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AN ANALYSIS OF A METHOD FOR SOLVING SINGULAR INTEGRAL EQUATIONS

PETER LINZ

Abstract.

We establish convergence rates for a method of approximate solution of certain singular integral equations. The method considered involves an expansion of the kernel of the equation in terms of Chebyshev polynomials.

1. Introduction.

We consider here the numerical solution of the singular integral equation

$$\int_{-1}^{1} \frac{\varphi(t)}{x-t}\,dt + \lambda \int_{-1}^{1} k(x,t)\varphi(t)\,dt = g(x), \qquad -1<x<1\,, \tag{1}$$

subject to a normalization condition

$$\int_{-1}^{1} \varphi(t)\,dt = N\,. \tag{2}$$

The first integral in (1) is to be understood as a principal value integral. We will assume that equations (1) and (2) define a unique $\varphi(x)$; this will be the case except for some special values of λ which will be omitted from consideration.

The specified conditions imply that $\varphi(x)$ is unbounded near $x=\pm 1$ ana is proportional to $(1-x^2)^{-\frac{1}{2}}$ near there (see [3]). To simplify matters, we introduce

$$y(x) = \sqrt{1-x^2}\,\varphi(x)$$

and re-write (1) and (2) as

$$\int_{-1}^{1} \frac{y(t)}{(x-t)\sqrt{1-t^2}}\,dt + \lambda \int_{-1}^{1} \frac{k(x,t)y(t)}{\sqrt{1-t^2}}\,dt = g(x)\,, \tag{3}$$

$$\int_{-1}^{1} \frac{y(t)}{\sqrt{1-t^2}}\,dt = N\,. \tag{4}$$

A technique frequently proposed for the approximate computation of the solution involves an expansion of the unknown $y(x)$ in terms of some orthogonal polynomials; typically Chebyshev or Jacobi polynomials when (3) is treated in the form given ([2], [3], [4]) or trigonometric polynomials if one considers an

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equivalent form obtained after changing variables ([1], [7]). This technique appears to be successful, at least in some simple examples, although a thorough error analysis has not yet been provided. Some discussion of the error analysis has been given in ([2], [4], [7]); in this paper we continue this analysis by establishing rates of convergence for the method.

Let us begin by defining our notation. Our expansion functions will be the normalized Chebyshev polynomials of the first kind

$$T_0(x) = \frac{1}{\sqrt{\pi}}, \quad T_i(x) = \sqrt{\frac{2}{\pi}}\cos(i\cos^{-1}x), \tag{5}$$

and those of the second kind

$$U_i(x) = \sqrt{\frac{2}{\pi}}\frac{\sin((i+1)\cos^{-1}x)}{\sin(\cos^{-1}x)}. \tag{6}$$

We will have occasion to use various types of inner products defined by

$$(f,g)_2 = \int_{-1}^{1} f(t)g(t)\,dt, \tag{7}$$

$$(f,g)_T = \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} f(t)g(t)\,dt, \tag{8}$$

$$(f,g)_U = \int_{-1}^{1} (1-t^2)^{\frac{1}{2}} f(t)g(t)\,dt. \tag{9}$$

The corresponding norms will be denoted by $\|\ \|_2$, $\|\ \|_T$ and $\|\ \|_U$, respectively. We also use the standard notation for the infinity norm:

$$\|f\|_\infty = \sup_{x\in\mathscr{D}} |f(x)|$$

for any function bounded in $\mathscr{D}$.

The following standard results will be needed:

$$(T_i, T_j)_T = \delta_{ij}, \tag{10}$$

$$(U_i, U_j)_U = \delta_{ij}, \tag{11}$$

$$\int_{-1}^{1} (1-t^2)^{\frac{1}{2}} \frac{U_i(t)}{x-t}\,dt = \pi T_{i+1}(x), \tag{12}$$

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} \frac{T_i(t)}{x-t}\,dt = (\delta_{0i}-1)\pi U_{i-1}(x), \tag{13}$$

where again these integrals are to be understood in the principal value sense.

