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A NUMERICAL METHOD FOR THE INTEGRATION OF OSCILLATORY FUNCTIONS

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Abstract.

A new method for the calculation of the integrals

$$I_1(m) = \int_a^b f(x) \sin mx\,dx \text{ and } I_2(m) = \int_a^b f(x) \cos mx\,dx$$

is presented. The function $f(x)$ is approximated by a sum of Chebyshev polynomials. The Chebyshev coefficients are then used to calculate a Neumann series approximation for $I_1(m)$ and $I_2(m)$.

The numerical examples demonstrate that this method is very accurate and efficient.

1. Introduction.

The numerical calculation of integrals of the form

$$\int_a^b f(x) \sin mx\,dx \tag{1}$$

and

$$\int_a^b f(x) \cos mx\,dx \tag{2}$$

occurs in many problems of physics and engineering. If m is large, special integration methods must be used for the evaluation of (1) and (2), since, in view of the strongly oscillatory character of the integrand, classical quadrature formulas require too much computation work [1]. The best known formulas are based on the piecewise approximation by polynomials of $f(x)$ on the integration interval. The resulting integrals are then integrated exactly. Usually, the degree of the approximating polynomials is low. The methods given by Filon [2], Flinn [3], Buyst and Schotsmans [4], Tuck [5], Einarsson [6], Van de Vooren and Van Linde [7] are of this type. Miklosko [8] proposed to use an interpolatory

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quadrature formula which has as nodes the zeros of the Chebyshev polynomials. However, the order of his formula is limited, since the calculation of the weights for high order formulas is affected by a large loss of significant figures. Other formulas are based on the integration between the zeros of $\cos mx$ or $\sin mx$ [1], [9], [10], [11], [12]. Some additional references are [13]–[16].

If $f(x)$ is a very smooth function on the interval $[a,b]$, it would be more interesting to approximate $f(x)$ by a polynomial $Q_N(x)$ of high degree N, in the whole integration interval. The integrals (1) and (2) are then approximated by

$$\int_a^b f(x) \begin{Bmatrix} \sin mx \\ \cos mx \end{Bmatrix} dx = \int_a^b Q_N(x) \begin{Bmatrix} \sin mx \\ \cos mx \end{Bmatrix} dx . \tag{3}$$

If $a = -1$ and $b = +1$, Bakhvalov and Vasileva [17] proposed to approximate $f(x)$ by a sum of Legendre polynomials

$$Q_N(x) = \sum_{k=0}^{N} C_k P_k(x) . \tag{4}$$

The coefficients C_k in (4) are determined by interpolation in the zeros of the Legendre polynomials. The expression (3) becomes

$$\int_a^b f(x) \sin mx dx \cong \sum_{k=0}^{[(N-1)/2]} C_{2k+1}(-1)^k \sqrt{2\pi/m}\, J_{2k+3/2}(m) \tag{5}$$

$$\int_a^b f(x) \cos mx dx \cong \sum_{k=0}^{[N/2]} C_{2k}(-1)^k \sqrt{2\pi/m}\, J_{2k+\frac{1}{2}}(m) \tag{6}$$

where $J_n(x)$ is the Bessel function of the first kind and order n.

The advantages of Chebyshev polynomials in approximation problems are well known [18] and thus it would be interesting to use these polynomials to approximate $f(x)$. Bakhvalov and Vasileva [17] have examined this question and their conclusion is that the resulting formulas are more complicated than with Legendre polynomials and that there is a serious effect of rounding off in the computation. We describe here a new method which can be considered as a modification of the method of Bakhvalov and Vasileva and which uses Chebyshev polynomials in place of Legendre polynomials, without a loss of significance during the computations.

2. Description of the method.

By a simple change of integration variable, it can be shown that the calculation of integrals (1) and (2) can be reduced to the evaluation of the integrals

$$S(m) = \int_{-1}^{+1} f(x) \sin mx\, dx \tag{7}$$

and

$$C(m) = \int_{-1}^{+1} f(x) \cos mx\, dx \tag{8}$$

where m may have a different value than before. Consider now the following Chebyshev expansion

$$\sqrt{1-x^2} f(x) = \sum_k{}' c_k T_k(x) \tag{9}$$

the single prime denoting that the first term is taken with factor $\frac{1}{2}$.

Since in most cases the series (9) is slowly convergent, coefficients c_k cannot be efficiently calculated using the orthogonal property of summation of the Chebyshev polynomials. We then use the following procedure.

