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ON THE NUMERICAL STABILITY OF SPLINE FUNCTION APPROXIMATIONS TO SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract.

A procedure, using spline functions of degree m, deficiency $k-1$, for obtaining approximate solutions to nonlinear Volterra integral equations of the second kind is presented. The paper is an investigation of the numerical stability of the procedure for various values of m and k.

1. Introduction.

Recently, we presented in [3] and [4] a method for the construction of a continuous approximation to solutions of nonlinear Volterra integral equations of the second kind. The constructed approximation is a spline of degree m in the continuity class C^{m-1}, C^1 in [3] and [4], respectively. It is observed in [3] that the method is divergent for $m > 2$. On the other hand, it is observed in [4] that by relaxing the spline continuity requirements an apparently stable method is obtained. The present paper is a study of these observations. Specifically, we discuss the numerical stability of the method for a spline of degree m, deficiency $k-1$.

The method is described in the next section. In section 3, where the stability problem is discussed, it is shown, among other things, that the method for splines with full continuity is divergent for all $m > 2$. Finally, in section 4, we give several numerical results illustrating some of the main conclusions of the paper.

2. Description of the method.

Consider the Volterra integral equation

$$(1) \qquad f(x) = g(x) + \int_0^x K(x,y,f(y))\,dy, \quad 0 \leqq x \leqq Q\,.$$

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Given two positive integers k and N, we subdivide $(0,Q)$ into subintervals of equal length

$$h = Q/kN . \tag{2}$$

Choose a quadrature formula

$$\int_0^1 u(x)\,dx \doteqdot \sum_{i=1}^{n} w_i u(x_i); \quad 0 \leqq x_i \leqq 1, \quad \text{for all} \quad i , \tag{3}$$

of degree $\geqq m$. Further, denote by $R_p(u)$ the repeated rule arising out of (3), viz.,

$$R_p(u) = h \sum_{i=1}^{n} \sum_{j=0}^{p-1} w_i u(jh + hx_i) . \tag{4}$$

Now for x in $[krh, k(r+1)h]$, $r = 0(1)N-1$, define the function $S(x)$ by

$$\begin{cases} S(x) = \sum_{j=0}^{m-k} \frac{1}{j!}(x-krh)^j S_{kr}^{(j)} + \sum_{j=1}^{k} (x-krh)^{m-k+j} a_j^{(r)}, \\ S_0^{(j)} = \left[\frac{d^j}{dx^j} S(x)\right]_{x=0} = f_0^{(j)}, \quad j = 0,1,\ldots,m-k , \end{cases} \tag{5}$$

where the parameters $a_j^{(r)}$ are determined according to the relations

$$S_{kr+j} = g_{kr+j} + R_{kr+j}\big(K((kr+j)h, y, S(y))\big), \quad j = 1,2,\ldots,k . \tag{6}$$

It is not difficult to show that the above construction defines $S(x)$ uniquely as a spline function of degree m, deficiency $(k-1)$ (see [1] for a definition of deficient splines). An outline of the proof may be found in [3] and [4]. Observe that $S(x)$ is in the continuity class C^{m-k}.

Henceforward, we shall refer to an m – spline deficiency $(k-1)$ method simply as an (m,k) – method.

3. Numerical Stability.

In this section we study the behaviour of the method as applied to the integral equation

$$f(x) = 1 + \lambda \int_0^x f(y)\,dy , \tag{7}$$

λ being a constant with negative real part.

DEFINITION 1. An (m,k) – method is said to be stable if all solutions $\{S_{kr}\}$ remain bounded, as $r \to \infty$, $h \to 0$ while $x_{kr} = krh$ remains fixed.

Some of the methods we discuss below however, possess an even stronger property which, in the terminology of Dahlquist [2] for ordinary differential equations, is called A – stability.

DEFINITION 2. An (m,k) – method is called A – stable if all solutions $\{S_{kr}\}$ tend to zero, as $r \to \infty$, when the method is applied with fixed positive h to any integral equation of the form (7).

