x_n, x_{i+1/2}) + \frac{(h/2)^4}{5!} k^{04}(x_n, x_{i+1/2}) + \right. \left. + \dots \right) Y_{i+1/2} = f(x_n), \tag{3.11}$$

where $k^{0j}(x_n, x_{i+\alpha}) = [\partial^j k(x_n, t)/\partial t^j]_{t = x_{i+\alpha}}$.

Using previous results we have

$$h \sum_{i=0}^{n-1} k(x_n, x_{i+1/2}) y(x_{i+1/2}) = f(x_n) - [M_1 h^2 \partial(k(x_n, t)y(t))/\partial t]_0^{x_n} - [M_2 h^4 \partial^3(k(x_n, t)y(t))/\partial t^3]_0^{x_n} - \dots. \tag{3.12}$$

From equations (3.11) and (3.12) we derive

$$h \sum_{i=0}^{n-1} \left(k(x_n, x_{i+1/2}) + \frac{(h/2)^2}{3!} k^{02}(x_n, x_{i+1/2}) + \frac{(h/2)^4}{5!} k^{04}(x_n, x_{i+1/2}) + \dots \right) \varepsilon_{i+1/2}$$

$$= h^2 \left([-M_1 \partial(k(x_n, t)y(t))/\partial t]_0^{x_n} + h \sum_{i=0}^{n-1} \frac{k^{02}(x_n, x_{i+1/2})}{2^2 3!} y(x_{i+1/2}) \right) \tag{3.13}$$

$$+ h^4 \left([-M_2 \partial^3(k(x_n, t)y(t))/\partial t^3]_0^{x_n} + h \sum_{i=0}^{n-1} \frac{k^{04}(x_n, x_{i+1/2})}{2^4 5!} y(x_{i+1/2}) \right) + \dots$$

where $\varepsilon_{i+1/2} = y(x_{i+1/2}) - Y_{i+1/2}$.

On the right-hand side of equation (3.13), each quadrature term is replaced by an integral minus error terms, and we obtain the following:

$$h \sum_{i=0}^{n-1} \left(k(x_n, x_{i+1/2}) + \frac{(h/2)^2}{3!} k^{02}(x_n, x_{i+1/2}) + \frac{(h/2)^4}{5!} k^{04}(x_n, x_{i+1/2}) + \right. \left. + \dots \right) \varepsilon_{i+1/2}$$

$$= h^2 \left([-M_1 \partial(k(x_n, t)y(t))/\partial t]_0^{x_n} + \int_0^{x_n} \frac{k^{02}(x_n, t)}{2^2 3!} y(t)dt \right) \tag{3.14}$$

$$+ h^4 \left([-M_2 \partial^3(k(x_n, t)y(t))/\partial t^3]_0^{x_n} - \left[\frac{M_1}{2^2 3!} \partial(k^{02}(x_n, t)y(t))/\partial t \right]_0^{x_n} \right.$$

$$\left. + \int_0^{x_n} \frac{k^{04}(x_n, t)}{2^4 5!} y(t)dt \right) + \dots.$$

We note that $|\varepsilon_{1/2}| \leq ch^2$ and hence the method is convergent with order at least two as a result of the following lemma.

LEMMA. *Let $\varepsilon_{i+1/2}$ be defined by*

$$h\sum_{i=0}^{n-1}\left(k(x_n, x_{i+1/2})+\frac{(h/2)^2}{3!}k^{02}(x_n, x_{i+1/2})+\frac{(h/2)^4}{5!}k^{04}(x_n, x_{i+1/2})+\dots\right)\varepsilon_{i+1/2}$$
$$= h^2q_1(x_n)+O(h^4) \quad \text{for} \quad n = 2, 3, \dots \quad \text{where}$$

$k(x,t)$ satisfies conditions C1 and C2,
$k(x,t)$ has a convergent Taylor series in the second variable, $0 \le t \le x \le a$,
$q_1(x)$ is differentiable on $(0, a]$,
and $|\varepsilon_{1/2}| \le c_1h^2$, c_1 a constant.
Then $|\varepsilon_{n+1/2}| \le c_2h^2$, for $n \ge 1$, where c_2 is a constant.

PROOF (McAlevey [22], see also Linz [20]).
Similarly it may be shown that the rectangular product methods converge with order of at least one.

4. Asymptotic error expansions.

We now derive the asymptotic error expansion when (1.1) is approximated by the midpoint product rule.

