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PERTURBATION BOUNDS FOR THE LDL^H AND LU DECOMPOSITIONS

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Abstract.

Let A and $A + \Delta A$ be Hermitian positive definite matrices. Suppose that $A = LDL^H$ and $(A + \Delta A) = (L + \Delta L)(D + \Delta D)(L + \Delta L)^H$ are the LDL^H decompositons of A and $A + \Delta A$, respectively. In this paper upper bounds on $\|\Delta D\|_F$ and $\|\Delta L\|_F$ are presented. Moreover, perturbation bounds are given for the LU decomposition of a complex $n \times n$ matrix.

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1. Introduction.

Let A be a positive definite Hermitian matrix. Then there exists a unit lower triangular matrix L, (that is a lower triangular matrix with ones on the diagonal), and a diagonal matrix D such that

$$A = LDL^H. \tag{1.1}$$

This decomposition is known as the LDL^H-decomposition [2, p. 137-138]. In this paper we derive the perturbation bounds (2.10) on $\|\Delta L\|_F$ and $\|\Delta D\|_F$ when A is perturbed by a Hermitian ΔA. We have not found any perturbation bounds for this problem in the literature. It should however be mentioned that in applications where the LDL^H- decompositon is used, one can usually also use the Cholesky decomposition. Perturbation bounds for the Cholesky decomposition are given in [3].

The LU-decomposition [2, p. 92-102] is so well-known that no detailed presentation is necessary. Surprisingly, at least as far as we know, no perturbation bounds on the factors L and U have been published. This paper presents such bounds in formula (3.3).

The following notations are used: $\|\cdot\|_F$ denotes the Frobenius norm; $\|\cdot\|_2$ denotes

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the matrix norm induced by the Euclidean vector norm; the superscript H denotes the complex conjugate.

2. The LDL^H-decomposition.

We will first find bounds for the differentials. In the following lemma $\|dL\|_F \le C(A)\|dA\|_F$, where C is a function of A, means that for any differentiable parametrization $A(t)$, we have $\|dL/dt\|_F \le C(A)\|dA/dt\|_F$. (This is also the meaning of a norm of a differential in [3]).

LEMMA 2.1 *Assume that A is an $n \times n$ Hermitian and positive definite matrix, and dA is an $n \times n$ Hermitian differential. Consider L and D in the LDL^H-decomposition as functions of the elements of A. Then we have the following bounds for the differentials.*

$$\|dL\|_F \le (1/\sqrt{2})\,\|A\|_2^{1/2}\|A^{-1}\|_2^{3/2}\|dA\|_F \tag{2.1a}$$

$$\|dD\|_F \le \|A\|_2\|A^{-1}\|_2\|dA\|_F. \tag{2.1b}$$

PROOF. It is known that L and D are differentiable functions of the elements of A. Differentiating $A = L\,D\,L^H$ we get

$$dA = dL\,D\,L^H + L\,dD\,L^H + L\,D\,dL^H. \tag{2.2}$$

Multiply (2.2) from the left by $D^{-1/2}L^{-1}$ and from the right by $L^{-H}D^{-1/2}$, to get

$$\begin{aligned} D^{-1/2}L^{-1}dAL^{-H}D^{-1/2} &= D^{-1/2}L^{-1}dLD^{1/2} + D^{-1/2}dDD^{-1/2} \\ &\quad + D^{1/2}dL^HL^{-H}D^{-1/2}. \end{aligned} \tag{2.3}$$

($D^{1/2}$ is the diagonal matrix where the diagonal elements are the nonnegative square roots of the diagonal elements of D. The inverse of $D^{1/2}$ is denoted by $D^{-1/2}$). Let $A = GG^H$ be the Cholesky decomposition of A, then $G = LD^{1/2}$. Hence, the relation (2.3) can be rewritten as

$$G^{-1}dAG^{-H} = G^{-1}dLD^{1/2} + D^{-1/2}dDD^{-1/2} + D^{1/2}dL^HG^{-H}. \tag{2.4}$$

Consequently, we have

$$2\|G^{-1}dLD^{1/2}\|_F^2 + \|D^{-1/2}dDD^{-1/2}\|_F^2 = \|G^{-1}dAG^{-H}\|_F^2 \le \|A^{-1}\|_2^2\|dA\|_F^2. \tag{2.5}$$

The inequality (2.5) gives

$$\|G\|_2^{-1}\|D^{-1/2}\|_2^{-1}\|dL\|_F \le \|G^{-1}dLD^{1/2}\|_F \le (1/\sqrt{2})\|A^{-1}\|_2\|dA\|_F \tag{2.6}$$

and

$$\|D\|_2^{-1}\|dD\|_F \le \|D^{-1/2}dDD^{-1/2}\|_F \le \|A^{-1}\|_2\|dA\|_F. \tag{2.7}$$

By combining (2.6), (2.7) with $\|D^{-1}\|_2 \le \|A^{-1}\|_2$ and $\|D\|_2 \le \|A\|_2$, we get (2.1a) and (2.1b) respectively. ■

Lemma 2.1 can now be used to derive perturbation bounds for the LDL^H-decomposition. To simplify one of the bounds we use the function ω defined in (2.11). This function is defined in the same way as in [3].