It is convenient at this stage to introduce some more notations. We write for $r = 0(1)N-1$

$$(8)\qquad \begin{cases} \boldsymbol{a}_r = (h^{m-k+1}a_1^{(r)}, h^{m-k+2}a_2^{(r)}, \ldots, h^m a_k^{(r)})^T, \\ \boldsymbol{S}_r = (S_{kr}, \dfrac{h}{1!}S_{kr}^{(1)}, \ldots, \dfrac{h^{m-k}}{(m-k)!}S_{kr}^{(m-k)})^T ; \end{cases}$$

and we further denote by B the $k \times k$ matrix whose (i,j) – element is

$$(9)\qquad b_{ij} = (1 - \frac{\lambda h i}{m-k+j+1})i^{m-k+j-1}; \quad i,j = 1(1)k ,$$

and by C the $k \times (m-k+1)$ matrix whose (i,j) – element is

$$(10)\qquad c_{ij} = \begin{cases} \lambda h;\ j = 1;\ i = 1(1)k; \\ \left(\dfrac{\lambda h i}{j} - 1\right) i^{j-1};\ j = 2(1)m-k+1;\ i = 1(1)k . \end{cases}$$

In this notation, we find from (6), (4), (5) and in view of the fact that the quadrature rule (3) is of degree $\geqq m$

$$(11)\qquad B\boldsymbol{a}_r = C\boldsymbol{S}_r, \quad r = 0,1,\ldots,N .$$

Also direct differentiation of the first of relations (5) gives

$$(12)\qquad \boldsymbol{S}_{r+1} = D\boldsymbol{S}_r + E\boldsymbol{a}_r$$

where D is the $(m-k+1) \times (m-k+1)$ upper triangular matrix whose (i,j) – element is

$$(13)\qquad d_{ij} = \begin{cases} \dbinom{j}{i} k^{j-i}, \quad i = 0,1,\ldots,m-k, \quad j \geqq i , \\ 0, \quad j < i , \end{cases}$$

and E is the $(m-k+1) \times k$ matrix with (i,j) – element is

(14) $e_{ij} = \binom{m-k+j}{i} k^{m-k+j-i}, \quad i = 0,1,\ldots,m-k;\ j = 1,2,\ldots,k\,.$

Elimination of a_r between (11) and (12) yields

$$S_{r+1} = AS_r\,, \tag{15}$$

where

$$A = D+EB^{-1}C\,. \tag{16}$$

We now investigate seven special cases.

I. $k=1$. This choice of k corresponds to m-splines with full continuity, i.e., $S(x) \in C^{m-1}[0,Q]$. We denote by A_0 the matrix A with $h=0$, and by $\mu^{(0)}$ and μ the eigenvalues of A_0 and A respectively. It then follows that

$$\mu = \mu^{(0)}+O(h)\,. \tag{17}$$

Concerning $\mu^{(0)}$ we have the following simple result.

LEMMA 1. *For $m \geqq 3$, there is at least one $\mu^{(0)}$ with*

$$|\mu^{(0)}| > 1\,. \tag{18}$$

PROOF. Using the binomial expansion we find the trace

$$\operatorname{tr}(A_0) = m+2-2^m\,, \tag{19}$$

and the result follows since (see [5])

$$\mu_1^{(0)}+\ldots+\mu_m^{(0)} = \operatorname{tr}(A_0)\,, \tag{20}$$

and since $\mu=1$ is an eigenvalue of A_0 for all m.

We have thus established

THEOREM 1. *The $(m,1)$-method is divergent for $m \geqq 3$.*

II. $k=2$. Denoting by a_{ii} the diagonal elements of A_0, we find

$$a_{11} = 1,$$
$$a_{ii} = 1+\binom{m-1}{i-2}\left(2^{m-i+1}-\frac{m+i-1}{i-1}\right), \quad i = 2(1)m-1\,.$$

Hence

$$\operatorname{tr}(A_0) = 3^{m-1}-3\cdot 2^{m-1}+m+2\,,$$

in view of which we have

THEOREM 2. *The* $(m,2)$*-method is divergent for* $m \geqq 4$.

