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A MODIFICATION OF MILLER'S RECURRENCE ALGORITHM

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Abstract.

Miller's recurrence algorithm for tabulating the subdominant solution of a second-order difference equation is modified so as to take the asymptotic behaviour of the solution into account. The asymptotic solutions of various types of equations are listed, and a method is given for estimating the error in the tabulated solution.

1. Introduction.

The general homogeneous linear difference equation of the second order may be written as

$$A_n u_{n-1} + B_n u_n + C_n u_{n+1} = 0 , \tag{1}$$

where A_n, B_n, C_n are known functions of n. If the difference equation has two independent solutions $u_n = \theta_n$, $u_n = \varphi_n$, neither being identically zero, such that

$$\frac{\theta_n}{\varphi_n} \to \infty \quad \text{as } n \to \infty , \tag{2}$$

then θ_n is said to be a *dominant* solution and φ_n a *subdominant* or *minimal* solution. This paper is concerned with the numerical tabulation of subdominant solutions.

We suppose that φ_n is normalised so that $\varphi_0 = 1$. (This would not be possible if $\varphi_0 = 0$; in this case we consider the equivalent problem obtained by replacing A_n, B_n, C_n by A_{n+1}, B_{n+1}, C_{n+1}.) We may take θ_n as any other solution of the difference equation, and we take it as the solution satisfying the initial conditions

$$\theta_0 = 0, \quad \theta_1 = 1 . \tag{3}$$

It is well-known that tabulation of φ_n by forward recurrence is unsuccessful, owing to the inevitable intrusion of a multiple of θ_n. The procedure for tabulating φ_n by backward recurrence was formulated by

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Miller [1] and has been studied in detail by subsequent authors, notably Olver [2, 3, 4], Oliver [5, 6, 7], Gautschi [8, 9] and Shintani [10]. Basically the procedure is to take

$$v_N = 1, \quad v_{N+1} = 0 \tag{4}$$

$$v_{n-1} = \frac{1}{A_n}(-B_n v_n - C_n v_{n+1}), \quad N-1 \geqq n \geqq 1 . \tag{5}$$

An approximation f_n to φ_n is then given by

$$f_n = \frac{v_n}{v_0} . \tag{6}$$

Error analyses of this method are given by the authors cited above, and the method has been successfully used in many circumstances, e.g. in the tabulation of Bessel functions.

2. Modification of the method.

Sometimes—particularly when the ratio θ_n/φ_n increases slowly as n increases—the method can be improved. This is best illustrated by means of an example. Consider the integral

$$I_n = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-t} \frac{t^{n-\frac{1}{2}}}{(t+\frac{1}{2})^n} dt , \tag{7}$$

which occurs in a previous paper by the present author [11]. It is easily shown that I_n is the subdominant solution of the difference equation

$$(n-\tfrac{1}{2})u_{n-1} - 2nu_n + nu_{n+1} = 0 . \tag{8}$$

The errors (rounded to 8 decimal places) in the values of I_1, I_2, I_3, I_4, I_5 when I_n is tabulated using Miller's algorithm with $N = 10(5)30$ are shown in the upper half of Table 1. It can be shown that, for large n,

$$I_n = K \exp[-(2n)^{\frac{1}{2}} + O(n^{-\frac{1}{2}})] , \tag{9}$$

and hence that

$$\frac{I_n}{I_{n+1}} = 1 + \frac{1}{(2n)^{\frac{1}{2}}} + \frac{1}{4n} + O(n^{-3/2}) . \tag{10}$$

We propose therefore to replace equations (4) above by

$$v_N = 1 + \frac{1}{(2N)^{\frac{1}{2}}} + \frac{1}{4N}, \quad v_{N+1} = 1 , \tag{11}$$

Table 1. *Error in the tabulation of I_n by Miller's algorithm (upper part) and the modified algorithm (lower part).*

			Error × 10^8				
			$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
Miller		10	−38060	−76120	−1 23695	−1 83957	−2 59680
		15	−5427	−10854	−17638	−26230	−37028
	N	20	−1035	−2070	−3364	−5002	−7062
		25	−239	−477	−776	−1153	−1628
		30	−63	−126	−205	−305	−430
Modified		10	−206	−412	−669	−995	−1404
		15	−18	−36	−58	−87	−123
	N	20	−2	−5	−8	−12	−17
		25	0	−1	−1	−2	−3
		30	0	0	0	0	−1

and then to tabulate v_n and f_n using equations (5) and (6) as before. The errors in the tabulation when this modified approach is used are shown in the lower half of Table 1, and there is clearly a substantial improvement in accuracy.

