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ON A CLASS OF CYCLIC METHODS FOR THE NUMERICAL INTEGRATION OF STIFF SYSTEMS OF O.D.E.s

J. R. CASH

Abstract.

A class of cyclic linear multistep methods suitable for the approximate numerical integration of stiff systems of first order ordinary differential equations is developed. Particular attention is paid to the problem of deriving schemes which are almost A-stable, self starting, have relatively high orders of accuracy and contain a built in error estimate. These requirements demand that the linear multistep methods which are used are solved iteratively rather than directly in the usual way and an efficient method for doing this is suggested. Finally the algorithms are illustrated by application to a particular test problem.

Introduction.

Recently [1] the present author has introduced a class of highly stable iterative integration procedures suitable for the approximate numerical integration of first order systems of ordinary differential equations of the form

$$dx/dt = f(x),\ x(t_0) = x_0,\ x \in \mathrm{R}^s\ . \tag{1.1}$$

One of the main virtues of these procedures is that they are able to achieve relatively high orders of accuracy while maintaining both an infinite region of absolute stability and a single step nature. A combination of these three characteristics is of course not possible with conventional linear multistep methods. The algorithms which were proposed in [1] were described as "implicit predictor-corrector schemes" since, although the "predictor" is implicit and must be iterated to convergence, the method of use of these schemes has distinct similarities to the way in which conventional explicit predictor-corrector methods are applied. If, for example, we use one of the integration procedures proposed in [1] to integrate from the nth step point t_n to t_{n+1} $(=t_n+h_n)$ it is necessary to first of all predict the solution values at the step points $(t_{n+1}, t_{n+2}, \ldots, t_{n+j})$ where $j \geqq 2$. In some cases this will result in a disproportionately large number of predicted values being required and this is obviously an undesirable situation. Two other problems which need to be investigated and which have not been discussed so far are firstly how our analysis may be extended to the multistep case and secondly how we may obtain a computable estimate of the local truncation error suitable for use with a step control procedure. The main purpose of the

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present paper is to examine these problems in some detail with the aim of suggesting ways in which they may be overcome. In the next section it will be shown that all of these three problems may be overcome to some extent if our basic algorithm is allowed to consist of more than one integration scheme with these schemes being applied in a well defined cyclic order. It has been found that in order to be able to derive a jth order method we need to use a composite procedure consisting of j distinct schemes and this allows a Milne-type error estimate to be available at every jth integration step.

In section 2 we present some particular algorithms of orders 2, 3 and 4 and finally in section 3 some numerical results are presented.

2. Cyclic iterative methods.

In a fairly recent paper [2] Donelson and Hansen have proposed a class of what they call cyclic composite multistep methods for the numerical integration of non-stiff systems of ordinary differential equations. Basically speaking their idea is to use not one but several integration procedures, applied in a given fixed order, in the hope of obtaining improved stability characteristics. It turns out, in fact, that a cyclic composite multistep method may be stable even though the k-step schemes which it uses are not all zero-stable. For numerous extensions of their analysis and a detailed account of the main implications of their approach the reader is referred to Stetter [3]. It is the purpose of this section to show that it is also possible to use iterative schemes of the type discussed in [1] in a cyclic fashion in such a way that the resulting algorithms offer certain computational advantages over more conventional ones currently in use. In order to explain our ideas we shall consider first of all a very simple scheme which we do not propose tor practical computation but which is useful for demonstrative purposes.

Scheme 1.

We shall denote our overall integration procedure by C1C2 and it will consist of two distinct schemes C1 and C2 applied in a cyclic fashion. We shall assume that the finally accepted approximation x_n to $x(t_n)$ has been computed and we consider first of all a predictor-corrector scheme, C1, which we shall use to calculate x_{n+1}. As a predictor we use the trapezoidal rule

$$x_{n+i}^{(0)}-x_{n+i-1}^{(0)} = \tfrac{1}{2}h\{f_{n+i}^{(0)}+f_{n+i-1}^{(0)}\} \qquad i=1,2,\ \ x_n^{(0)}=x_n\ ,$$