2. Solution of the Equation with $\lambda=0$.

We begin by considering the simple case $\lambda=0$. This will serve to elucidate the method as well as to provide some results needed in the treatment of the full problem. In operator notation we write (3) as

$$K^0 y = g \,, \tag{14}$$

which has a known solution

$$y(x) = -\frac{1}{\pi^2}\int_{-1}^{1} \frac{g(t)}{(x-t)\sqrt{1-t^2}}\,dt + c \,, \tag{15}$$

where the constant c is uniquely determined by the normalization condition (4) (cf. [5], [8])*. From this, and the properties of principal value integrals we know that if $g \in C^{(1)}[-1,1]$, then (14) subject to (4) has a unique continuous solution.

Although (15) gives an explicit representation of the solution of (14) this does not completely solve the problem, since it requires the evaluation of some complicated integrals. One can, without any loss of efficiency, consider the approximate solution of (14) directly. We look for an approximate solution of the form

$$y_n(x) = \sum_{i=0}^{n} \alpha_i T_i(x) \,. \tag{16}$$

The unknown coefficients α_i can be chosen in a variety of ways; the technique due to Erdogan is essentially a Galerkin-type method, in which the coefficients are chosen such that the residual

$$r_n(x) = \sum_{i=0}^{n} \alpha_i \int_{-1}^{1} \frac{T_i(t)}{(x-t)\sqrt{1-t^2}}\,dt - g(x)$$

satisfies the orthogonality relation

$$(r_n, U_i)_U = 0, \quad i=0,1,\ldots,n-1 \,.$$

Using (10)–(13) and satisfying (4) exactly we get

$$\alpha_0 = \frac{N}{\sqrt{\pi}} \,, \tag{17}$$

$$\alpha_i = -\pi^{-1}(g, U_{i-1})_U, \quad i=1,\ldots,n \,. \tag{18}$$

Let us now consider the convergence of $y_n(x)$ towards $y(x)$. To do so we need the following

* Throughout this paper we will use c to denote a constant of unspecified value; it may generally take on different values even within a sequence of steps.

LEMMA 1. *Let* $g \in C^{(p+1)}(-1,1)$. *Then*

$$|(g, U_i)_U| \leqq ci^{-(p+1)}\|g^{(p+1)}\|_\infty , \tag{19}$$

$$|(g, T_i)_T| \leqq ci^{-(p+1)}\|g^{(p+1)}\|_\infty . \tag{20}$$

PROOF. Let $P^*_{i-1}(x)$ be the min-max approximation polynomial to $g(x)$. Then, by Jackson's theorem ([6], p. 23) we know that

$$\|g - P^*_{i-1}\|_\infty \leqq ci^{-(p+1)}\|g^{(p+1)}\|_\infty .$$

Now

$$(g, U_i)_U = (P^*_{i-1}, U_i)_U + (g - P^*_{i-1}, U_i)_U$$

and, since $(P^*_{i-1}, U_i)_U = 0$, we have by the Schwarz inequality

$$|(g, U_i)_U| \leqq \|g - P^*_{i-1}\|_U \|U_i\|_U \leqq c\|g - P^*_{i-1}\|_\infty .$$

Thus (19) follows and the bound (20) can be derived in exactly the same manner.