THEOREM 2.2 *Let A be an $n \times n$ Hermitian and positive definite matrix with decomposition $A = LDL^H$. If ΔA is an $n \times n$ Hermitian matrix satisfying*

$$\|\Delta A\|_2 < 1/\|A^{-1}\|_2, \tag{2.8}$$

then there exists a unique LDL^H-decomposition

$$A + \Delta A = (L + \Delta L)(D + \Delta D)(L + \Delta L)^H, \tag{2.9}$$

where we have the bounds

$$\|\Delta L\|_F \leq \frac{1}{\sqrt{2}} \frac{(\|A\|_2\|A^{-1}\|_2)^{3/2}}{1 - \|A^{-1}\|_2\|\Delta A\|_2} \cdot \frac{\|\Delta A\|_F}{\|A\|_2}. \tag{2.10a}$$

and

$$\|\Delta D\|_F \leq ((\sigma_1/\sigma_n + 1)\omega(\|\Delta A\|_2/\sigma_n) - 1)\|\Delta A\|_F \leq \frac{\|A\|_2\|A^{-1}\|_2}{1 - \|A^{-1}\|_2\|\Delta A\|_2} \cdot \|\Delta A\|_F \tag{2.10b}$$

where σ_1 and σ_n are the largest and smallest singular value of A respectively, and

$$\omega(\varepsilon) = \frac{1}{\varepsilon}\ln\frac{1}{1-\varepsilon}, \qquad 0 < \varepsilon < 1. \tag{2.11}$$

PROOF: The existence of a unique LDL^H-decomposition follows from the existence of a unique Cholesky decomposition [3]. We will use Lemma 2.1 and integral-techniques [1,3] to derive (2.10). Let

$$A(t) = A + t\Delta A, \quad 0 \leq t \leq 1. \tag{2.12}$$

Then, $A(t)$ is positive definite, and we have the LDL^H-decomposition

$$A(t) = L(t)D(t)L(t)^H. \tag{2.13}$$

In particular $L(0) = L$ and $L(1) = L + \Delta L$. Hence, we get from (2.1a) the inequality

$$\|\Delta L\|_F = \|L(1) - L(0)\|_F = \|\textstyle\int_0^1 dL(t)dt\|_F \leq \int_0^1 \|dL(t)\|_F dt \leq \frac{1}{\sqrt{2}}\int_0^1 \frac{\sigma_1^{1/2}(t)}{\sigma_n^{3/2}(t)}\|\Delta A\|_F dt, \tag{2.14}$$

where $\sigma_1(t)$ and $\sigma_n(t)$ are the largest and smallest singular values of $A(t)$, respectively.

From a wellknown inequality for singular values [2, p. 428] we have

$$\sigma_1(t) \le \sigma_1 + t\|\Delta A\|_2, \quad \sigma_n(t) \ge \sigma_n - t\|\Delta A\|_2. \tag{2.15}$$

Thus

$$\|\Delta L\|_F \le \frac{\|\Delta A\|_F}{\sqrt{2}} \int_0^1 \frac{\sqrt{(\sigma_1 + t\|\Delta A\|_2)}}{(\sigma_n - t\|\Delta A\|_2)^{3/2}}\, dt \le$$

$$\frac{\|\Delta A\|_F}{\sqrt{2}} \int_0^1 \frac{\sqrt{(\sigma_1\sigma_n)}}{(\sigma_n - t\|\Delta A\|_2)^2}\, dt = \frac{\|\Delta A\|_F}{\sqrt{2}} \cdot \sqrt{\left(\frac{\sigma_1}{\sigma_n}\right)} \cdot \frac{1}{\sigma_n - \|\Delta A\|_2} = \tag{2.16}$$

$$= \frac{1}{\sqrt{2}} \frac{(\|A\|_2\|A^{-1}\|_2)^{3/2}}{1 - \|A^{-1}\|_2\|\Delta A\|_2} \cdot \frac{\|\Delta A\|_F}{\|A\|_2}.$$

Similar to (2.14), we get the inequality

$$\|\Delta D\|_F \le \int_0^1 \|dD(t)\|_F\, dt. \tag{2.17}$$

By combining (2.1b), (2.15) and (2.17) we get

$$\|\Delta D\|_F \le \|\Delta A\|_F \int_0^1 \frac{\sigma_1 + t\|\Delta A\|_2}{\sigma_n - t\|\Delta A\|_2}\, dt = \|\Delta A\|_F \left((\sigma_1 + \sigma_n) \int_0^1 \frac{dt}{\sigma_n - t\|\Delta A\|_2} - 1 \right) =$$

$$((\sigma_1/\sigma_n + 1)\omega(\|\Delta A\|_2/\sigma_n) - 1)\|\Delta A\|_F \le \frac{\sigma_1}{\sigma_n - \|\Delta A\|_2} \|\Delta A\|_F = \tag{2.18}$$

$$\frac{\|A\|_2\|A^{-1}\|_2}{1 - \|A^{-1}\|_2\|\Delta A\|_2} \cdot \|\Delta A\|_F. \qquad \blacksquare$$

3. The LU-decomposition.

We have the following bounds for the LU-decomposition.

