BIT 17 (1977), 128–133

A SIXTH ORDER TRIDIAGONAL FINITE DIFFERENCE METHOD FOR NON-LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract.

We present a new sixth order finite difference method for the second order differential equation $y''=f(x,y)$ subject to the boundary conditions $y(a)=A$, $y(b)=B$. An interesting feature of our method is that each discretization of the differential equation at an interior grid point is based on *five* evaluations of f; the classical second order method is based on one and the well-known fourth order method of Noumerov is based on three evaluations of f. In case of linear differential equations, our finite difference scheme leads to tridiagonal linear systems. We establish, under appropriate conditions, the sixth order convergence of the finite difference method. Numerical examples are considered to demonstrate computationally the sixth order of the method.

1. Introduction.

We consider the second order nonlinear differential equation

$$y'' = f(x,y) \tag{1a}$$

subject to the boundary conditions

$$y(a) = A, \quad y(b) = B . \tag{1b}$$

Here, $-\infty < a \leqq x \leqq b < \infty$, A and B are finite constants. We assume that for $x \in [a,b]$ and $-\infty < y < \infty$, (i) $f(x,y)$ is continuous, (ii) $\partial f/\partial y$ exists and is continuous, and (iii) $\partial f/\partial y \geqq 0$. These conditions assure us that the boundary value problem (1) has a unique solution; see Henrici [1].

In the classical second order method ([1], eqn. (7 – 9)), each discretization of the differential equation at an interior grid point is based on *one* evaluation of f. In the well-known fourth order method of Noumerov [2] (or, see [1], eqn. (7–10)), each discretization of the differential equation at an interior grid point is based on *three* evaluations of f. It is thus natural to ask if there is a sixth order method in which each discretization is based on five evaluations of f; the present paper describes such a method.

Received Nov. 25, 1976.

We present a sixth order finite difference method for the second order nonlinear boundary value problem (1). An interesting feature of the present method is that each discretization of the differential equation at an interior grid point is based on *five* evaluations of f. In case the differential equation (1a) is linear, our method leads to tridiagonal linear systems. In Section 2 we describe the finite difference method; we also obtain the local truncation error. In Section 3 we establish, under appropriate conditions, $O(h^6)$-convergence of the finite difference method. In Section 4 we consider numerical examples to illustrate the method and demonstrate computationally the sixth order of the method; we also compare our method with extrapolation of the fourth order method of Noumerov.

2. The finite difference method.

We first consider the derivation of the method and obtain its local truncation error.

Let N be a positive integer, $h=(b-a)/N$ and $x_k=a+kh, k=0(1)N$. The values of the exact solution $y(x)$ at grid points x_k are denoted by y_k; similarly, $f_k=f(x_k,y_k)$. We also set $x_{k+\frac{1}{2}}=x_k+h/2, k=0(1)N-1, y_{k+\frac{1}{2}}=y(x_{k+\frac{1}{2}})$ and $f_{k+\frac{1}{2}}=f(x_{k+\frac{1}{2}},y_{k+\frac{1}{2}})$. Using Taylor expansion method we easily derive

$$\text{(2a)}\qquad \begin{aligned} y_{k+\frac{1}{2}} = {} & \tfrac{1}{2}(y_{k+1}+y_k)-(h^2/384)(34f_k+19f_{k+1}-5f_{k-1})+ \\ & +(5h^5/768)y_k^{(5)}+(13h^6/15360)y^{(6)}(\xi_k)\,, \end{aligned}$$

$$\text{(2b)}\qquad \begin{aligned} y_{k-\frac{1}{2}} = {} & \tfrac{1}{2}(y_{k-1}+y_k)-(h^2/384)(34f_k+19f_{k-1}-5f_{k+1})- \\ & -(5h^5/768)y_k^{(5)}+(13h^6/15360)y^{(6)}(\eta_k) \end{aligned}$$

and

$$\text{(3)}\quad \delta^2 y_k = (h^2/60)(f_{k+1}+26f_k+f_{k-1}+16(f_{k+\frac{1}{2}}+f_{k-\frac{1}{2}}))-(h^8/120960)\cdot y^{(8)}(\zeta_k)\,,$$