THEOREM 4.1. *Let $\varepsilon_{i+1/2}$ be defined by the following equation:*

$$h\sum_{i=0}^{n-1}\left(k(x_n, x_{i+1/2})+\frac{(h/2)^2}{3!}k^{02}(x_n, x_{i+1/2})+\frac{(h/2)^4}{5!}k^{04}(x_n, x_{i+1/2})+\dots\right)\varepsilon_{i+1/2}$$
$$= h^2q_1(x_n)+h^4q_2(x_n)+h^6q_3(x_n)+\dots$$

where

(i) $q_i(x) \in C^\infty[0, a], a > 0$ *for* $i \ge 1$,
(ii) $q_i(0) = 0$, *for* $i \ge 1$,
(iii) *$k(x,t)$ satisfies conditions C1 and C2 of section 1,*
(iv) *$k(x,t)$ has a convergent Taylor series in the second variable, $0 \le t \le x \le a$.*

Then

$$h\sum_{i=0}^{n-1}\left(k(x_n, x_{i+1/2})+\frac{(h/2)^2}{3!}k^{02}(x_n, x_{i+1/2})+\frac{(h/2)^4}{5!}k^{04}(x_n, x_{i+1/2})+\dots\right)\varepsilon^{(m)}_{i+1/2}$$
$$= h^2q_1^{(m)}(x_n)+h^4q_2^{(m)}(x_n)+h^6q_3^{(m)}(x_n)+\dots \quad \textit{where}$$

$$\varepsilon^{(m)}_{i+1/2} = \frac{\varepsilon_{i+1/2}}{h^{2m}} - \frac{e_1(x_{i+1/2})}{h^{2m-2}} - \frac{e_2(x_{i+1/2})}{h^{2m-4}} - \dots - e_m(x_{i+1/2})$$

and the $e_p(t)$, $1 \le p \le m$, *are defined by*

$$\int_0^x k(x,t)e_p(t)dt = q_1^{(p-1)}(x).$$

The $q_j^{(p)}(x)$, $1 \le p \le m$, $j \ge 1$, *are defined by*

$$q_j^{(p)}(x) = q_{j+1}^{(p-1)}(x) + \\ + \sum_{i=1}^{j}\left[\frac{M_i}{2^{2(j-i)}(2(j-i)+1)!}\partial^{2i-1}(k^{0,2(j-i)}(x,t)e_p(t))/\partial t^{2i-1}\right]_0^x \\ - \int_0^x \frac{k^{0,2j}(x,t)}{2^{2j}(2j+1)!}e_p(t)dt.$$

where the M_j, $j \ge 1$ *are constants, and* $q_j^{(0)}(x) = q_j(x)$. *Moreover* $q_j^{(p)}(0) = 0$ *and* $q_j^{(p)}(x) \in C^\infty[0,a]$.

Proof. The case $m = 1$: we have

$$h\sum_{i=0}^{n-1}\left(k(x_n, x_{i+1/2}) + \frac{(h/2)^2}{3!}k^{02}(x_n, x_{i+1/2}) + \frac{(h/2)^4}{5!}k^{04}(x_n, x_{i+1/2}) + \dots\right)\varepsilon_{i+1/2} \\ = h^2 q_1(x_n) + h^4 q_2(x_n) + h^6 q_3(x_n) + \dots, \tag{4.1}$$

and dividing by h^2 we get

$$h^{-1}\sum_{i=0}^{n-1}\left(k(x_n, x_{i+1/2}) + \frac{(h/2)^2}{3!}k^{02}(x_n, x_{i+1/2}) + \frac{(h/2)^4}{5!}k^{04}(x_n, x_{i+1/2}) + \dots\right)\varepsilon_{i+1/2} \\ = q_1(x_n) + h^2 q_2(x_n) + h^4 q_3(x_n) + \dots. \tag{4.2}$$

Defining $e_1(t)$ to be the solution of

$$\int_0^x k(x,t)e_1(t)dt = q_1(x), \tag{4.3}$$

we note, that under the given conditions for $k(x,t)$ and $q_1(x)$, it follows that $e_1(t) \in C^\infty[0,a]$. Discretising equation (4.3) gives

$$h\sum_{i=0}^{n-1} k(x_n, x_{i+1/2})e_1(x_{i+1/2}) = q_1(x_n) \\ - \sum_{i=1}^{\infty}[M_i h^{2i}\partial^{2i-1}(k(x_n,t)e_1(t))/\partial t^{2i-1}]_0^{x_n}. \tag{4.4}$$