The idea of choosing v_N and v_{N+1} so that v_N/v_{N+1} is approximately equal to φ_N/φ_{N+1} was suggested by Gautschi [9, pp. 38–40], who showed that, in general, this leads to an improvement in accuracy. In the above example, our knowledge of the asymptotic form of the solution, given by (9), enabled us to approximate the ratio φ_N/φ_{N+1} to a reasonably good degree of accuracy. In general if we can find an expression k_n which approximates φ_n/φ_{n+1} we propose to replace equations (4) by

$$v_N = k_N, \quad v_{N+1} = 1 , \tag{12}$$

and then proceed as before. The main purpose of the present paper is to indicate a suitable expression for k_n for various types of equation; and to show how, with this choice of k_n, the errors in the tabulation may be estimated.

3. Asymptotic form of solutions.

We assume that, after suitable manipulation, the difference equation (1) can be written in the form

$$\begin{aligned}(a_0+a_1n^{-1}+a_2n^{-2}+\ldots)u_{n-1}+(b_0+b_1n^{-1}+b_2n^{-2}+\ldots)u_n \\ +(c_0+c_1n^{-1}+c_2n^{-2}+\ldots)u_{n+1} = 0 ,\end{aligned} \tag{13}$$

in which at least one of a_0, b_0, c_0 is non-zero. This is possible if A_n, B_n, C_n, considered as functions of a complex variable n, are regular or have poles of finite order at infinity. If $a_0 \neq 0$, the equation is known as a "Poincaré difference equation"; the principal theorems relating to such equations are summarized by Gautschi [9, p. 33 *et seq.*], who also gives a comprehensive bibliography of the original papers on the subject.

The asymptotic forms of both φ_n and θ_n (apart from constant multipliers K and L) are given for various types of equation in the list below. These are derived by assuming expansions for u_{n+1}/u_n, u_{n-1}/u_n in descending powers of n and substituting into (13); the work is elementary but tedious. It is assumed that a_i, b_i, c_i are real; if this is not the case, the solutions are still of the same form, but φ_n is not necessarily subdominant. For each type of equation, an expression for k_n is suggested, such that

$$k_n = \frac{\varphi_n}{\varphi_{n+1}}[1+O(n^{-h})],$$

where $h=2$ for an equation of Type 1, 3, 4, 5 or 6 and $h=\frac{3}{2}$ for an equation of Type 2, 7 or 8. For the sake of convenience we also list here a quantity Q_N which is required for the error estimate described in the next section.

Type 1: $a_0 \neq 0$, $c_0 \neq 0$; the equation $c_0x^2+b_0x+a_0=0$ has separate roots α,β with $|\alpha|<|\beta|$.

$$\varphi_n = K\alpha^n[n^p+O(n^{p-1})],$$
$$\theta_n = L\beta^n[n^q+O(n^{q-1})],$$

where

$$p = \frac{a_1+b_1\alpha+c_1\alpha^2}{a_0-c_0\alpha^2}, \qquad q = \frac{a_1+b_1\beta+c_1\beta^2}{a_0-c_0\beta^2}.$$

Take

$$k_n = \frac{1}{\alpha}\left(1-\frac{p}{n}\right), \qquad Q_N = \frac{\alpha\beta}{\beta-\alpha}.$$

If α and β are complex conjugates, there is no subdominant solution.