III. $k=3$. Here, we find

$$\operatorname{tr}(A_0) = m-4^{m-2}+8(5/2)^{m-2}-5\cdot 2^{m-1}+2 ,$$

thus proving

THEOREM 3. *The* $(m,3)$*-method is divergent for* $m \geqq 5$.

IV. $k=m$. This is the other extreme of case I above and $S(x)$ is only in the continuity class C. In order to deal with this case, we start by effecting a slight change in notation. We shall now understand by $\boldsymbol{S}_{r+1}$ the $m \times 1$ column vector

$$\boldsymbol{S}_{r+1} = (S_{mr+1}, S_{mr+2}, \ldots, S_{m(r+1)})^T, \quad r = 0(1)N-1 ,$$

whence (12) is now replaced by

$$\boldsymbol{S}_{r+1} = (1,1,\ldots,1)^T S_{mr} + \hat{E}\boldsymbol{a}_r \tag{21}$$

where the elements of $\hat{E}$ are given by

$$\hat{e}_{ij} = i^j; \quad i,j = 1(1)m . \tag{22}$$

We further find

$$\begin{cases} \hat{B}\boldsymbol{a}_r = \hat{C}S_{mr}, \\ \boldsymbol{S}_{r+1} = \hat{A}S_{mr}, \\ \hat{A} = (1,1,\ldots,1)^T + \hat{E}\hat{B}^{-1}\hat{C} , \end{cases} \tag{23}$$

where the elements of $\hat{B}$ and $\hat{C}$ are, in terms of those of B and C, (9) and (10), given by

$$\begin{cases} \hat{b}_{ij} = ib_{ij}, \quad i,j = 1(1)m, \\ \hat{c}_{ij} = ic_{ij}, \quad i = 1(1)m; \quad j = 1 . \end{cases} \tag{24}$$

Note that C is an $m \times 1$ matrix in this case.
Observing that

$$\hat{B} = \hat{E}+O(h), \qquad \hat{C} = O(h) ,$$

and hence

$$\hat{A} = (1,\ldots,1)^T + \hat{C} + O(h^2) ,$$

we obtain from (24), (10) and (23)

$$S_{mr+t} = \big(1+t\lambda h+O(\lambda^2h^2)\big)S_{mr}, \quad t = 1(1)m . \tag{25}$$

In particular, for $m=1,2,3$ we find

$$(26)\qquad S_{m(r+1)} = \begin{cases} \dfrac{1+\lambda h/2}{1-\lambda h/2} S_r, & m = 1, \\[2ex] \dfrac{1+\lambda h+\lambda^2h^2/3}{1-\lambda h+\lambda^2h^2/3} S_{2r}, & m = 2, \\[2ex] \dfrac{1+3\lambda h/2+11\lambda^2h^2/12+\lambda^3h^3/4}{1-3\lambda h/2+11\lambda^2h^2/12-\lambda^3h^3/4} S_{3r}, & m = 3\,. \end{cases}$$

The rational functions are regular in the left half plane and since the absolute value is 1 on the imaginary axis it is <1 to the left of it by the maximum principle. Hence we have established

THEOREM 4. *The (m,m)-method is stable for all m. In particular, it is A-stable for $m \leqq 3$.*

V. *The $(2,1)$-method.* We proposed in [3] the application of quadratic splines in conjunction with the trapezoidal rule. Here the degree of the quadrature is 1 and we can not therefore use (11). However, direct computation yields

$$S_{r+1} = \frac{1+\lambda h/2}{1-\lambda h/2} S_r\,,$$

as a consequence of which we have

THEOREM 5. *The $(2,1)$-method, where the quadrature rule (3) is the trapezoidal rule, is A-stable.*

Observe that the use of the trapezoidal rule here is essential. For, using the *mid-point* rule, for instance, gives

$$\mathbf{S}_{r+1} = \frac{1}{1-\lambda h/4}\begin{pmatrix} 1+3\lambda h/4 & \lambda h/4 \\ 2\lambda h & 3\lambda h/4-1 \end{pmatrix} \mathbf{S}_r\,,$$

and although the method is stable, we no longer have A-stability. The reason why the choice of the quadrature rule affects the stability behaviour in the case of theorem 5 and not in theorem 4 is the more stringent continuity condition in the former.