where $\xi_k,\ \eta_k,\ \zeta_k\in(x_{k-1},x_{k+1})$. Defining

$$\text{(4)}\qquad y_{k\pm\frac{1}{2}} = \bar{y}_{k\pm\frac{1}{2}}+\tau_k(\pm h)\,,$$

where the meaning of $\bar{y}_{k\pm\frac{1}{2}}$ and $\tau_k(\pm h)$ follows from (2), and

$$\text{(5)}\qquad \delta^2 y_k = (h^2/60)(f_{k+1}+26f_k+f_{k-1}+16(\bar{f}_{k+\frac{1}{2}}+\bar{f}_{k-\frac{1}{2}}))+T_k(h)\,,$$

where we have set $\bar{f}_{k\pm\frac{1}{2}}=f(x_{k\pm\frac{1}{2}},\bar{y}_{k\pm\frac{1}{2}})$, from (3) and (5) it follows that

$$\text{(6)}\quad T_k(h) = -(h^8/120960)y^{(8)}(\zeta_k)+(4h^2/15)(f_{k+\frac{1}{2}}-\bar{f}_{k+\frac{1}{2}}+f_{k-\frac{1}{2}}-\bar{f}_{k-\frac{1}{2}})\,.$$

We next show that $T_k(h)=O(h^8)$. With the help of (4) we obtain

$$\text{(7)}\qquad f_{k\pm\frac{1}{2}} = \bar{f}_{k\pm\frac{1}{2}}+\tau(\pm h)\hat{F}_{k\pm\frac{1}{2}}\,,$$

where $F(x,y)=(\partial/\partial y)f(x,y), \hat{F}_{k\pm\frac{1}{2}}=F(x_{k\pm\frac{1}{2}},\hat{y}_{k\pm\frac{1}{2}})$, $\min(y_{k\pm\frac{1}{2}},\bar{y}_{k\pm\frac{1}{2}})<\hat{y}_{k\pm\frac{1}{2}}<\max(y_{k\pm\frac{1}{2}},\bar{y}_{k\pm\frac{1}{2}})$. With the help of the mean-value theorem, we may write

$$\hat{F}_{k+\frac{1}{2}}-\hat{F}_{k-\frac{1}{2}} = h\hat{F}'(c_k)\,, \tag{8}$$

where $F'=dF/dx, \hat{F}'(c_k)=F'(c_k,\hat{y}(c_k)), c_k\in(x_{k-1},x_{k+1})$. With the help of (8), from (7) we obtain

$$f_{k+\frac{1}{2}}+f_{k-\frac{1}{2}} = \bar{f}_{k+\frac{1}{2}}+\bar{f}_{k-\frac{1}{2}}+(h^6/15360)t_k \tag{9}$$

where

$$t_k = 100y_k^{(5)}\hat{F}'(c_k)+13(\hat{F}_{k+\frac{1}{2}}y^{(6)}(\xi_k)+\hat{F}_{k-\frac{1}{2}}y^{(6)}(\eta_k))\,.$$

Finally, from (6) and (9) we obtain

$$T_k(h) = (h^8/1209600)(21t_k-10y^{(8)}(\zeta_k))\,. \tag{10}$$

We have thus established the following result.

THEOREM 1. *Let $y\in C^8[a,b]$ and let $F(x,y)$ exist for $x\in[a,b]$ and $-\infty<y<\infty$. Also, let $F'(x,y(x))$ exist for $x\in[a,b]$. Then, the local truncation error $T_k(h)$ in* (5) *is given by* (10).

Next, we discuss the finite difference method. At each $x_k, k=1(1)N-1$, the differential equation (1a) is discretized by (5); note also that, from (1b), $y_0=A, y_N=B$.