and as a corrector we use the scheme

$$x_{n+1}-x_n = hf_{n+1}-\tfrac{1}{2}\{x_{n+2}^{(0)}-2x_{n+1}^{(0)}+x_n^{(0)}\} \equiv hf_{n+1}-\tfrac{1}{2}\Delta^2 x_n^{(0)},$$

$$\text{where } \Delta x_n \equiv x_{n+1}-x_n\ .$$

We now make a few remarks regarding the computational aspects of this predictor-corrector scheme and these remarks will hold generally for all the schemes which we shall propose in this section. We note first of all that our "predictor" is implicit and to solve for the required solution $x_{n+i}^{(0)}$ we use a quasi-Newton iteration procedure iterated to convergence. It is important to note that associated with our algorithm there are two distinct iterations namely the predictor-corrector scheme itself (which we call the primary iteration) and the quasi-Newton scheme (which we call the secondary iteration) which is used to generate the required solution of the algebraic equations occuring at each step point. The way in which we apply our quasi-Newton schemes is very similar to that described by Gear [4] in that we keep the Jacobian matrix fixed and re-evaluate it only if the scheme fails to converge to the prescribed degree of precision in four iterations. If the scheme still fails to converge in four iterations we halve h and re-start from the step point t_n. The only remaining problem which we have to consider is that of finding a suitable initial iterate for use with our iteration schemes and for our predictor this is provided by the explicit Euler rule and for our corrector $x_{n+1}^{(0)}$ serves as our initial iterate. It may be shown that the scheme C1 is L-stable and has order 2. In order to obtain the required solution at the step point t_{n+2} we use the backward differentiation scheme (denoted by C2)

$$x_{n+2}-(4/3)x_{n+1}+(1/3)x_n = (2/3)h\mathring{x}_{n+2} .$$

Note that we already have the iterate $x_{n+2}^{(0)}$ available, which we hope is a reasonable approximation to x_{n+2}, and we can use this as the first iterate in the quasi-Newton scheme used to solve for x_{n+2}. We now examine the region of absolute stability of our composite scheme ClC2. Applying scheme C1 to the scalar test equation $\mathring{x}=\lambda x$, where for the remainder of this paper λ will denote a complex constant with negative real part, we obtain the expression

(2.1)
$$x_{n+1}/x_n = R_1(h\lambda) = (1-q/2-7q^2/16)/(1-3q/2+9q^2/16-q^3/16), \qquad q=h\lambda .$$

If we now apply scheme C2 to our scalar test equation we obtain the expression

(2.2) $$x_{n+2}/x_n = R_2(h\lambda) = (4R_1(h\lambda)/3-1/3)/(1-2q/3) .$$

By examining the magnitude of the rational function $R_2(h\lambda)$ we may easily determine the region of absolute stability of our cyclic scheme and it may be shown that scheme C1C2 is L-stable and of order 2. We note, however, that this scheme as it stands requires three coefficient matrices to be factorised for each block step ($=2$ integration steps of length h) and this would seem to be an area where an increase in computational efficiency could be obtained. We may of course be able to use a quasi-Newton iteration scheme which allows these coefficient matrices to be kept piecewise constant over several successive integration steps but in general we shall try to avoid the possibility of having more than two matrices to factorise for each block step. In general when considering conventional linear multistep methods for use as predictors it would seem to be advisable to use the trapezoidal rule since this is the most accurate A-

stable one step scheme. When considering possible correctors, however, we have a very wide choice and we shall need to bear in mind that we wish to keep the number of matrix factorisations to a minimum. We shall not consider any modifications to scheme 1 since we are not proposing it for practical computation but we shall instead go on to consider the problem of obtaining an estimate of the local truncation error committed in integrating from t_n to t_{n+2}. If we again consider our backward differentiation scheme we have

$$x_{n+2}-(4/3)x_{n+1}+(1/3)x_n = (2/3)h\mathring{x}_{n+2}\,. \tag{2.3}$$

The analytic solution $x(t_n)$ satisfies

$$x(t_{n+2})-(4/3)x(t_{n+1})+(1/3)x(t_n) = (2/3)hf(t_{n+2},x(t_{n+2})) + \alpha_1 h^3 x_n''' + O(h^4) \tag{2.4}$$