The function $f(x)$ is approximated by the truncated Chebyshev series

$$f(x) \cong \sum_{i=0}^{N}{}' a_i T_i(x) . \tag{10}$$

Since $f(x)$ is supposed to be smooth on $[-1,1]$, formula (10) gives a good approximation of $f(x)$, even with a small value of N. The coefficients a_i can be approximated by the formulae [18], [19]

$$a_i \cong \frac{2}{N} \sum_{j=0}^{N}{}'' f\left(\cos \frac{\pi j}{N}\right) \cos \frac{\pi j i}{N} \tag{11}$$

where the double prime indicates that both the first and last terms of the sum are taken with factor $\frac{1}{2}$, or

$$a_i \cong \frac{2}{N+1} \sum_{j=0}^{N} f(\cos x_j) \cos (i x_j) \tag{12}$$

where

$$x_j = \frac{2j+1}{N+1} \cdot \frac{\pi}{2} .$$

If desired, the fast Fourier transform algorithm (see Cooley and Tuckey [20]) can be used in order to compute (11) very efficiently.

The values of N must be determined in the same way as for the Clenshaw–Curtis quadrature method [21]. Thus, if for a value of N, the approximation (10) is found to be insufficient, N must be doubled. Only N additional function evaluations are then necessary.

For using formulae (11) or (12), $f(x)$ is required at non-equidistant points. This does not present any difficulty if $f(x)$ is known analytically. If $f(x)$ is given as a table with equidistant or even arbitrary non-equidistant values of x, which is a rather general situation when the function originates from physical experiments, Clenshaw's curve fitting method [22] can be applied in order to construct an approximation of $f(x)$ in the form (10).

Since

$$c_k = \frac{2}{\pi} \int_{-1}^{+1} f(x) T_k(x)\, dx \tag{13}$$

we have

$$c_k \cong \frac{2}{\pi} \sum_{i=0}^{N}{}' a_i \int_{-1}^{+1} T_k(x) T_i(x)\, dx \tag{14}$$

or

$$c_{2k} \cong -\frac{2}{\pi} \sum_{i=0}^{[N/2]}{}' a_{2i} \left[\frac{1}{(2i+2k)^2-1} + \frac{1}{(2i-2k)^2-1}\right] \tag{15}$$

and

$$c_{2k+1} \cong -\frac{2}{\pi} \sum_{i=0}^{[(N-1)/2]} a_{2i+1} \left[\frac{1}{(2i+2k+2)^2-1} + \frac{1}{(2i-2k)^2-1}\right]. \tag{16}$$

The formulas (7) and (9) yield

$$S(m) \cong \sum_{k=0}^{\infty}{}' c_k \int_{-1}^{+1} \frac{\sin mx T_k(x)}{\sqrt{1-x^2}}\, dx \tag{17}$$

or

$$S(m) \cong \sum_{k=0}^{\infty} c_{2k+1}(-1)^k \pi J_{2k+1}(m)\,. \tag{18}$$

In the same manner, we have

$$C(m) \cong \sum_{k=0}^{\infty}{}' c_{2k}(-1)^k \pi J_{2k}(m)\,. \tag{19}$$

We obtain thus Neumann series approximations for the integrals $S(m)$ and $C(m)$.

From the Riemann–Lebesgue lemma, we know that c_k tends to zero for $k \to \infty$, if $f(x)$ is absolutely integrable from -1 to 1. We know also that

$$J_n(m) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{em}{2n}\right)^n \quad n \to \infty . \tag{20}$$

Thus series (18) and (19) are convergent and the terms decrease very rapidly for increasing k and $k > m/2$. Thus, if series (18) and (19) are truncated after M terms, where M is only a little larger than $m/2$, the truncation error may be completely neglected. It is also evident that the choice of M depends not only on the value of m, but also on the desired accuracy of this result. This problem, however, is already discussed by many authors [23–27]. In the following error analysis we assume that the errors result only from the truncation of series (10). We neglect also the roundoff errors.