We thus have the following discretisation for $e_1(t)$,

$$\begin{aligned} h\sum_{i=0}^{n-1}\left(k(x_n,x_{i+1/2})+\frac{(h/2)^2}{3!}k^{02}(x_n,x_{i+1/2})+\frac{(h/2)^4}{5!}k^{04}(x_n,x_{i+1/2})+\dots\right)e_1(x_{i+1/2}) \\ = q_1(x_n)+h^2\left[[-M_1\partial(k(x_n,t)e_1(t))/\partial t]_0^{x_n}+h\sum_{i=0}^{n-1}\frac{k^{02}(x_n,x_{i+1/2})}{2^2 3!}e_1(x_{i+1/2})\right] \\ +h^4\left[[-M_2\partial^3(k(x_n,t)e_1(t))/\partial t^3]_0^{x_n}+h\sum_{i=0}^{n-1}\frac{k^{04}(x_n,x_{i+1/2})}{2^4 5!}e_1(x_{i+1/2})\right]+\dots. \end{aligned} \tag{4.5}$$

Rewriting (4.5) replacing summation terms by an integral minus error terms gives

$$\begin{aligned} h\sum_{i=0}^{n-1}\left(k(x_n,x_{i+1/2})+\frac{(h/2)^2}{3!}k^{02}(x_n,x_{i+1/2})+\frac{(h/2)^4}{5!}k^{04}(x_n,x_{i+1/2})+\dots\right)e_1(x_{i+1/2}) \\ = q_1(x_n)+h^2\left[[-M_1\partial(k(x_n,t)e_1(t))/\partial t]_0^{x_n}+\int_0^{x_n}\frac{k^{02}(x_n,t)}{2^2 3!}e_1(t)dt\right] \\ +h^4\left[[-M_2\partial^3(k(x_n,t)e_1(t))/\partial t^3]_0^{x_n}-\left[\frac{M_1}{2^2 3!}\partial(k^{02}(x_n,t)e_1(t))/\partial t\right]_0^{x_n}\right. \\ \left.+\int_0^{x_n}\frac{k^{04}(x_n,t)}{2^4 5!}e_1(t)dt\right]+\dots. \end{aligned} \tag{4.6}$$

Subtracting (4.6) from (4.2) gives

$$\begin{aligned} h\sum_{i=0}^{n-1}\left(k(x_n,x_{i+1/2})+\frac{(h/2)^2}{3!}k^{02}(x_n,x_{i+1/2})+\frac{(h/2)^4}{5!}k^{04}(x_n,x_{i+1/2})+\dots\right)\varepsilon_{i+1/2}^{(1)} \\ = h^2\left[q_2(x_n)+[M_1\partial(k(x_n,t)e_1(t))/\partial t]_0^{x_n}-\int_0^{x_n}\frac{k^{02}(x_n,t)}{2^2 3!}e_1(t)dt\right] \\ +h^4\left[q_3(x_n)+[M_2\partial^3(k(x_n,t)e_1(t))/\partial t^3]_0^{x_n}\right. \\ \left.+\left[\frac{M_1}{2^2 3!}\partial(k^{02}(x_n,t)e_1(t))/\partial t\right]_0^{x_n}-\int_0^{x_n}\frac{k^{04}(x_n,t)}{2^4 5!}e_1(t)dt\right]+\dots \\ = h^2q_1^{(1)}(x_n)+h^4q_2^{(1)}(x_n)+h^6q_3^{(1)}(x_n)+\dots. \end{aligned} \tag{4.7}$$

The general term is given by

$$\begin{aligned} q_j^{(1)}(x) = q_{j+1}(x)+\sum_{i=1}^{j}\left[\frac{M_i}{2^{2(j-i)}(2(j-i)+1)!}\right. \\ \left.\times\ \partial^{2i-1}(k^{0,2(j-i)}(x,t)e_1(t))/\partial t^{2i-1}\right]_0^x-\int_0^x\frac{k^{0,2j}(x,t)}{2^{2j}(2j+1)!}e_1(t)dt. \end{aligned}$$

It follows from the smoothness conditions on $q_j(x)$, $k(x,t)$ and $e_1(t)$ that $q_j^{(1)}(x) \in C^\infty[0,a]$. Moreover from their definition $q_j^{(1)}(0) = 0$, $j \geq 1$. This completes the proof for $m = 1$.