THEOREM 3.1 *Assume that the $n \times n$-matrix has the LU-decomposition*

$$A = LU \tag{3.1}$$

where L is unit lower triangular and U is upper triangular. Also assume that the perturbed matrix $A + \Delta A$ has the LU-decomposition

$$A + \Delta A = (L + \Delta L)(U + \Delta U) \tag{3.2}$$

where $L + \Delta L$ is unit lower triangular and $U + \Delta U$ is upper triangular. Finally assume *that* $\|L^{-1}\Delta A U^{-1}\|_2 < 1$. *Then the following inequalities are satisfied:*

$$\|\Delta L\|_F \le \frac{\|L\|_2 \|L^{-1}\Delta A\, U^{-1}\|_F}{1 - \|L^{-1}\Delta A\, U^{-1}\|_2} \tag{3.3a}$$

$$\|\Delta U\|_F \le \frac{\|U\|_2 \|L^{-1}\Delta A\, U^{-1}\|_F}{1 - \|L^{-1}\Delta A\, U^{-1}\|_2}. \tag{3.3b}$$

PROOF. Note that

$$L\Delta U + \Delta L(U + \Delta U) = \Delta A. \tag{3.4}$$

Left-multiply with L^{-1} and right-multiply with $(U + \Delta U)^{-1}$ to get

$$L^{-1}\Delta L + \Delta U(U + \Delta U)^{-1} = L^{-1}\Delta A(U + \Delta U)^{-1}. \tag{3.5}$$

Note that ΔL is strictly lower triangular and hence $L^{-1}\Delta L$ is strictly lower triangular. The matrix $\Delta U(U + \Delta U)^{-1}$ is upper triangular and hence

$$\|L^{-1}\Delta L\|_F \le \|L^{-1}\Delta A(U + \Delta U)^{-1}\|_F \tag{3.6a}$$

and

$$\|\Delta U(U + \Delta U)^{-1}\|_F \le \|L^{-1}\Delta A(U + \Delta U)^{-1}\|_F. \tag{3.6b}$$

Since $(U + \Delta U)^{-1} = U^{-1} - U^{-1}\Delta U(U + \Delta U)^{-1}$ we have

$$\begin{aligned} \|L^{-1}\Delta A(U + \Delta U)^{-1}\|_F &\le \|L^{-1}\Delta A U^{-1}\|_F + \|L^{-1}\Delta A U^{-1}\Delta U(U + \Delta U)^{-1}\|_F \le \\ &\|L^{-1}\Delta A\, U^{-1}\|_F + \|L^{-1}\Delta A\, U^{-1}\|_2 \|\Delta U(U + \Delta U)^{-1}\|_F \le \\ &\|L^{-1}\Delta A\, U^{-1}\|_F + \|L^{-1}\Delta A\, U^{-1}\|_2 \|L^{-1}\Delta A(U + \Delta U)^{-1}\|_F \end{aligned} \tag{3.7}$$

or

$$\|L^{-1}\Delta A(U + \Delta U)^{-1}\|_F \le \frac{\|L^{-1}\Delta A\, U^{-1}\|_F}{1 - \|L^{-1}\Delta A\, U^{-1}\|_2}. \tag{3.8}$$

By combining (3.6), (3.8) and $\|\Delta L\|_F \le \|L\|_2 \|L^{-1}\Delta L\|_F$ we get (3.3a).

The proof of the bound (3.3b) is similar and therefore we take it short. We use

$$(L + \Delta L)^{-1}\Delta L + \Delta U\, U^{-1} = (L + \Delta L)^{-1}\Delta A U^{-1}. \tag{3.9}$$

Similarly to (3.6-8) we get

$$\|U^{-1}\Delta U\|_F \le \|(L + \Delta L)^{-1}\Delta A U^{-1}\|_F \le \frac{\|L^{-1}\Delta A\, U^{-1}\|_F}{1 - \|L^{-1}\Delta A\, U^{-1}\|_2}. \tag{3.10}$$

By combining (3.10) with $\|\Delta U\|_F \le \|U\|_2 \|U^{-1}\Delta U\|_F$ we get (3.3b). ■

One disadvantage with the bounds (3.3) is that they include the factors L and U. My guess is however that it is not possible to get any simple bound on ΔL and ΔU that only includes norms of A and ΔA. A simple row-permutation can completely change the perturbation sensitivity. Consider first the LU-decomposition of

$$\begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}. \tag{3.11}$$

Next consider the LU-decomposition of

$$\begin{bmatrix} 0.1 & 1 \\ 1 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 1 \\ 0 & -9.9 \end{bmatrix}. \tag{3.12}$$

The example (3.12) is more sensitive to perturbations than (3.11), and there is no difference in the Euclidean and Frobenius norms of the original matrices, $\begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0.1 & 1 \\ 1 & 0.1 \end{bmatrix}$.

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