If we denote the error on $S(m)$ and $C(m)$ by $\varepsilon_S(m)$ resp. $\varepsilon_C(m)$, we have

$$\varepsilon_S(m) = \sum_{i=N+1}^{\infty} a_i \int_{-1}^{+1} \sin mx T_i(x)\, dx . \tag{21}$$

Since

$$\int_{-1}^{+1} \sin mx\, T_{2i+1}(x)\, dx = -\frac{2 \sin m}{(2i+1)^2} + O(i^{-4}), \quad i \to \infty \tag{22}$$

we have for large N ($N \gg 2m$)

$$\varepsilon_S(m) \cong -2 \sin m \sum_{i=[(N+1)/2]}^{\infty} \frac{a_{2i+1}}{(2i+1)^2} \tag{23}$$

and, in the same way

$$\varepsilon_C(m) \cong -\frac{\cos m}{2} \sum_{i=[N/2]+1}^{\infty} \frac{a_{2i}}{i^2} . \tag{24}$$

From these expressions, we conclude that for $N \to \infty$, the integration method converges faster than the Chebyshev series for $f(x)$ itself. This explains the very good results of this method, even for singular integrals, as will be showed in the examples.

It is also important to note that, although this method does not impose any restriction on the values of m, (unlike some other algorithms which require m to be an integer times π), the accuracy depends on m, because of the factor $\sin m$ in (23) and $\cos m$ in (24).

3. Numerical examples.

In this section we give some numerical examples. For the evaluation of the Neumann series, we have used the method of Babuskova [27].

We calculate

$$\int_0^{2\pi} f_i(x) \sin mx\, dx, \quad i = 1(1)4 \tag{25}$$

for several values of m, where

$$\begin{aligned} f_1(x) &= x \cos x \\ f_2(x) &= x \cos 50x \\ f_3(x) &= \frac{x}{\sqrt{1-x^2/4\pi^2}} \\ f_4(x) &= \log x\,. \end{aligned}$$

The function $f_1(x)$ and $f_2(x)$ are analytical and we can thus expect very accurate results. The functions $f_3(x)$ and $f_4(x)$ have a singularity on the integration interval but the results, although less accurate, will still be useful, for reasons explained above.

Davis and Rabinowitz [1] have also calculated these integrals using other methods, namely Filon's method and a method based on Lobatto quadrature. Here we compare our method with Filon's rule (however with more function evaluation than in [1]) and with a Gaussian rule

$$\int_{-\pi}^{\pi} f(x) \sin x\, dx \cong \sum_{j=1}^{L} w_j[f(x_j) - f(-x_j)]\,. \tag{26}$$

The integrals (25) can be evaluated by applying m times formula (26). The number of function evaluations is thus $2mL$. Abscissas x_j and weights w_j of (26) are tabulated for $L = 1(1)9$ in [11]. This Gaussian rule is more efficient than Lobatto's rule, as is pointed out in [12]. The calculations were carried out in double precision on an IBM 360/44 computer. In Tables I–IV, the exact values and the errors (by error we mean |exact value—computed value|) of the different methods are shown. The numbers between brackets in the tables give the number of function evaluations. The number of terms in (12) and (19) is automatically determined by the computer program so that the addition of other terms gives no improvement.

Table I. $\int_0^{2\pi} x \cos x \sin mx\, dx.$

		Errors		
m	Exact	Filon's rule	Gauss rule	New method
1	-1.5707963267948966	9×10^{-11} (1003)	2×10^{-15} (12)	5×10^{-15} (30)
2	-4.1887902047863910	1×10^{-10} (1003)	1×10^{-15} (16)	9×10^{-15} (30)
4	-1.6755160019145564	4×10^{-11} (1003)	3×10^{-15} (24)	3×10^{-15} (30)
16	-0.3942390780975427	1×10^{-11} (1003)	7×10^{-15} (64)	1×10^{-14} (30)
64	-0.0981987447275930	2×10^{-12} (1003)	7×10^{-16} (256)	3×10^{-15} (30)
256	-0.0245440671189132	1×10^{-12} (1003)	2×10^{-13} (512)	1×10^{-15} (30)

Table II. $\int_0^{2\pi} x \cos 50x \sin mx\, dx.$

		Errors		
m	Exact	Filon's rule	Gauss rule	New method
1	0.002514279834805	4×10^{-7} (1003)	/	3×10^{-14} (200)
2	0.005034603611522	7×10^{-6} (1003)	7×10^{-4} (36)	1×10^{-14} (200)
4	0.010117850736199	2×10^{-6} (1003)	1×10^{-3} (72)	9×10^{-15} (200)
16	0.044799895238357	7×10^{-6} (1003)	2×10^{-14} (256)	6×10^{-15} (200)
64	-0.251957305551061	4×10^{-5} (1003)	3×10^{-14} (512)	9×10^{-11} (200)
256	-0.025517092433498	2×10^{-6} (1003)	4×10^{-12} (1024)	2×10^{-12} (200)