The general case follows by a similar inductive argument. ■

COROLLARY. Since

$$h \sum_{i=0}^{n-1} \left(k(x_n, x_{i+1/2}) + \frac{(h/2)^2}{3!} k^{02}(x_n, x_{i+1/2}) + \frac{(h/2)^4}{5!} k^{04}(x_n, x_{i+1/2}) + \ldots \right) \varepsilon_{i+1/2}^{(m)}$$
$$= h^2 q_1^{(m)}(x_n) + h^4 q_2^{(m)}(x_n) + h^6 q_3^{(m)}(x_n) + \ldots$$

and noting $|\varepsilon_{1/2}^{(m)}| \leq c_1 h^2, c_1$ a constant, it follows from the Lemma of section 3 that $|\varepsilon_{n+1/2}^{(m)}| \leq c_2 h^2$ for $n \geq 1$, where c_2 is a constant, that is

$$\left| \frac{\varepsilon_{n+1/2}}{h^{2m}} - \frac{e_1(x_{n+1/2})}{h^{2m-2}} - \frac{e_2(x_{n+1/2})}{h^{2m-4}} - \ldots - e_m(x_{n+1/2}) \right| \leq c_2 h^2.$$

Hence it follows that

$$\varepsilon_{n+1/2} = h^2 e_1(x_{n+1/2}) + h^4 e_2(x_{n+1/2}) + \ldots + h^{2m} e_m(x_{n+1/2}) + O(h^{2m+2}),$$

where m is arbitrary. Thus we have $\varepsilon_{n+1/2} \sim \sum h^{2j} e_j(x_{n+1/2})$. (Linz [20] established $\varepsilon_{n+1/2} = h^2 e_1(x_{n+1/2}) + O(h^3)$).

In like manner, asymptotic error expansions may be derived for the left and right rectangular product methods producing all powers of h. These results are summarised in the following theorem:

THEOREM 4.2. *If the same smoothness conditions hold for* (1.1) *as in Theorem* 4.1 *then*

(*i*) *for the left rectangular product rule* $\varepsilon_n \sim \sum h^j e_j(x_n)$,

(*ii*) *for the right rectangular product rule* $\varepsilon_{n+1} \sim \sum h^j e_j(x_{n+1})$.

(See McAlevey [22]).

5. Inexact moments.

In the preceding sections of this paper, we have assumed the moment integral has been exact. In the absence of exact moments, equation (2.2) is replaced by

$$\sum_{i=0}^{n-1} \hat{m}_i(x_n) \hat{Y}_{i+\alpha} = f(x_n) \tag{5.1}$$

where $\hat{m}_i(x_n)$ is the inexact moment integral, and $\hat{Y}_{i+\alpha}$ the approximating values generated by the rule.

In practise the moment integral may be approximated by a quadrature rule of arbitrary order. Linz [20] has shown that when the midpoint product method has inexact moments, calculated by a rule of degree q then

$$\hat{\varepsilon}_{n+1/2} = h^2 e_1(x_{n+1/2}) + h^q \hat{e}_1(x_{n+1/2}) + \dots \qquad \text{where}$$

$$\hat{\varepsilon}_{i+1/2} = y(x_{i+1/2}) - \hat{Y}_{i+1/2}.$$

Clearly q is to be chosen as high as possible, to ensure it produces only higher order contributions to the asymptotic error expansion of $\hat{\varepsilon}_{n+1/2}$.

A general theory for an arbitrary quadrature rule is not presented here due to its cumbersome nature. Instead we consider an example, showing how an asymptotic error expansion may be generated for inexact moments.

EXAMPLE. We consider the midpoint method (3.10) from a different viewpoint. We may regard the ordinary midpoint method as being the result of the midpoint product method with an inexact moment, namely $hk(x_n, x_{i+1/2})$.

Using the result that the inexact moment may be expressed as the exact moment minus error terms, we have

$$hk(x_n, x_{i+1/2}) = \left(hk(x_n, x_{i+1/2}) + \frac{h^3}{2^2 3!} k^{02}(x_n, x_{i+1/2}) + \frac{h^5}{2^4 5!} k^{04}(x_n, x_{i+1/2}) + \dots\right)$$
$$- \left([M_1 h^2 \partial k(x_n, t)/\partial t]_{x_i}^{x_{i+1}} + [M_2 h^4 \partial^3 k(x_n, t)/\partial t^3]_{x_i}^{x_{i+1}} + \dots\right).$$

Thus (5.1) is replaced by

$$(5.2) \qquad h \sum_{i=0}^{n-1} \Bigg\{ k(x_n, x_{i+1/2}) + h^2 \left(\frac{1}{2^2 3!} - M_1\right) k^{02}(x_n, x_{i+1/2})$$
$$+ h^4 \left(\frac{1}{2^4 5!} - \frac{M_1}{2^2 3!} - M_2\right) k^{04}(x_n, x_{i+1/2})$$
$$+ h^6 \left(\frac{1}{2^6 7!} - \frac{M_1}{2^4 5!} - \frac{M_2}{2^2 3!} - M_3\right) k^{06}(x_n, x_{i+1/2}) + \dots \Bigg\} \hat{Y}_{i+1/2} = f(x_n).$$

