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A NOTE ON THE STACK SIZE OF REGULARLY DISTRIBUTED BINARY TREES

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Abstract.

Assume that in one unit of time a node is stored in the stack or is removed from the top of the stack during postorder-traversing of a binary tree. If all binary trees are equally likely the average stack size after t units of time and the variance is computed as a function of the proportion $\varrho = t/n$.

1. Introduction.

Let $T(n)$, $n \in \mathrm{N}$, be the set of all *extended binary trees* ([6]) with n leaves and $T \in T(n)$. The *stack size* $S(T)$ is recursively defined by

$$S(T) := \text{IF } |T| = 1 \text{ THEN } 1 \text{ ELSE IF } S(T_1) > S(T_2) \text{ THEN } S(T_1) \text{ ELSE } S(T_2)+1;$$

where $|T|$ is the number of nodes of the tree T and T_1 (T_2) is the left (right) subtree of T. $S(T)$ is the maximum number of nodes stored in the stack during postorder-traversing of $T \in T(n)$. In [4] it is implicitly shown that the average stack size of a binary tree $T \in T(n)$ is asymptotically given by $\sqrt{(\pi n)} - 1/2 + O(\ln n/\sqrt{n})$ assuming that all n-node trees are equally likely. The variance is computed in [5] and is asymptotically given by

$$(\pi/3-1)\pi n + \tfrac{1}{12} - \tfrac{1}{18}\pi^2 + \tfrac{1}{12}\pi + O(\ln n/n^{\frac{1}{2}-\varepsilon}) \text{ for all } \varepsilon > 0 .$$

In this paper we consider an analogous problem. Evaluating a binary tree $T \in T(n)$ in postorder we assume that in one unit of time a node is stored in the stack or is removed from the top of the stack. Considering all trees $T \in T(n)$ equally likely we shall compute the average number of nodes $R_1(n,t)$ stored in the stack after t units of time as a function of the proportion $\varrho = t/n$. Moreover, we give an asymptotic equivalent for the sth moment $R_s(n,t)$ with respect to the origin, and for the variance.

2. The average stack size after t units of time.

Obviously, each path from $(t,k) = (1,1)$ to $(t,k) = (2n-1,1)$ in Figure 1 corresponds to the evaluation of a binary tree $T \in T(n)$ in postorder ([6; p. 316]); for example, the marked path in Figure 1 corresponds to the following tree $T \in T(6)$.

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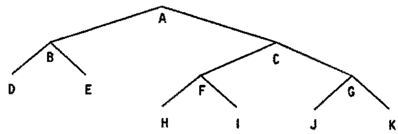

If we reach the point (i,j) then we have exactly j nodes in the stack after i units of time.

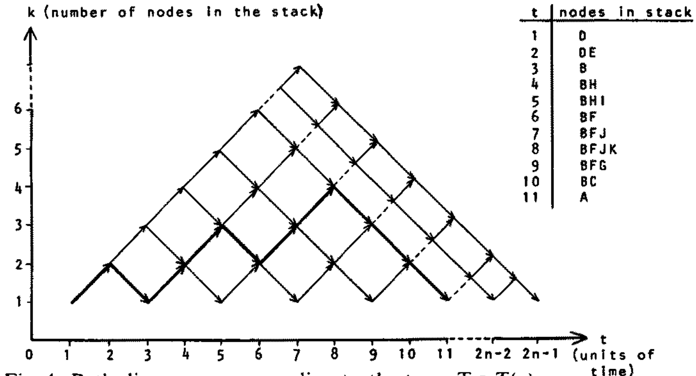

Fig. 1. Path diagram corresponding to the trees $T \in T(n)$.

Now, let $H(n,k,t)$ be the number of binary trees $T \in T(n)$ having exactly k nodes in the stack after t units of time. An inspection of Figure 1 shows that this number is the product of

(a) the number of paths from $(1,1)$ to (t,k), which is

$$\frac{k}{t}\binom{t}{(t+k)/2}, \quad \text{and}$$

(b) the number of paths from (t,k) to $(2n-1,1)$, which is

$$\frac{k}{2n-t}\binom{2n-t}{n-(t+k)/2}.$$

These enumeration results of the number of paths are well-known ([3]). Hence

$$H(n,k,t) = \frac{k^2}{t(2n-t)}\binom{t}{(t+k)/2}\binom{2n-t}{n-(t+k)/2}. \tag{1}$$