Table III. $\int_0^{2\pi} \frac{x}{\sqrt{1 - x^2/4\pi^2}} \sin mx\, dx.$

		Errors	
m	Exact	Gauss rule	New method
1	-13.17038298	1×10^{-2} (18)	3×10^{-4} (50)
2	-9.58285045	5×10^{-3} (36)	5×10^{-4} (50)
4	-6.87607126	4×10^{-3} (72)	1×10^{-3} (50)
10	-4.38761102	3×10^{-3} (180)	4×10^{-5} (200)
20	-3.11175260	2×10^{-3} (360)	8×10^{-5} (200)
30	-2.54325962	2×10^{-3} (540)	1×10^{-4} (200)

Table IV. $\int_0^{2\pi} \log x \sin mx\, dx$.

		Errors	
m	Exact	Gauss rule	New method
1	−2.437653393	1×10^{-4} (18)	1×10^{-6} (50)
2	−1.557178276	8×10^{-5} (36)	3×10^{-6} (50)
4	−0.950738911	4×10^{-5} (72)	5×10^{-6} (50)
10	−0.471793074	1×10^{-5} (180)	5×10^{-8} (200)
20	−0.270544415	7×10^{-6} (360)	1×10^{-7} (200)
30	−0.193877275	5×10^{-6} (540)	3×10^{-7} (200)

For $f_3(x)$ and $f_4(x)$, Filon's rule is only applicable if we ignore the singularities by setting arbitrarily $f_3(2\pi)=0$ and $f_4(0)=0$ (or any other value). Since the results, thus obtained, are very bad (only 2 to 3 accurate significant figures for 1003 function evaluations) they are not shown in the tables III and IV.

We conclude that our method is superior to Filon's rule and also, for strongly oscillating integrals (large m) or singular integrals, to the Gauss rule.

Finally, we give an example where $f(x)$ is given as a table of function values $f(x_k)$ for equidistant values of the argument

$$x_k = -1+0.1k \qquad k = 0(1)20\,.$$

The function values are generated by calculating

$$f(x_k) = \frac{1}{x_k+3}$$

and, in order to simulate experimental errors, normally distributed random errors of 0.1% are superimposed. Using Clenshaw's curve fitting technique, we approximate $f(x)$ by a linear combination of Chebyshev polynomials (10) with $N=4$, and use (16) and (18) to calculate

$$\int_{-1}^{+1} \frac{1}{x+3} \sin mx\, dx\,.$$

In Table V, the results are compared with those of Filon's rule. The superiority of our method is evident.

Table V. $\int_{-1}^{1} \frac{\sin mx}{x+3} dx$

m	Exact	Filon's rule	New method ($N=4$)
1	-0.071675	-0.076150	-0.071682
2	-0.103085	-0.109804	-0.103102
4	-0.025117	-0.027653	-0.025132
10	-0.019120	-0.021167	-0.019140
20	0.004379	0.005748	0.004386
30	0.001627	0.000935	0.001616

4. Calculation of other types of integrals.

The same method can be used for the calculation of other types of integrals with a strongly varying factor in the integrand, for example

$$I_1(a) = \int_{-1}^{+1} e^{ax} f(x)\, dx \tag{27}$$

and

$$I_2(a) = \int_{-1}^{+1} e^{ax^2} f(x)\, dx\,. \tag{28}$$

These integrals are difficult to calculate with classical methods if $|a|$ is large.

We have the following formulas

$$I_1(a) \cong {\sum_{k=0}^{M}}' c_k \pi I_k(a) \tag{29}$$

and

$$I_2(a) \cong 2e^{a/2} {\sum_{k=0}^{M}}' c_{2k} \pi I_k(a/2) \tag{30}$$

where the coefficients c_k are given by (15) and (16) and where $I_k(x)$ is the modified Bessel function of the first kind and order k.

5. Conclusion.

We have presented a new method for the evaluation of integrals with strongly oscillating integrand. Our method is as economical as accurate and has the following advantages:

1. The function evaluations are common for all values of m and are also common for both integrals (1) and (2). This is particularly advantageous for the calculation of the coefficients of Fourier series, where both integrals (1) and (2) must be calculated for various values of m.
2. It is easy to write an "automatic integrator" for the calculation of integrals with highly oscillatory integrands, based on our method (for the definition of automatic integrator, see Davis and Rabinowitz [1]).