where α_1 is the principal error constant associated with (2.3). Usually the procedure adopted at this stage is to assume that x_n and x_{n+1} are exact and then to derive an estimate of the error in x_{n+2} on this assumption. In our case, however, it does not seem reasonable to assume that x_{n+1} is exact — what we really need to do is to calculate an approximation to the error in x_{n+2} on the assumption that x_n only is exact. We may do this by using as a corrector the scheme

$$x_{n+2}-(4/3)x_{n+1}+(1/3)x_n = h\{(4/3)\mathring{x}_{n+1}-(2/3)\mathring{x}_n\}\,. \tag{2.5}$$

The true solution of this scheme satisfies

$$x(t_{n+2})-(4/3)x(t_{n+1})+(1/3)x(t_n) = h\{(4/3)f(t_{n+1},x(t_{n+1})) - (2/3)f(t_n,x(t_n))\} + \alpha_2 h^3 x_n''' + O(h^4) \tag{2.6}$$

where α_2 is the principal error constant associated with (2.5). Denoting the solution obtained using (2.3) by x_{n+2} and that obtained from (2.5) by $x_{n+2}^{(0)}$ we have

$$x(t_{n+2})-x_{n+2}^{(0)}-(4/3)(x(t_{n+1})-x_{n+1}) = \alpha_2 h^3 x_n''' + O(h^4)\,. \tag{2.7}$$

Similarly subtracting (2.3) from (2.4) we have

$$x(t_{n+2})-x_{n+2}-(4/3)(x(t_{n+1})-x_{n+1}) = \alpha_1 h^3 x_n''' + O(h^4)\,. \tag{2.8}$$

Ignoring the $O(h^4)$ terms we have from (2.7) and (2.8).

$$\alpha_1(x_{n+2}-x_{n+2}^{(0)})/(\alpha_1-\alpha_2) = \alpha_1 h^3 x_n''' \tag{2.9}$$

and the left hand side of this expression serves as a computable estimate of the local truncation error of our cyclic scheme.

Scheme 2.

We now consider the derivation of a third order cyclic scheme comprising of three distinct iterative integration procedures. As our first scheme, which we denote by C1, we use an implicit predictor-corrector scheme where as a predictor we use

$$x_{n+i}^{(0)} - x_{n+i-1}^{(0)} = (h/2)\{f(t_{n+i}, x_{n+i}^{(0)}) + f(t_{n+i-1}, x_{n+i-1}^{(0)})\} \qquad i=1,2,3,\; x_n^{(0)} = x_n\,,$$

and as a corrector we use

$$x_{n+1} - x_n = (h/2)\{\mathring{x}_{n+1} + \mathring{x}_n\} - (1/12)\Delta^3 x_n^{(0)}\,.$$

When applied to our scalar test equation $\mathring{x} = \lambda x$ this scheme yields an expression of the form

$$x_{n+1}/x_n = R_1(h\lambda)$$

where

$$R_1(h\lambda) = \alpha - (\alpha-1)^3/(12(1-q/2)), \qquad \alpha = (1+q/2)/(1-q/2),\; q = h\lambda\,.$$

In order to calculate an approximation to x_{n+2} we use the iteration scheme C2 given by

$$x_{n+2} - x_{n+1} = (h/2)\{\mathring{x}_{n+2} + \mathring{x}_{n+1}\} - (\Delta^3 x_n^{(0)})/12\,.$$

Two important practical points which we note are

1) The predicted values $\{x_n^{(0)}\}$ computed for use with scheme C1 are also used with scheme C2 and so it is not necessary to compute any additional predicted values.
2) For both scheme C1 and C2 the coefficient matrix of our secondary iteration scheme takes the same form i.e. $I - hJ/2$ where J is some suitable approximation to the $s \times s$ Jacobian matrix $\partial f/\partial x$. In practice it has often been found to be possible to keep this coefficient matrix fixed for both steps and this generally results in a considerable saving in computational effort.