VI. $k=m-1$. For $m=3$ (the cubic spline-deficiency -1 method proposed in 4 in conjunction with Simpson's rule) we find for the matrix A of (15)

$$(27)\qquad A = \begin{pmatrix} 1+2\lambda h+2\lambda^2h^2 & \lambda^2h^2/8 \\ 5\lambda^2h^2/2 & 1-\lambda h/2+\lambda^2h^2/8 \end{pmatrix} + O(\lambda^3h^3)\,.$$

For general m, we have from (9)

$$B = B^* + O(\lambda h)$$

where

$$b_{ij}^* = i^j ,$$

and if we denote by e_1 the first row of E, we easily find in view of (14) that

$$e_1 B^{-1} = (0, 0, \ldots, 0, m-1) + O(\lambda h) .$$

This result together with (10) yields

$$e_1 B^{-1} C = [(m-1)\lambda h + O(\lambda^2 h^2) \quad -(m-1) + O(\lambda h)]$$

which, in view of (13), gives for the first row of A

$$(1 + (m-1)\lambda h + O(\lambda^2 h^2) \quad O(\lambda h)) ;$$

and on invoking (15) and (8) we immediately obtain

$$S_{(m-1)(r+1)} = (1 + (m-1)\lambda h + O(\lambda^2 h^2)) S_{(m-1)r} + O(\lambda^2 h^2) . \tag{28}$$

We have thus established

THEOREM 6. *The $(m, m-1)$-method is stable for all $m > 1$.*

VII. *The remaining cases: a conjecture.* In view of the results presented above in I-VI, we are led to conjecture that the (m, k)-method is

(i) A-stable, for $k = m$, for all m;
(ii) divergent, for $k \leqq m-2$, for all $m > 2$.

4. Numerical illustrations.

We give below the results of applying various (m, k)-methods to the following two examples.

1. $$f(x) = 2 - e^x + \int_0^x e^{x-y} f(y)\,dy, \quad 0 \leqq x \leqq 4;$$

2. $$f(x) = e^x - \int_0^x e^{x-y} f(y)\,dy, \quad 0 \leqq x \leqq 4$$

Both equations have $f(x) = 1$ for their solution; and while the first has a positive kernel the second has a negetive one.

The results of Tables 1, 2 and 3 are illustrations of theorems 1, 4 and 6, respectively. The entries of Table 2 and 3 give maximum absolute error over $[0, 4]$.

Table 1. *Absolute error in solution of example* 2 *by a* (3,1)*-method.*

h \ x	0.4	1.6	2.8	4
0.4	5×10^{-2}	3×10^{-1}	20	1×10^{3}
0.2	8×10^{-3}	7	2×10^{4}	8×10^{7}
0.1	1×10^{-3}	3×10^{4}	3×10^{11}	3×10^{18}

Table 2. *Maximum absolute error in solution of examples* 1 *and* 2 *by a* (2,2)*-method.*

h	0.8	0.4	0.2
Example 1	9×10^{-2}	1×10^{-2}	8×10^{-4}
Example 2	6×10^{-4}	4×10^{-5}	2×10^{-6}

Table 3. *Maximum absolute error in solution of examples* 1 *and* 2 *by a* (4,3)*-method.*

h	0.8	0.4	0.2
Example 1	2×10^{-3}	3×10^{-6}	6×10^{-8}
Example 2	6×10^{-7}	9×10^{-9}	1×10^{-9}

The application of a (4,1)- and a (4,2)-method give results similar to those of Table 1 and we therefore do not report them here.

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