Type 2: $a_0 \neq 0$, $c_0 \neq 0$; the equation $c_0x^2+b_0x+a_0=0$ has a double root α such that $c_1\alpha^2+b_1\alpha+a_1 \neq 0$.

$$\varphi_n = K\alpha^n \exp(-An^{\frac{1}{2}})[n^p+O(n^{p-\frac{1}{2}})],$$
$$\theta_n = L\alpha^n \exp(An^{\frac{1}{2}})[n^p+O(n^{p-\frac{1}{2}})],$$

where

$$A = 2\sqrt{\left(-\frac{a_1+b_1\alpha+c_1\alpha^2}{a_0}\right)}, \quad p = \frac{1}{4}+\frac{a_1-c_1\alpha^2}{2a_0}.$$

Take

$$k_n = \frac{1}{\alpha}\left(1+\frac{A}{2n^{\frac{1}{2}}}+\frac{B}{n}\right), \quad Q_N = \frac{\alpha N^{\frac{1}{2}}}{A},$$

where

$$B = \frac{b_1}{b_0}-\frac{a_1}{a_0}-\frac{1}{4}.$$

There is no subdominant solution unless A is real.

Type 3: $a_0 \neq 0$, $c_0 \neq 0$; the equation $c_0x^2+b_0x+a_0=0$ has a double root α such that $c_1\alpha^2+b_1\alpha+a_1=0$.

$$\varphi_n = K\alpha^n \frac{\Gamma(n)}{\Gamma(n+\lambda_1)}[1+O(n^{-1})],$$

$$\theta_n = L\alpha^n \frac{\Gamma(n)}{\Gamma(n+\lambda_2)}[1+O(n^{-1})],$$

where $\lambda_1 > \lambda_2$ are the roots of

$$a_0\lambda^2+(a_0+a_1-c_1\alpha^2)\lambda+(a_2+b_2\alpha+c_2\alpha^2) = 0.$$

Take

$$k_n = \frac{1}{\alpha}\left(1+\frac{\lambda_1}{n}\right), \quad Q_N = \frac{\alpha N a_0}{a_1-c_1\alpha^2+2a_0(1+\lambda_1)}.$$

The asymptotic form for θ_n fails if $\lambda_1=\lambda_2$ or if λ_1 and λ_2 differ by an integer, but the expressions for k_n and Q_N still apply.

Type 4: $a_0 \neq 0$, $b_0 \neq 0$; $c_0=0$, $c_1 \neq 0$.

$$\varphi_n = K\alpha^n[n^p+O(n^{p-1})],$$
$$\theta_n = L\beta^n(n-1)![n^q+O(n^{q-1})],$$

where

$$\alpha = -\frac{a_0}{b_0}, \quad \beta = -\frac{b_0}{c_1},$$

$$p = \frac{a_1+b_1\alpha+c_1\alpha^2}{a_0}, \quad q = \frac{a_0+b_1\beta+c_2\beta^2}{b_0\beta}.$$

Take

$$k_n = \frac{1}{\alpha}\left(1-\frac{p}{n}\right), \qquad Q_N = \alpha\,.$$

Type 5: $b_0 \neq 0,\ c_0 \neq 0;\ a_0 = 0,\ a_1 \neq 0.$

$$\varphi_n = \frac{K\alpha^n}{n!}[n^p + O(n^{p-1})]\,,$$

$$\theta_n = L\beta^n[n^q + O(n^{q-1})]\,,$$

where

$$\alpha = -\frac{a_1}{b_0}, \qquad \beta = -\frac{b_0}{c_0},$$

$$p = \frac{a_2 + b_1\alpha + c_0\alpha^2}{a_1}, \qquad q = \frac{a_1 + b_1\beta + c_1\beta^2}{b_0\beta}.$$

Take

$$k_n = \frac{n+1}{\alpha}\left(1-\frac{p}{n}\right), \qquad Q_N = \frac{\alpha}{N}.$$

Type 6: $b_0 \neq 0$; the first non-zero terms in the sequences $(a_0, a_1, a_2, \ldots)$, $(c_0, c_1, c_2, \ldots)$ are a_r, c_s respectively, where $r+s \geqq 2$. (Either r or s may be zero.)

$$\varphi_n = \frac{K\alpha^n}{(n!)^r}[n^p + O(n^{p-1})]\,,$$

$$\theta_n = L\beta^n[(n-1)!]^s[n^q + O(n^{q-1})]\,,$$

where

$$\alpha = -\frac{a_r}{b_0}, \quad \beta = -\frac{b_0}{c_s}, \quad p = \frac{a_{r+1} + b_1\alpha}{a_r}, \quad q = \frac{b_1 + c_{s+1}\beta}{b_0}.$$