If we now apply scheme C2 to our scalar test equation we obtain

$$x_{n+2}/x_n = R_2(h\lambda)$$

where

$$R_2(h\lambda) = \alpha R_1(h\lambda) - (\alpha-1)^3/(12(1-q/2))\,.$$

For our third and final iteration scheme we shall consider the third order backward differentiation scheme given by

$$x_{n+3} - 18x_{n+2}/11 + 9x_{n+1}/11 - 2x_n/11 = 6h\mathring{x}_{n+3}/11$$

which we denote by scheme C3. We note that we already have an initial approximation $x_{n+3}^{(0)}$ to x_{n+3} and hopefully this will provide a good initial iterate for use with our quasi-Newton iteration scheme. Applying scheme C3 to our scalar test equation we obtain

$$x_{n+3}/x_n = R_3(h\lambda)$$

where

(2.10) $$R_3(h\lambda) = (18R_2(h\lambda)/11 - 9R_1(h\lambda)/11 + 2/11)/(1 - 6q/11).$$

We note that scheme C1C2C3 requires a maximum of only 2 matrix factorisations for each block step if our quasi-Newton iteration scheme converges. Before deriving a procedure for estimating the local truncation error in our solution we first of all consider another possible scheme C1$\overline{\text{C2}}$C3 where scheme $\overline{\text{C2}}$ is given by

$$x_{n+2} - (12/11)x_{n+1} + x_n/11 = h\{6\mathring{x}_{n+2}/11 + 4\mathring{x}_{n+1}/11\} - 4\Delta^3 x_n^{(0)}/33.$$

We note that in common with scheme C1C2C3 considered earlier, the composite scheme C1$\overline{\text{C2}}$C3 requires only two matrix factorisations at the maximum per block step if the relevant quasi-Newton secondary iteration schemes converge and also it requires only one set of predicted values $\{x_n^{(0)}\}$. Applying scheme C1$\overline{\text{C2}}$C3 to $\mathring{x} = \lambda x$ we again obtain an expression of the form

$$x_{n+3}/x_n = R_3(h\lambda)$$

where

(2.11) $$R_3(h\lambda) = (18R_2(h\lambda)/11 - 9R_1(h\lambda)/11 + 2/11)/(1 - 6q/11)$$

and where

$$R_2(h\lambda) = \{(12/11 + 4q/11)R_1(h\lambda) - 1/11 - 4(\alpha - 1)^3/33\}/(1 - 6q/11).$$

It is clear that our expressions for the characteristic roots have now become rather too complicated to allow an analytic investigation of the stability properties of our block schemes and so we have to rely on a numerical method. Since all of the schemes which we have considered yield an expression of the form $x_{n+3}/x_n = R_3(h\lambda)$ it is easy to plot the locus of q in the complex left hand half plane which

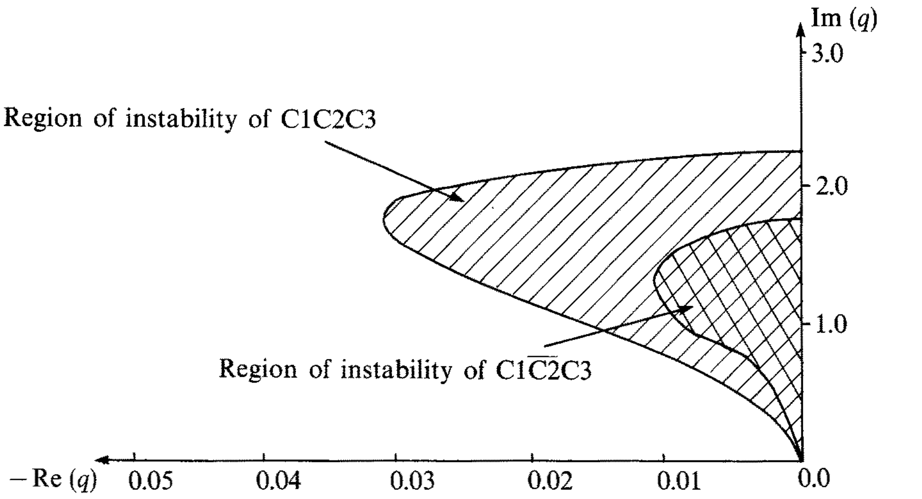

Figure 1.