Let $\boldsymbol{D}=(d_{ij})_{i,j=1}^{N-1}$ denote the tridiagonal matrix with

$$d_{k,k+1} = -1,\quad k=1(1)N-2,\quad d_{k,k-1} = -1,\quad k=2(1)N-1,$$
$$d_{k,k} = 2,\quad k=1(1)N-1\,. \tag{11}$$

For $k=1(1)N-1$, let

$$g_k(y_{k-1},y_k,y_{k+1}) = (h^2/60)(f_{k+1}+26f_k+f_{k-1}+16(\bar{f}_{k+\frac{1}{2}}+\bar{f}_{k-\frac{1}{2}})) - (A\delta_{k,1}+B\delta_{k,N-1})\,, \tag{12}$$

where $\delta_{ij}=1$ if $i=j$ and 0 if $i\neq j$. Let $\boldsymbol{Y}=(y_1,\ldots,y_{N-1})^T$, $\boldsymbol{G}(\boldsymbol{Y})=(g_1,\ldots,g_{N-1})^T$ and $\boldsymbol{T}(h)=(T_1(h),\ldots,T_{N-1}(h))^T$. Then, the finite difference method can be written in the matrix form

$$\boldsymbol{DY}+\boldsymbol{G}(\boldsymbol{Y})+\boldsymbol{T}(h) = \boldsymbol{0}\,. \tag{13}$$

Thus, the method consists in obtaining an approximation $\tilde{\boldsymbol{Y}}=(\tilde{y}_1,\ldots,\tilde{y}_{N-1})^T$ for $\boldsymbol{Y}$ by solving the $(N-1)\times(N-1)$ system

$$\boldsymbol{D}\tilde{\boldsymbol{Y}}+\boldsymbol{G}(\tilde{\boldsymbol{Y}}) = \boldsymbol{0}\,. \tag{14}$$

In case the differential equation (1a) is linear, (14) is a tridiagonal linear system and can be solved by an adaptation of Gauss elimination; in the case of nonlinear differential equations, the system (14) can be solved by the Newton-Raphson method. For details, see Henrici [1].

3. Convergence of the finite difference method.

We next establish convergence of the finite difference method showing that, under appropriate conditions, the global truncation error of the method is $O(h^6)$.

Let $e_k = \tilde{y}_k - y_k$, $k = 1(1)N-1$, and $\boldsymbol{E} = (e_1, \ldots, e_{N-1})^T$. Let $\sim$ over an expression mean that all the y_k's in that expression are replaced by $\tilde{y}_k$; then, we may write

$$\tilde{f}_k - f_k = e_k U_k, \quad k = 0(1)N, \tag{15}$$

and

$$\tilde{\bar{f}}_{k+\frac{1}{2}} - \bar{f}_{k+\frac{1}{2}} = (\tilde{\bar{y}}_{k+\frac{1}{2}} - \bar{y}_{k+\frac{1}{2}}) U_{k+\frac{1}{2}}, \quad k = 0(1)N-1, \tag{16}$$

for suitable U_k and $U_{k+\frac{1}{2}}$. Using the definition of $\bar{y}_{k\pm\frac{1}{2}}$, with the help of (12) we may write

$$\boldsymbol{G}(\tilde{\boldsymbol{Y}}) - \boldsymbol{G}(\boldsymbol{Y}) = \boldsymbol{ME} \tag{17}$$

where $\boldsymbol{M} = (m_{ij})_{i,j=1}^{N-1}$ is the tridiagonal matrix with

$$m_{k,k} = (h^2/60)\big(26U_k + 8U_{k+\frac{1}{2}}(1-(17h^2/96)U_k) + 8U_{k-\frac{1}{2}}(1-(17h^2/96)U_k)\big), \quad k = 1(1)N-1,$$

$$m_{k,k\pm1} = (h^2/60)\big(U_{k\pm1} + 8U_{k\pm\frac{1}{2}}(1-(19h^2/192)U_{k\pm1}) + U_{k\mp\frac{1}{2}}((5h^2/24)U_{k\pm1})\big), \quad k = \begin{cases} 1(1)N-2, \\ 2(1)N-1. \end{cases}$$

With the help of (17), from (13) and (14) we obtain the error equation:

$$(\boldsymbol{D} + \boldsymbol{M})\boldsymbol{E} = \boldsymbol{T}(h). \tag{18}$$

Let $U = \max(\partial/\partial y) f(x, y(x))$ for $x \in [a, b]$; then, $0 \leqq U_k, U_{k+\frac{1}{2}} \leqq U$. The case $U = 0$ being trivial, in the following we shall assume that $U > 0$.