Since the solution $y(x)$ is continuous it has a Fourier expansion

$$y(x) = \sum_{i=0}^{\infty} \beta_i T_i(x) ,$$

where

$$\beta_i = (y, T_i)_T .$$

Hence

$$\begin{aligned}
\beta_i &= \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} y(t) T_i(t)\, dt \\
&= \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} y(t) \int_{-1}^{1} \frac{(1-s^2)^{\frac{1}{2}}}{t-s} U_{i-1}(s)\, ds \\
&= \frac{1}{\pi} \int_{-1}^{1} (1-s^2)^{\frac{1}{2}} U_{i-1}(s) \int_{-1}^{1} \frac{y(t)\, dt}{(1-t^2)^{\frac{1}{2}}(t-s)} \\
&= -\frac{1}{\pi} \int_{-1}^{1} (1-s^2)^{\frac{1}{2}} U_{i-1}(s) g(s)\, ds \\
&= -\frac{1}{\pi}(g, U_{i-1})_U .
\end{aligned}$$

The interchange of the order of integration is justified by standard results on principal value integrals ([8], p. 170). Thus we have

$$y(x) - y_n(x) = -\frac{1}{\pi} \sum_{i=n+1}^{\infty} (g, U_{i-1})_U T_i(x) . \tag{21}$$

THEOREM 1. *If* $g \in C^{(1)}(-1,1)$ *then*

$$\lim_{n\to\infty} \|y-y_n\|_T = 0 . \tag{22}$$

PROOF. Parseval's identity gives

$$\|y-y_n\|_T^2 = \sum_{i=n+1}^{\infty} (g, U_{i-1})_U^2$$

and applying (19) with $p=0$ the result follows.

We can strengthen this result to get point-wise convergence as well as an order of convergence by making additional smoothness assumptions on g.

THEOREM 2. *If* $g \in C^{(p+1)}(-1, 1)$, $1 \leqq p < \infty$, *then*

$$\lim_{n\to\infty} \|y-y_n\|_\infty = 0 \tag{23}$$

and furthermore, there exists some c *such that*

$$\|y-y_n\|_\infty \leqq cn^{-p} . \tag{24}$$

PROOF. From (19) and (21) we have

$$\|y-y_n\|_\infty \leqq \frac{1}{\pi} \sum_{i=n+1}^{\infty} \|T_i\|_\infty |(g, U_{i-1})_U| \leqq \sum_{i=n}^{\infty} \frac{c}{i^{p+1}} ,$$

proving both (23) and (24).

3. Solution for General Values of λ.

To discuss the general case we need a result on the behavior of the solution of (3) with respect to small perturbations in $k(x,t)$ and $g(x)$.

LEMMA 2. *Let* $k_n(x,t)$ *and* $g_n(t)$ *be two sequences of functions such that*

$$\lim_{n\to\infty} \|k(x,t)-k_n(x,t)\|_\infty = 0 ,$$

$$\lim_{n\to\infty} \|g(x)-g_n(x)\|_\infty = 0 .$$

Let $y(x)$ *satisfy equations* (3) *and* (4), *and let* $y_n(x)$ *satisfy similar equations in which* $k(x,t)$ *is replaced by* $k_n(x,t)$ *and* $g(x)$ *is replaced by* $g_n(x)$. *Then, for sufficiently large* n,

$$\|y-y_n\|_\infty \leqq c(\|\Delta K_n^*\|_\infty + \|\Delta g_n^*\|_\infty) , \tag{25}$$

where

$$\Delta K_n^*(x) = \frac{1}{\pi^2}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\left\{\int_{-1}^{1}(k(t,s)-k_n(t,s))y(s)\,ds\right\}dt\,,$$

$$\Delta g_n^*(x) = \frac{1}{\pi^2}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}(g(t)-g_n(t))\,dt\,.$$

PROOF. Let us write (3) in operator form as

$$K^0y+\lambda Ky = g\,.$$

The corresponding equation for y_n is

$$K^0y_n+\lambda K_ny_n = g_n$$

and setting $\varepsilon_n=y-y_n$, we have

$$K^0\varepsilon_n+\lambda K_n\varepsilon_n = \lambda(K_n-K)y+g-g_n\,. \tag{26}$$