is such that $|R_3(h\lambda)|<1$ and this is shown in Figure 1 for each of the methods which we have considered. Since the region of instability is symmetric about the negative real axis we in fact only show its intersection with the region Re $(q)<0$, Im $(q)>0$. We now consider the derivation of a Milne type procedure for estimating the local truncation error of our third order block schemes. We denote the solution obtained at the step point t_{n+3} using the cyclic scheme in question by x_{n+3} and the solution obtained at t_{n+3} using the scheme $x_{n+3}-(18/11)x_{n+2}+(9/11)x_{n+1}-(2/11)x_n=h(18\mathring{x}_{n+2}/11-18\mathring{x}_{n+1}/11+6\mathring{x}_n/11)$ by $x^{(0)}_{n+3}$. It may be shown that a Milne type error estimate of the local truncation error of our cyclic scheme is given by $T.E.=-(x^{(0)}_{n+3}-x_{n+3})/4$. We again emphasise the fact that both the third order schemes which we have suggested require only one set of predicted values to be calculated and as a result a considerable saving in computational effort is obtained compared with the schemes proposed in [1]. The main reason for finishing off our cyclic scheme with a backward differentiation scheme was in an attempt to give the whole scheme the correct asymptotic behaviour for large q. If, however, we are prepared to accept $A(\beta)$-stability rather than $L(\beta)$-stability we may make a further saving in computational effort by using as our third scheme $\hat{C3}$

$$x_{n+3}-x_{n+2} = (h/2)\{\mathring{x}_{n+3}+\mathring{x}_{n+2}\}-(\Delta^3 x_n^{(0)})/12 .$$

The region of absolute stability of our new composite scheme C1C2$\hat{C3}$ may be found in the way previously explained and may be shown to have a small region of instability close to the imaginary axis.

Scheme 3

Finally in this section we consider the derivation of a fourth order cyclic scheme consisting of four distinct iteration procedures. As our first predictor-corrector scheme, which we denote by C1, we use the predictor

$$x^{(0)}_{n+i}-x^{(0)}_{n+i-1} = (h/2)\{f^{(0)}_{n+i}+f^{(0)}_{n+i-1}\} \qquad i=1,2,3,4$$

and as a corrector we use

$$x_{n+1}-x_n = (h/2)\{\mathring{x}_{n+1}+\mathring{x}_n\}-\tfrac{1}{12}\{\Delta^3 x_n^{(0)}-\Delta^4 x_n^{(0)}\} .$$

It is easy to show that this scheme has order 4 and when applied to the scalar test equation $\mathring{x}=\lambda x$ it yields an expression of the form

$$x_{n+1}/x_n = R_1(h\lambda)$$

where

$$R_1(h\lambda) = \alpha-\{-\alpha^4+5\alpha^3-9\alpha^2+7\alpha-2\}/(12(1-q/2)) ,$$

$$q=h\lambda,\ \alpha=(1+q/2)/(1-q/2) .$$

The region of absolute stability of this scheme may easily be determined using a numerical method and it may be shown to be A-stable. In order to calculate our approximation x_{n+2} we consider the use of another predictor-corrector scheme where we again impose the restrictions that firstly we do not want to calculate any more predicted values and secondly we want to allow ourselves the possibility of not having to factorise any more coefficient matrices. With these restrictions in mind the scheme, C2, which we use to calculate x_{n+2} is

$$x_{n+2}-x_{n+1} = (h/2)\{\mathring{x}_{n+2}+\mathring{x}_{n+1}\}-\tfrac{1}{12}\Delta^3 x_n^{(0)} .$$

Applying this scheme to our scalar test equation we obtain the expression

$$x_{n+2}/x_n = R_2(h\lambda)$$

where

$$R_2(h\lambda) = \alpha R_1(h\lambda)-(\alpha^3-3\alpha^2+3\alpha-1)/(12(1-q/2)) .$$