Take

$$k_n = \frac{(n+1)^r}{\alpha}\left(1-\frac{p}{n}\right), \quad Q_N = \frac{\alpha}{N^r}.$$

Type 7: $a_0 \neq 0,\ b_0 = c_0 = 0,\ c_1 \neq 0.$

$$\varphi_n = K(n!)^{\frac{1}{2}}(-\alpha)^n \exp(-An^{\frac{1}{2}})[n^p + O(n^{p-\frac{1}{2}})]\,,$$
$$\theta_n = L(n!)^{\frac{1}{2}}\alpha^n \exp(An^{\frac{1}{2}})[n^p + O(n^{p-\frac{1}{2}})]\,,$$

where

$$\alpha^2 = -\frac{a_0}{c_1}, \quad A = \frac{b_1\alpha}{a_0}, \quad p = -\frac{1}{4} + \frac{a_1}{2a_0} - \frac{c_2}{2c_1},$$

the sign of α being chosen so that $A > 0$. (There is no subdominant solution unless α is real.)

Take

$$k_n = -\frac{1}{\alpha(n+1)^{\frac{1}{2}}}\left(1+\frac{A}{2n^{\frac{1}{2}}}+\frac{\frac{1}{8}A^2-p}{n}\right), \quad Q_N = -\tfrac{1}{2}\alpha N^{\frac{1}{2}}.$$

Type 8: $c_0 \neq 0$; $a_0 = b_0 = 0$, $a_1 \neq 0$.

$$\varphi_n = \frac{K(-\alpha)^n}{(n!)^{\frac{1}{2}}}\exp(-An^{\frac{1}{2}})[n^p+O(n^{p-\frac{1}{2}})],$$

$$\theta_n = \frac{L\alpha^n}{(n!)^{\frac{1}{2}}}\exp(An^{\frac{1}{2}})[n^p+O(n^{p-\frac{1}{2}})],$$

where

$$\alpha^2 = -\frac{a_1}{c_0}, \quad A = \frac{b_1\alpha}{a_1}, \quad p = \frac{1}{4}+\frac{a_2}{2a_1}-\frac{c_1}{2c_0},$$

the sign of α being chosen so that $A > 0$ (there is no subdominant solution unless α is real).

Take

$$k_n = -\frac{(n+1)^{\frac{1}{2}}}{\alpha}\left(1+\frac{A}{2n^{\frac{1}{2}}}+\frac{\frac{1}{8}A^2-p}{n}\right), \quad Q_N = -\frac{\alpha}{2N^{\frac{1}{2}}}.$$

Type 9: $a_0 \neq 0$, $b_0 = c_0 = c_1 = 0$ *or* $c_0 \neq 0$, $a_0 = a_1 = b_0 = 0$.

The equation can be transformed into one of the above eight types by putting $u_n = (n!)^p U_n$ for some positive or negative integer p. (Some of the above types can also be transformed into one another by the same means.)

4. Error analysis.

Once f_n has been tabulated it is necessary to have some estimate of the error E_n defined by

(14) $$E_n = f_n - \varphi_n .$$

Now E_n is a solution of the difference equation (1), and also $E_n = 0$ since $f_0 = \varphi_0 = 1$. It follows that

(15) $$E_n = E_1\theta_n .$$

Since θ_n is easily tabulated by forward recurrence the error in any f_n can be found once the error in f_1 is known. Thus for the difference equation (8) we have

$$\theta_1 = 1,\ \theta_2 = 2,\ \theta_3 = \tfrac{13}{4},\ \theta_4 = \tfrac{29}{6},\ \theta_5 = \tfrac{655}{96}$$

and it will be seen that the entries in each row of Table 1 are in the ratios $1:2:\frac{13}{4}:\frac{29}{6}:\frac{655}{96}$. So our problem is to find an estimate of E_1.