From the general theory of integral equations ([5], p. 338) we know that $\varepsilon_n(x)$ satisfies an integral equation of the form

$$\varepsilon_n(x)+\lambda\int_{-1}^{1}q_n(x,t)\varepsilon_n(t)\,dt = -\lambda\Delta K_n^*(x)+\Delta g_n^*(x)+c = G_n(x)+c\,,$$

where

$$q_n(x,t) = \frac{1}{\pi^2\sqrt{1-t^2}}\int_{-1}^{1}\frac{k_n(s,t)\sqrt{1-s^2}}{s-x}ds\,.$$

By making the further transformation

$$x = \sin\theta, \qquad t = \sin\theta'$$

$$e_n(\theta) = \varepsilon_n(\sin\theta), \qquad \tilde{G}_n(\theta) = G_n(\sin\theta)$$

we obtain

$$e_n(\theta)+\lambda\int_{-\pi/2}^{\pi/2}\tilde{q}_n(\theta,\theta')e_n(\theta')\,d\theta' = \tilde{G}_n(\theta)+c\,, \tag{27}$$

where the kernel

$$\tilde{q}_n(\theta,\theta') = \frac{1}{\pi^2}\int_{-1}^{1}\frac{k_n(s,\sin\theta')\sqrt{1-s^2}}{(s-\sin\theta)}ds \tag{28}$$

is now bounded. Equation (27) is therefore a Fredholm equation which we write in operator form as

$$(I+\lambda\tilde{Q}_n)e_n = \tilde{G}_n+c\,. \tag{29}$$

Let $\tilde{q}(\theta,\theta')$ denote the kernel derived in the same way as $\tilde{q}_n(\theta,\theta')$, but with $k(x,t)$ replacing $k_n(x,t)$; let $\tilde{Q}$ denote the operator corresponding to $\tilde{q}(\theta,\theta')$. Then from (28)

$$\lim_{n\to\infty} \|q-q_n\|_\infty = 0 ,$$

and it follows from the standard Fredholm theory that, except for those λ which are eigenvalues of $\tilde{Q}$, the operator $(I+\lambda\tilde{Q}_n)$ has a bounded inverse for sufficiently large n. We can then write

$$e_n = (I+\lambda\tilde{Q}_n)^{-1}(\tilde{G}_n+c) ,$$

and, since y and y_n satisfy (4), we also have,

$$\int_{-\pi/2}^{\pi/2} e_n(\theta)\,d\theta = 0 .$$

Thus

$$c = \frac{\int_{-\pi/2}^{\pi/2} (I+\lambda\tilde{Q}_n)^{-1}\tilde{G}_n(\theta)\,d\theta}{\int_{-\pi/2}^{\pi/2} (I+\lambda\tilde{Q}_n)^{-1}\,d\theta} . \tag{30}$$

Finally, since $(I+\lambda\tilde{Q}_n)^{-1} \to (I+\lambda\tilde{Q})^{-1}$ we get

$$c = O(\|\tilde{G}_n\|_\infty) ,$$

$$\|\varepsilon_n\|_\infty = \|e_n\|_\infty = O(\|\tilde{G}_n\|_\infty) = O(\|G_n\|_\infty) ,$$

completing the proof.

To obtain the approximate solution to (3) we set

$$g_n(x) = \sum_{i=0}^{n-1} (g, U_i)_U U_i(x)$$

and

$$k_n(x,t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n} \beta_{ij} U_i(x) T_j(t)$$

where

$$\beta_{ij} = \big(U_i, (k(x,\cdot), T_j)_T\big)_U .$$

Corresponding to (3) and (4) we then have the approximating system

$$\int_{-1}^{1} \frac{y_n(t)}{(x-t)\sqrt{1-t^2}}\,dt + \lambda \int_{-1}^{1} \frac{k_n(x,t)y_n(t)}{\sqrt{1-t^2}}\,dt = g_n(x) , \tag{31}$$