It may easily be shown that the scheme C1C2 is A-stable. Following this line of approach we may calculate the approximation x_{n+3} using the predictor-corrector scheme C3 which is given by

$$x_{n+3}-x_{n+2} = (h/2)(\mathring{x}_{n+3}+\mathring{x}_{n+2})-(\Delta^3 x_n^{(0)}+\Delta^4 x_n^{(0)})/12 ,$$

as corrector with the same predictor as before being used. Applying this scheme to our scalar test equation we obtain

$$x_{n+3}/x_n = R_3(h\lambda)$$

where

$$R_3(h\lambda) = \alpha R_2(h\lambda)-\alpha(\alpha-1)^3/(12(1-q/2)) .$$

It is easy to show using a numerical method that the scheme C1C2C3 is A-stable. We now consider two possible schemes $\hat{C}4$ and $\overline{C4}$ to calculate the solution x_{n+4} and so complete our block step forward. If we demand that our overall scheme should be $L(\beta)$-stable for β close to $\pi/2$ it would seem to be necessary to use an $L(\beta)$-stable scheme $\hat{C}4$. One obvious candidate is the fourth order backward differentiation scheme

$$x_{n+4}-48x_{n+3}/25+36x_{n+2}/25-16x_{n+1}/25+3x_n/25 = 12h\mathring{x}_{n+4}/25 .$$

Applying this scheme to our scalar test equation we obtain

$$x_{n+4}/x_n = R_4(h\lambda)$$

where

$$R_4(h\lambda) = (48R_3(h\lambda)/25-36R_2(h\lambda)/25+16R_1(h\lambda)/25-3/25)/(1-12q/25) .$$

The region of absolute stability of our composite scheme C1C2C3$\hat{C}4$ may now be found in the usual way using a numerical method and the region of instability is

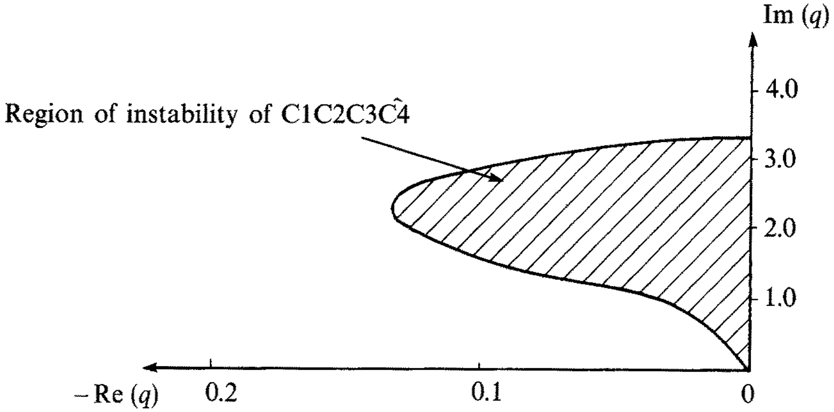

Figure 2.

plotted in Figure 2. As an alternative we may use a different scheme $\overline{C4}$ to calculate x_{n+4} which gives rise to a procedure $C1C2C3\overline{C4}$ which is $A(\beta)$-stable (but not $L(\beta)$-stable) but which provides us with the possibility of only having to factorise one coefficient matrix for each block step. The scheme which we use is

$$x_{n+4}-x_{n+3} = (h/2)(\mathring{x}_{n+4}+\mathring{x}_{n+3})-(\Delta^3 x_n^{(0)})/12-(\Delta^4 x_n^{(0)})/6 .$$

If we now apply our composite scheme $C1C2C3\overline{C4}$ to the scalar test equation $\mathring{x} = \lambda x$ we obtain an expression of the form

$$x_{n+4}/x_n = R_4(h\lambda)$$

where

$$R_4(h\lambda) = \alpha R_3(h\lambda)-(2\alpha^4-7\alpha^3+9\alpha^2-5\alpha+1)/(12(1-q/2)) ,$$
$$\alpha = (1+q/2)/(1-q/2) .$$