Obviously, we have the conditions $k \leqq t \leqq 2n-1$ and $(k+t) \equiv 0 \bmod 2$. Now, let $|T(n)| = t(n)$. It is well-known ([6]) that

$$t(n) = \frac{1}{n}\binom{2n-2}{n-1}. \tag{2}$$

Considering all binary trees $T \in T(n)$ equally likely the quotient $p(n,k)$

$=H(n,k,t)/t(n)$ is the probability that there are k nodes in the stack after t units of time during postorder-traversing of a binary tree $T \in T(n)$. Therefore, the sth moment is given by

$$R_s(n,t) = t(n)^{-1} \sum_{k=1}^{t} k^s H(n,k,t) . \tag{3}$$

Supposing $b \in \mathbf{Z}$ and using the explicit expression for $H(n,k,t)$ given in (1) we obtain by a simple calculation

$$\begin{aligned} &R_s(n,2t+b) \\ &= 2\frac{2n-1}{(2t+b)(2n-2t-b)} \sum_{k\geqq 0} (2k+b)^{s+2} \binom{2t+b}{t-k}\binom{2n-2t-b}{n-t+k}\Big/\binom{2n}{n} \end{aligned} \tag{4}$$

where $b \in \{0,1\}$. In order to compute $R_s(n,t)$ as a function of the proportion of the units of time t to the whole number of units of time $2n$ we have to compute an asymptotic equivalent for the sum

$$h_{b,s}(n,t) = \sum_{k\geqq 0} k^s \binom{2t+b}{t-k}\binom{2n-2t-b}{n-t+k}\Big/\binom{2n}{n}, \quad s\geqq 0 \tag{5}$$

because

$$R_s(n,2t+b) = 2\frac{2n-1}{(2t+b)(2n-2t-b)} \sum_{\mu=0}^{s+2} \binom{s+2}{\mu} 2^\mu b^{s+2-\mu} h_{b,\mu}(n,t) . \tag{6}$$

In [5] a method is given for the computation of a closed expression for $h_{0,3}(n,t)$; this method works also for $h_{0,s}(n,t)$ where s is odd, but not for even s. In this paper we shall give another procedure for the computation of an asymptotic equivalent to $h_{b,s}(n,t)$ effective for all $s \geqq 0$.

Let $\varrho_b = (t+b/2)/n$, $x = (k+b/2)/(\varrho_b n)$ and for $i \geqq 1$

$$a_i = \frac{\varrho_b}{i(2i-1)}\left[1+\left(\frac{\varrho_b}{1-\varrho_b}\right)^{2i-1}\right], \quad b_i = \frac{1}{i}\left[1+\left(\frac{\varrho_b}{1-\varrho_b}\right)^{2i}\right],$$

$$c_0 = \frac{1}{8}\left[1-\frac{1}{\varrho_b(1-\varrho_b)}\right], \quad c_i = \frac{1}{6\varrho_b}\left[1+\left(\frac{\varrho_b}{1-\varrho_b}\right)^{2i+1}\right].$$

For $\varrho_b = \text{const.}$ we have by Stirling's approximation

$$\begin{aligned} \binom{2t+b}{t-k}\binom{2n-2t-b}{n-t+k}\Big/\binom{2n}{n} &= (\pi n \varrho_b(1-\varrho_b))^{-\frac{1}{2}} \exp(-n(a_1x^2+a_2x^4+\ldots)+ \\ &+\tfrac{1}{2}(b_1x^2+b_2x^4+\ldots)+\frac{1}{n}(c_0+c_1x^2+c_2x^4+\ldots)+O(x^2n^{-3})) \end{aligned}$$

when $-1<x<+1$, and $\binom{2t+b}{t-k}\binom{2n-2t-b}{n-t+k}\Big/\binom{2n}{n} = O(n^{-\frac{1}{2}}\exp(-n^{2\varepsilon}))$

when $k \geqq \sqrt{\{\varrho_b(1-\varrho_b)\}}\, n^{\varepsilon+\frac{1}{2}} - b/2$, for all fixed $\varepsilon > 0$. Therefore the sum of all terms for $k \geqq \sqrt{\{\varrho_b(1-\varrho_b)\}}\, n^{\varepsilon+\frac{1}{2}} - b/2$ in (5) is negligible, being $O(n^{-m})$ for all $m > 0$. Hence we may take $x = O(n^{-\frac{1}{2}+\varepsilon})$ in (5). Therefore

$$\binom{2t+b}{t-k}\binom{2n-2t-b}{n-t+k}\Big/\binom{2n}{n} = (\pi n \varrho_b(1-\varrho_b))^{-\frac{1}{2}} \exp\left(-n(a_1x^2+a_2x^4)+\tfrac{1}{2}b_1x^2+c_0/n+O(n^{-2+\varepsilon})\right).$$