Now, let

$$h_0^2 = ((321)^{\frac{1}{2}} - 9)/(2U).$$

For $h < h_0$, it is easy to see that

$$|m_{k,k+1}| < 1, \quad k = 1(1)N-2, \quad |m_{k,k-1}| < 1, \quad k = 2(1)N-1,$$

and hence, $D + M$ is irreducible. If S_k denotes the sum of the elements in the kth row of $\boldsymbol{D}+\boldsymbol{M}$, then for $h<h_0$, $S_1>0$, $S_k \geqq 0$, $k=2(1)\,N-2$, $S_{N-1}>0$, and hence, $\boldsymbol{D}+\boldsymbol{M}$ is monotone. It therefore follows that $(\boldsymbol{D}+\boldsymbol{M})^{-1}$ exists and $(\boldsymbol{D}+\boldsymbol{M})^{-1} \geqq \boldsymbol{0}$. Since $\boldsymbol{D}$ is also irreducible and monotone and for $h<h_0$, $\boldsymbol{M} \geqq \boldsymbol{0}$, it follows that $\boldsymbol{D}+\boldsymbol{M} \geqq \boldsymbol{D}$, and hence, $(\boldsymbol{D}+\boldsymbol{M})^{-1} \leqq \boldsymbol{D}^{-1}$. Consequently, from (18) we have for $h<h_0$,

$$|e_k| \leqq \textstyle\sum_{j=1}^{N-1}(\boldsymbol{D}^{-1})_{kj}|T_j(h)|, \quad k=1(1)N-1\,. \tag{19}$$

We now have the following result establishing that the global error of the finite difference method is $O(h^6)$. In the following, $\|\boldsymbol{E}\|=\max\{|e_k| : k=1(1)N-1\}$.

THEOREM 2. *Let the hypothesis of Theorem* 1 *hold. In addition, let* $|y^{(k)}| \leqq Y^{(k)}$, $|F| \leqq F^{(0)}$ *and* $|F'| \leqq F^{(1)}$ *for suitable constants* $Y^{(k)}, F^{(0)}$ *and* $F^{(1)}$. *Then, for* $h<h_0$,

$$\|\boldsymbol{E}\| \leqq Ch^6 \tag{20}$$

where

$$C = \left((b-a)^2/1209600\right)\left(5Y^{(8)}+21(50Y^{(5)}F^{(1)}+13Y^{(6)}F^{(0)})\right)\,.$$

PROOF. From (10) we have

$$\max\nolimits_{1\leqq j\leqq N-1}|T_j(h)| \leqq 2Ch^8/(b-a)^2\,. \tag{21}$$

Since

$$\textstyle\sum_{j=1}^{N-1}(\boldsymbol{D}^{-1})_{kj} \leqq (b-a)^2/2h^2, \quad k=1(1)N-1\,, \tag{22}$$

see Henrici [1], p. 375, (20) follows from (19) with the help of (21) and (22).

4. Numerical illustrations.

To illustrate our method and to demonstrate computationally its sixth order of convergence, we consider two examples; the first is a linear and the second is a nonlinear boundary value problem.

$$y'' = \left((1-x)y+1\right)/(1+x)^2; \quad y(0) = 1, \quad y(1) = 0.5\,, \tag{23}$$

with the exact solution $y(x)=1/(1+x)$.

$$y'' = \left((2-x)e^{2y}+1/(1+x)\right)/3\,; \; y(0) = 0, \quad y(1) = -\log 2, \tag{24}$$

with the exact solution $y(x) = \log(1/(1+x))$.

In Table 1 we show $\|\boldsymbol{E}\|$ computed for (23) and (24) for $N=4,8,16$. The numerical results verify the sixth order convergence of the method. For the sake of comparison we also solved (23) and (24) by extrapolation of the fourth order method of Noumerov; these results are also shown in Table 1. It will be observed from the numerical results that the present method and $O(h^4)$-extrapolation method are comparable in accuracy; note also that the present method and $O(h^4)$-extrapolation method are both based on five evaluations of f at each interior grid point.

Table 1.

Problem	N	Our Method	$O(h^4)$-Extrapol. Method
(23)	4	9.2 (−7)	6.5 (−7)
	8	1.7 (−8)	1.1 (−8)
	16	2.8 (−10)	
(24)	4	3.6 (−7)	9.8 (−8)
	8	6.3 (−9)	1.1 (−9)
	16	1.0 (−10)	

The error for different methods and step-lengths.

REFERENCES

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