The region of stability of this method may be found numerically in the usual way and the scheme $C1C2C3\overline{C4}$ may be shown to be "almost" A-stable with a small region of instability close to the imaginary axis. We now consider a procedure for estimating the local truncation error of our composite scheme $C1C2C3\hat{C}4$. We denote the solution obtained at the step point t_{n+4} using our cyclic scheme by x_{n+4} and that obtained using the fourth order explicit scheme

$$x_{n+4}-48x_{n+3}/25+36x_{n+2}/25-16x_{n+1}/25+3x_n/25$$
$$= h(48\mathring{x}_{n+3}-72\mathring{x}_{n+2}+48\mathring{x}_{n+1}-12\mathring{x}_n)/25$$

by $x_{n+4}^{(0)}$. It may be shown that a computable estimate of the local truncation error of our cyclic scheme is given by

$$T.E. = (x_{n+4}-x_{n+4}^{(0)})/5 .$$

We note that for this scheme an estimate of the local truncation error in our solution is obtained at every fourth step point and even if our approximate solution is satisfactory at this particular point it may be that approximations at previous points in the block are relatively poor. In many practical applications this will not matter because we are often only interested in computing the solution at the end point of the range of integration for stiff systems. In some cases, however, it would be useful to be able to estimate the error at intermediate points, especially as this should add to the robustness of the scheme, and the possibility of deriving an algorithm which will allow us to do this is at present being investigated. Similar remarks also apply to other block methods which have been derived in this section.

3. Numerical results.

In this section we illustrate the general approach described in section 2 by considering the numerical integration of the first order system of ordinary differential equations

$$\begin{aligned} \mathring{x} &= 0.01-(0.01+x+y)(x^2+1001x+1001), & x(0) &= 0 \\ \mathring{y} &= 0.01-(0.01+x+y)(1+y^2), & y(0) &= 0 \end{aligned} \qquad 0 \leqq t \leqq 100\,,$$

using the scheme C1C2C3$\hat{C}$4. This problem, which is very stiff initially but becomes less so for large t, has been suggested as a test problem by Bjurel et. al. [5]. Two runs of this problem were performed first with a local error tolerance of $\varepsilon=10^{-4}$ and then with an error tolerance of $\varepsilon=10^{-6}$. At every fourth step an estimate, *T.E.*, of the local truncation error was obtained using the procedure described in section 2 and, assuming that we are integrating from t_n to t_{n+4}, the following step control procedure was used:

1) If $|T.E.|>\varepsilon$ halve h and go back to t_n.
2) If $\varepsilon/50<|\mathrm{T.E.}|<\varepsilon$ keep h fixed.
3) If $\varepsilon/2500<|T.E.|<\varepsilon/50$ double h
4) otherwise take a value of h four times the old value.

The results obtained for the solution of this problem are summarised in Table 1. As can be seen an increasingly accurate solution is obtained at $t=100$ as stricter local error tolerances are imposed. The value of t given in Table 1 denotes the end point of the block step being considered and the value of h denotes the step length of integration used for that particular block step. As can be seen from the results presented the error estimate obtained always provides a reasonable estimate to the true error and so is suitable for use with a step control procedure (and this was also found to be the case for all step points not shown in Table 1). This general conclusion was also reached with all other test problems which have been run.

Table 1

Solution obtained at $t=100$ with	$\varepsilon=10^{-4}$	$x=-0.99154381$, $y=0.98322422$
	$\varepsilon=10^{-6}$	$x=-0.99163985$, $y=0.98333382$
	true solution	$x=-0.99164207$, $y=0.98333636$

Performance of error estimate

t	h	True error in x	True error in y	Estimated error in x	Estimated error in y
0.0004	0.0001	$0.834\cdot10^{-8}$	$0.703\cdot10^{-11}$	$0.956\cdot10^{-8}$	$0.856\cdot10^{-11}$
35.1652	6.5536	$0.584\cdot10^{-7}$	$0.584\cdot10^{-7}$	$0.442\cdot10^{-7}$	$0.408\cdot10^{-7}$
94.1476	1.6384	$0.203\cdot10^{-5}$	$0.203\cdot10^{-5}$	$0.353\cdot10^{-5}$	$0.149\cdot10^{-5}$
97.4224	0.8192	$0.130\cdot10^{-5}$	$0.129\cdot10^{-5}$	$0.118\cdot10^{-5}$	$0.888\cdot10^{-6}$

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DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE
QUEEN'S GATE, LONDON S.W. 7
ENGLAND