Using the definition of x, ϱ_b and the above explicit expressions for a_1, a_2, b_1, c_0 we obtain by an elementary computation

$$(7)\quad \binom{2n+b}{t-k}\binom{2n-2t-b}{n-t+k}\Big/\binom{2n}{n} = (\pi n \varrho_b(1-\varrho_b))^{-\frac{1}{2}} \exp\left(-\frac{(k+b/2)^2}{n\varrho_b(1-\varrho_b)}+A_{n,k,b}\right)$$

where

$$A_{n,k,b} = \frac{1}{8n} - \frac{1}{8n\varrho_b(1-\varrho_b)} + \frac{(1-\varrho_b)^2+\varrho_b^2}{2n^2\varrho_b^2(1-\varrho_b)^2}(k+\tfrac{1}{2}b)^2 - \\ - \frac{(1-\varrho_b)^3+\varrho_b^3}{6n^3\varrho_b^3(1-\varrho_b)^3}(k+\tfrac{1}{2}b)^4 + O(n^{-1.5+\varepsilon}).$$

We now consider the function

$$g_{l,b}(n,t) = \sum_{k \geqq 1} k^l \exp\left(-k^2/(n\varrho_b(1-\varrho_b))\right), \qquad l \text{ fixed}.$$

Again the terms for $k \geqq \sqrt{\{\varrho_b(1-\varrho_b)\}}\, n^{\varepsilon+\frac{1}{2}} - b/2$ are negligible. Hence, we can use (7) to express $h_{b,s}(n,t)$ in terms of $g_{l,b}(n,t)$:

$$(8)\quad h_{b,s}(n,t) = (\pi n \varrho_b(1-\varrho_b))^{-\frac{1}{2}} \left[\left(1 - \frac{\varrho_b^2-\varrho_b+2b^2+1}{8n\varrho_b(1-\varrho_b)}\right) g_{s,b}(n,t) - \right. \\ - \frac{b}{n\varrho_b(1-\varrho_b)} g_{s+1,b}(n,t) + \frac{2\varrho_b^2-2\varrho_b+b^2+1}{2n^2\varrho_b^2(1-\varrho_b)^2} g_{s+2,b}(n,t) - \\ \left. - \frac{3\varrho_b^2-3\varrho_b+1}{6n^3\varrho_b^3(1-\varrho_b)^3} g_{s+4,b}(n,t) + O(g_{s,b}(n,t)n^{-1.5+\varepsilon})\right].$$

Now we compute the asymptotic behavior of $g_{s,b}(n,t)$. With the wellknown formula

$$\exp(-x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(z) x^{-z}\, dz, \qquad x > 0, \qquad c > 0,$$

where $\Gamma(z)$ is the complete gamma function, we find

$$g_{l,b}(n,t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(z) n^z \varrho_b^z (1-\varrho_b)^z \zeta(2z-l)\, dz$$

with $c > (l+1)/2$. Here, $\zeta(z)$ is the Riemann zeta function. It can be shown by a well-known method that we can shift the integration line to the left as far as we please if we only take the residues into account. There are simple poles at $z = (l+1)/2$ and possibly at $z = -k$, $k \in \mathbb{N}_0$. A computation of these residues leads to

$$g_{l,b}(n,t) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}(l+1))[n\varrho_b(1-\varrho_b)]^{(l+1)/2} + \zeta(-l) - \frac{\zeta(-2-l)}{n\varrho_b(1-\varrho_b)} + O(n^{-2}) .$$