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REFERENCES

1. P. J. Davis and P. Rabinowitz, *Numerical Integration*, Blaisdell Publ. Co., London, 1967.
2. L. N. G. Filon, *On a quadrature formula for trigonometric integrals*, Proc. Roy. Soc. Edinburgh, 49 (1929), 38–47.
3. E. A. Flinn, *A modification of Filon's method of numerical integration*, J. Ass. Comp. Mach. 7 (1960), 181–184.
4. L. Buyst and L. Schotsmans, *A method of Gaussian type for the numerical integration of oscillating functions*, ICC Bull. 3 (1964), 210–214.
5. E. O. Tuck, *A simple Filon-trapezoidal rule*, Math. Comp. 21 (1967), 239–241.
6. B. Einarsson, *Numerical calculation of Fourier integrals with cubic splines*, BIT 8 (1968), 279–286.
7. A. I. van de Vooren and H. J. van Linde, *Numerical calculation of integrals with strongly oscillating integrand*, Math. Comp. 20 (1966), 232–245.
8. J. Miklosko, *Numerical integration with weight functions* $\cos kx$, $\sin kx$ *on* $[0, 2\pi/t]$, $t=1,2,\ldots$, Apl. Mat. 14 (1969), 179–194.
9. J. F. Price, *Discussion of quadrature formulas for use on digital computers*, Boeing Scientific Research Labs. Report D1-82-0052, 1960.
10. I. M. Longman, *A method for the numerical evaluation of finite integrals of oscillatory functions*, Math. Comp. 14 (1960), 53–59.
11. R. Piessens, *Gaussian quadrature formulas for the integration of oscillating functions*, Math. Comp. 24 (1970), microfiche card.
12. R. Piessens, *Gaussian quadrature formulas for the integration of oscillating functions*, ZAMM, vol. 50, 1970.
13. H. J. Stetter, *Numerical approximation of Fourier transforms*, Num. Math. 8 (1966), 235–249.
14. W. W. Clendenin, *A method for numerical calculation of Fourier integrals*, Num. Math. 8 (1966), 422–436.

15. Y. L. Luke, *On the computation of oscillating integrals*, Proc. Cambridge Philos. Soc. 50 (1954), 269–277.
16. J. N. Lyness, *The calculation of Fourier coefficients by the Möbius inversion of the Poisson formula*, Part I. *Functions whose early derivatives are continuous*, Math. Comp 24 (1970), 101–135. Part II. *Piecewise continuous functions and functions with poles near the interval* [0,1], Math. Comp. 25, 59–78.
17. N. S. Bakhvalov and L. G. Vasileva, *Evaluation of the integrals of oscillating functions by interpolation at nodes of Gaussian quadratures*, Z. Vycisl. mat. i mat. Fiz. 8 (1968), 175–181.
18. C. W. Clenshaw, *Chebyshev series for mathematical functions*, Math. Tables, vol. 5, National Physical Laboratory, London: H.M.S.O., 1962.
19. L. Fox and I. B. Parker, *Chebyshev polynomials in numerical analysis*, Oxford University Press, London, 1968.
20. J. W. Cooley and J. W. Tuckey, *An algorithm for the machine calculation of complex Fourier series*, Math. Comp. 19 (1965), 297–301.
21. C. W. Clenshaw and A. R. Curtis, *A method for numerical integration on an automatic computer*, Num. Math. 2 (1960), 197–205.
22. C. W. Clenshaw, *Curve fitting with a digital computer*, Comp. J. 2 (1960), 170–173.
23. J. C. P. Miller, *Handbook of Mathematical functions*, M. Abramowitz and I. Stegun Ed., chap. 19, Dover Publ., New York, 1965.
24. W. Gautschi, *Algorithm 236 – Bessel functions of the first kind*, Comm. ACM 7 (1964), 479–480.
25. W. Gautschi, *Computational aspects of three-term recurrence relations*, SIAM Rev. 9 (1967), 24–82.
26. F. Mechel, *Improvement in recurrence techniques for the computation of Bessel functions of integral orders*, Math. Comp. 22 (1968), 202–205.
27. R. Babuskova, *Über numerische Stabilität einiger Rekursionsformeln*, Apl. Mat. 9, 1964.

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