The asymptotic error expansion may now be established in the usual manner. However, the coefficients of the $k^{0,2j}(x_n, x_{i+1/2})$ are found to vanish (McAlevey [22]), and hence (5.2) reduces to

$$(5.3) \qquad h \sum_{i=0}^{n-1} k(x_n, x_{i+1/2}) \hat{Y}_{i+1/2} = f(x_n).$$

Thus we have the following equation for $\hat{\varepsilon}_{i+1/2}$

$$h\sum_{i=0}^{n-1} k(x_n, x_{i+1/2})\hat{\varepsilon}_{i+1/2} = -[M_1 h^2 \partial(k(x_n, t)y(t))/\partial t]_0^{x_n}$$
$$-[M_2 h^4 \partial^3(k(x_n, t)y(t))/\partial t^3]_0^{x_n} - \ldots.$$

Clearly this is the ordinary midpoint method.

6. Extrapolations to improve accuracy.

The asymptotic error expansions derived provide a justification for the use of extrapolation to improve accuracy. Consider the midpoint method. Solutions are computed with step lengths h and $3h$ (to make the grid points coincide). Then at a particular point Y_p we have

$$Y_p(h) = y(x_{n+1/2}) + h^2 e_1(x_{n+1/2}) + O(h^4), \tag{6.1}$$

$$Y_p(3h) = y(x_{n+1/2}) + 9h^2 e_1(x_{n+1/2}) + O(h^4). \tag{6.2}$$

An improved solution is given by

$$Y_p(3h, h) = 1/8(9Y_p(h) - Y_p(3h)), \tag{6.3}$$
$$\text{where } Y_p(3h, h) = y(x_{n+1/2}) + O(h^4).$$

Similar results are obtained for the rectangular product rules.

7. Numerical examples.

We now solve numerically by the midpoint method, the left rectangular and the right rectangular methods the equation:

$$\int_0^x (4 + x\cos 49t)y(t)dt =$$
$$= 4(1-\cos x) + \frac{x}{2}\{(1/50)(1-\cos(50x)) - 1/48)(1-\cos(48x))\}.$$

Solution: $y(t) = \sin t$.
Evaluation is at $x = 1/6$ with $\sin(1/6) = 0.16589\,61327$.

In the extrapolation tables of this section, the ratio column contains the ratios of errors of consecutive column entries.

In Table 1 we note h^2 and h^4 convergence. An extremely rapid improvement is achieved. (Note $3^2 = 9$ and $3^4 = 81$ are the limiting ratios since for the method a step length with a multiple of 3 is chosen.) Ratio columns then become swamped by rounding errors.

Table 1. *Method is the midpoint product. The sequence of step lengths is* (1/9, 1/81, 1/243, 1/729, 1/2187).

	ratio	h^2 extrap	ratio	h^4 extrap
.16442967				
.16564627	5.87	.16579835		
.16586647	8.42	.16589399	45.69	.16589519
.16589281	8.93	.16589610	71.33	.16589613
.16589576	8.97	.16589613		.16589613

In Tables 2 and 3 we note h and h^2 convergence. The improvement due to extrapolation is not as rapid as in Table 1.

Table 2. *Method is the left rectangular product. Sequence of step lengths is* (1/12, 1/24, 1/48, 1/96, 1/192, 1/384, 1/768).

	ratio	h extrap	ratio	h^2 extrap
.20674330				
.18654631	1.98	.16634932		
.17613236	2.02	.16571841		.16550811
.17101301	2.00	.16589366		.16595208
.16845861	2.00	.16590421		.16590773
.16717872	2.00	.16589883	2.99	.16589704
.16653779	2.00	.16589686	3.70	.16589620

Table 3. *Method is the right rectangular product. The sequence of step lengths is* (1/12, 1/24, 1/48, 1/96, 1/192, 1/384, 1/768).

	ratio	h extrap	ratio	h^2 extrap
.12442305				
.14510974	2.01	.16579643		
.15553971	2.01	.16596968		.16602743
.16073706	2.00	.16593441	1.92	.16592265
.16332220	2.00	.16590734	3.42	.16589832
.16461063	2.00	.16589906	3.83	.16589630
.16525375	2.00	.16589687	3.96	.16589614

8. Concluding remarks.

Many sophisticated high order methods have been suggested in the solution of the Volterra integral equation of the first kind. We have shown that under certain smoothness conditions low order product integration rules may be used to achieve solutions of arbitrarily high order.

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