$$\int_{-1}^{1} \frac{y_n(t)}{\sqrt{1-t^2}}\,dt = N\,. \tag{32}$$

The solution of these equations is given by

$$y_n(x) = \sum_{i=0}^{n} \alpha_i T_i(x)\,, \tag{33}$$

with

$$\alpha_0 = \frac{N}{\sqrt{\pi}}$$

and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)^T$ given by the matrix system

$$A_n\boldsymbol{\alpha} = \boldsymbol{g}_n\,. \tag{34}$$

Here $\boldsymbol{g}_n=((g,U_0)_U,\ldots,(g,U_{n-1})_U)^T$ and the elements of A_n are

$$a_{ij} = -\pi\delta_{ij}+\lambda\beta_{i-1,j}\,. \tag{35}$$

This can be verified by substitution. Convergence results then follow easily.

THEOREM 3. *If* $g \in C^{(p_1+1)}(-1,1)$, $p_1 \geqq 1$, *and* $k(x,t) \in C^{(p_2+1)}(-1,1)$, $p_2 \geqq 1$ (*with respect to both variables*) *then, for sufficiently large* n

$$\|y-y_n\|_\infty \leqq cn^{-p}, \qquad p = \min(p_1,p_2)\,. \tag{36}$$

PROOF. Since $k(x,t)$ is continuous

$$k(x,t)-k_n(x,t) = \sum_{\sigma} \beta_{ij}U_i(x)T_j(t)\,, \tag{37}$$

where $\sigma=\{0\leqq i<\infty, j\geqq n+1\}\cup\{n\leqq i<\infty, 0\leqq j\leqq n\}$. Thus

$$\begin{aligned}
\Delta K_n^*(x) &= \frac{1}{\pi^2}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\int_{-1}^{1}\sum_{\sigma}\beta_{ij}U_i(t)T_j(s)y(s)\,ds\,dt \\
&= \frac{1}{\pi^2}\int_{-1}^{1} y(s)\sum_{\sigma}\beta_{ij}T_j(s)\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}U_i(t)\,dt\,ds \\
&= \frac{1}{\pi}\int_{-1}^{1} y(s)\sum_{\sigma}\beta_{ij}T_j(s)T_{i+1}(x)\,ds\,,
\end{aligned}$$

where the interchanges can be justified by the theory of singular integrals. Now

$$\beta_{ij} = (U_i,(k(x,\cdot),T_j)_T)_U\,,$$

and setting $q_j(x)=(k(x,\cdot),T_j)_T$ we have from Lemma 1 that

$$|\beta_{ij}| \leqq ci^{-(p_2+1)}\|q_j^{(p_2+1)}(x)\|_\infty\,.$$

Applying the lemma again to $q_j^{(p_2+1)}(x)$ gives

$$|\beta_{ij}| \leqq c i^{-(p_2+1)} j^{-(p_2+1)},$$

and it follows that

$$\sum_{\sigma} |\beta_{ij}| \leqq \frac{c}{n^{p_2}},$$

and

$$\|\Delta K_n^*\| = O(n^{-p_2}):$$

A similar argument can be made for Δg_n^* and we obtain (36).

4. Conclusions.

We have shown that the method of orthogonal polynomial expansion is convergent for the solution of certain singular integral equations, and that the convergence rates are high if the functions in the equation have a high degree of smoothness. The arguments given here can be generalized in various ways. Systems of such integral equations can be treated in exactly the same manner. In practice, it is sometimes required to find solutions of (1) which behave like $\sqrt{1-x^2}$ near the ends. This requires that a subsidiary condition be imposed on g rather than on φ; the consequent analysis can be carried out in the manner indicated although the details will change.

Finally we comment that it may be possible, by a more thorough analysis, to extend the results to prove convergence for functions which do not possess all the smoothness assumed in our discussion. We have not investigated this since we consider such a result of marginal usefulness. This type of method is usually practical only when all functions are quite smooth; otherwise convergence becomes too slow.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616, USA