Such an estimate can be obtained by a method similar to that used by Gautschi [9, p. 40]. If we define P_N by

$$\frac{1}{P_N} = \frac{B_N}{A_N} + k_{N-1}\left(1 + \frac{\varepsilon_{N-1}}{\varepsilon_N} + \varepsilon_{N-1}\right), \tag{16}$$

where

$$\varepsilon_n = \frac{\varphi_n}{k_n \varphi_{n+1}} - 1 ,$$

and w_n by

$$w_0 = 1, \quad w_n = \frac{C_n}{A_n} w_{n-1} , \tag{17}$$

then it can be shown that

$$E_1 = w_N P_N \varphi_{N+1}(f_{N-1} - f_N k_{N-1}) . \tag{18}$$

Equation (18) is an exact expression for E_1; it cannot be evaluated since φ_{N+1} and P_N are unknown. But the unknown part of P_N consists of terms of small order, and P_N can be approximated by Q_N, where Q_N is listed for each type of equation in Section 3; the approximation is such that the ratio P_N/Q_N is of the form $1 + O(N^{-1})$ for an equation of Type 1, 3, 4, 5, 6 or of the form $1 + O(N^{-\frac{1}{2}})$ for an equation of Type 2, 7, 8. In order to estimate E_1 using equation (18) we replace P_N by Q_N and φ_{N+1} by its approximate value f_{N+1}, and so obtain the error estimate e_1 defined by

$$e_1 = w_N Q_N f_{N+1}(f_{N-1} - f_N k_{N-1}) . \tag{19}$$

This error estimate is believed to be new. It will be seen in the numerical examples below that it gives a good indication of the accuracy of the tabulation.

5. Numerical examples.

In Tables 2, 3, 4 we give the error E_1 (rounded to 10 decimal places) for the difference equations in the examples below. We also give the error estimate e_1 and, for the sake of comparison, the error when the unmodified form of Miller's algorithm is used.

EXAMPLE 1 (see Table 2)

$$(n-\tfrac{1}{2})u_{n-1}-2nu_n+nu_{n+1} = 0 . \tag{20}$$

This equation has already been considered in Section 2. It is of Type 2, so that

$$k_n = 1+\frac{1}{(2n)^{\frac{1}{2}}}+\frac{1}{4n}, \quad Q_N = \left(\frac{N}{2}\right)^{\frac{1}{2}} .$$

Table 2. *Errors in the value of* f_1 *for equation* (20).

N	Error × 10^{10}		
	Miller	Modified	Error Estimate
5	−456 25015	−5 81348	−10 92759
10	−38 05999	−20578	−29855
15	−5 42694	−1799	−2379
20	−1 03498	−244	−307
25	−23865	−43	−53
30	−6308	−9	−11
35	−1850	−2	−3

EXAMPLE 2 (see Table 3)

$$u_{n-1}-\left(2+\frac{1}{n^2}\right)u_n+u_{n+1} = 0 . \tag{21}$$

This equation is of Type 3, giving

$$k_n = 1+\frac{\sqrt{5}-1}{2n}, \quad Q_N = \tfrac{1}{4}(\sqrt{5}-1)N .$$

Table 3. *Errors in the value of* f_1 *for equation* (21).

N	Error × 10^{10}		
	Miller	Modified	Error Estimate
5	−867 03162	−12 74731	−17 62488
10	−224 19867	−1 42322	−1 65878
20	−52 85415	−15379	−16568
50	−7 27004	−799	−823
100	−1 57761	−85	−86
200	−33861	−9	−9

EXAMPLE 3 (see Table 4)

$$u_{n-1} - 2u_n - 2nu_{n+1} = 0 . \tag{22}$$

This equation is of Type 8, giving

$$k_n = (2n+2)^{\frac{1}{2}}\left(1+\frac{1}{(2n)^{\frac{1}{2}}}\right), \quad Q_N = \frac{1}{(8N)^{\frac{1}{2}}}.$$

This is one of a class of difference equations which arise in the tabulation of repeated integrals of the error function (cf. Gautschi [8]).

Table 4. *Errors in the value of f_1 for equation* (22).

N	Error $\times 10^{10}$		
	Miller	Modified	Error Estimate
5	+567 93065	+4 02462	+5 06241
10	−46 66301	−11922	−13154
15	+6 60233	+928	+987
20	−1 25273	−115	−121
25	+28781	+19	+20
30	−7586	−4	−4
35	+2221	+1	+1

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