Therefore we get from (8) by an elementary calculation for all $\varepsilon > 0$:

$$\begin{aligned} h_{b,s}(n,t) = {} & \tfrac{1}{2}\pi^{-\frac{1}{2}}[p_1 y^{s/2} + p_2 y^{(s-1)/2} + p_3 y^{(s-2)/2} + p_4 y^{-\frac{1}{2}} + \\ & + O(n^{(s-3)/2+\varepsilon})] \end{aligned}$$

where $y = n\varrho_b(1-\varrho_b)$ and:

$$\begin{aligned} p_1 &= \Gamma(\tfrac{1}{2}(s+1)), \\ p_2 &= -b\Gamma(\tfrac{1}{2}(s+2)), \\ p_3 &= -\tfrac{1}{8}\Gamma(\tfrac{1}{2}(s+1))[\varrho_b^2 - \varrho_b + 2b^2 + 1] + \tfrac{1}{2}\Gamma(\tfrac{1}{2}(s+3))[2\varrho_b^2 - 2\varrho_b + b^2 + 1] - \\ & \quad -\tfrac{1}{6}\Gamma(\tfrac{1}{2}(s+5))[3\varrho_b^2 - 3\varrho_b + 1], \\ p_4 &= 2\zeta(-s) . \end{aligned}$$

Returning to (6) we obtain with this approximation the following

LEMMA. *Let $b \in \{0,1\}$ and $\varrho_b = (t + \frac{1}{2}b)/n$. The* sth *moment is given for all $\varepsilon > 0$ by*

$$(a) \quad R_1(n, 2t+b) = \sqrt{(n/\pi)}\left[4\sqrt{\{\varrho_b(1-\varrho_b)\}} - \frac{5\varrho_b^2 - 5\varrho_b + 1}{2n\sqrt{\{\varrho_b(1-\varrho_b)\}}} + O(n^{-3/2+\varepsilon})\right]$$

$$(b) \quad R_2(n, 2t+b) = n\left[6\varrho_b(1-\varrho_b) - \frac{1}{n}(9\varrho_b^2 - 9\varrho_b + 2) + O(n^{-3/2+\varepsilon})\right]$$

$$\begin{aligned} (c) \quad R_s(n, 2t+b) = {} & \pi^{-\frac{1}{2}} 2^{s+1}\Gamma(\tfrac{1}{2}(s+3))\,[n\varrho_b(1-\varrho_b)]^{s/2} + \\ & + O(n^{(s-1)/2}) \quad \textit{for } s \geqq 3 . \end{aligned}$$

Since $R_1(n,t)$ is the average number of nodes in the stack after t units of time during postorder-traversing of a tree $T \in T(n)$ and the variance $\sigma^2(n)$ is given by $R_2(n,t) - R_1^2(n,t)$ we get with the preceding Lemma the following

THEOREM. *Assuming that all binary trees with n leaves are equally likely the average number of nodes $R_1(n,t)$ stored in the stack after t units of time during postorder-traversing of $T \in T(n)$ is given for all $\varepsilon > 0$ by*

$$R_1(n,t) = 4\sqrt{\{n\varrho(1-\varrho)/\pi\}} - \frac{5\varrho^2 - 5\varrho + 1}{2\sqrt{\{\pi n\varrho(1-\varrho)\}}} + O(n^{-1+\varepsilon})$$

where $\varrho = t/2n = const.$ *For all* $\varepsilon > 0$ *the variance is*

$$\sigma^2(n,t) = (6-16/\pi)\varrho(1-\varrho)n + (9-20/\pi)\varrho(1-\varrho) - (2-4/\pi) + O(n^{-\frac{1}{2}+\varepsilon}).$$

Figure 2 shows the graph of $R_1(n,t)$ and $\sigma(n,t)$ as functions of the proportion of the units of time t to the whole number of units of time $2n$ needed to traverse a tree $T \in T(n)$.

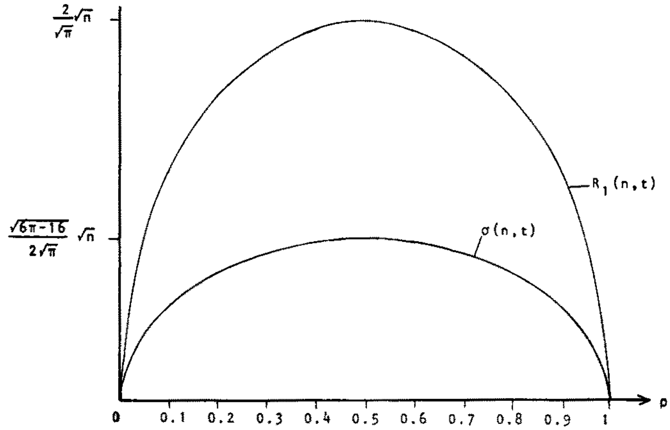

Figure 2. The average stack size and the standard deviation as functions of $\varrho = t/2n$.

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