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Kite-Free P- and Q-Polynomial Schemes

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Abstract. Let $Y = (X, \{R_i\}_{0 \le i \le d})$ denote a P-polynomial association scheme. By a kite of length i $(2 \le i \le d)$ in Y, we mean a 4-tuple $xyzu$ $(x, y, z, u \in X)$ such that $(x, y) \in R_1$, $(x, z) \in R_1$, $(y, z) \in R_1$, $(u, y) \in R_{i-1}$, $(u, z) \in R_{i-1}$, $(u, x) \in R_i$.
Our main result in this paper is the following.

Theorem. *Let Y be a P- and Q-polynomial association scheme. Suppose Y has diameter $d \ge 3$, and suppose Y has intersection number $a_1 \ne 0$.*
Then the following (i)–(iii) *are equivalent.*

(i) *Y has classical parameters (d, b, α, β), and either $b < -1$, or Y is a dual polar scheme or a Hamming scheme.*
(ii) *Y has no kites of length* 2 *and no kites of length* 3.
(iii) *Y has no kites of any length i $(2 \le i \le d)$.*

1. Introduction

It is shown by P. Terwilliger [Kite-Free Distance-Regular Graphs, preprint] that a P- and Q-polynomial scheme with classical parameters (d, b, α, β), such that $d \ge 3$ and $b < -1$, has no kites of any length i $(2 \le i \le d)$. In this paper we show that if Y is not a dual polar scheme or a Hamming scheme, then the converse is also true. Theorem 2.6 is our main result.

For the rest of this section, we recall some definitions and basic concepts concerning the theory of P- and Q-polynomial schemes. See Bannai and Ito[1], or Terwilliger[3] for more background information.

Let d denote a non-negative integer. A *symmetric, d-class assocation scheme* (or simply a *scheme*) is a configuration $Y = (X, \{R_i\}_{0 \le i \le d})$, where X is a non-empty set and R_0, R_1, …, R_d are non-empty subsets of the Cartesian product $X \times X$, possessing the following properties.

(i) $(x, y) \in R_0$ if and only if $x = y$ $\quad (x, y \in X)$.
(ii) $(x, y) \in R_i$ for exactly one i $\quad (0 \le i \le d)$, $(x, y \in X)$.
(iii) $R_i^t = R_i \quad (0 \le i \le d)$, where $R_i^t = \{(y, x) | (x, y) \in R_i\} \quad (0 \le i \le d)$.

(iv) For all integers i, j, k $(0 \le i,j,k \le d)$, and all x, $y \in X$ with $(x,y) \in R_k$, the number p_{ij}^k of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant that depends only on i, j, k.

(v) $p_{ij}^k = p_{ji}^k \quad (0 \le i,j,k \le d)$.

The elements of X, the constants p_{ij}^k, and the constant d are known as the *vertices*, the *intersection numbers*, and *the diameter*, of Y.

Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over the real number field with the rows and columns indexed by the elements of X. The *associate matrices* of Y are the matrices $A_0, A_1, \ldots, A_d \in \mathrm{Mat}_X(\mathbb{R})$, defined by the rule

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i \end{cases} \qquad 0 \le i \le d,$$

where x, $y \in X$.

Then by (i)–(v) we have

$$A_0 = I,$$
$$A_0 + A_1 + \cdots + A_d = J \quad (J = \textit{all 1's matrix}),$$
$$A_i^t = A_i \quad (0 \le i \le d),$$
$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \quad (0 \le i,j \le d),$$

and

$$A_i A_j = A_j A_i. \quad (0 \le i,j \le d).$$

The algebra M spanned by the associate matrices over the real number field $\mathbb{R}$ is a commutative semi-simple subalgebra of $\mathrm{Mat}_X(\mathbb{R})$, and is known as the *Bose-Mesner algebra* of Y. By [1, p59, p64], M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |X|^{-1} J,$$
$$E_i E_j = \delta_{ij} E_i \quad (0 \le i,j \le d),$$
$$E_0 + E_1 + \cdots + E_d = I,$$
$$E_i^t = E_i \quad (0 \le i \le d).$$

We refer to E_i as the ith *primitive idempotent* of Y $(0 \le i \le d)$.

Let $\circ$ denote entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \le i,j \le d),$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{R}$ $(0 \le i,j,k \le d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{ij}^k E_k \quad (0 \le i,j \le d).$$

A scheme Y is said to be *P-polynomial* with respect to the ordering $A_0, A_1, \ldots, A_d$ of the associate matrices if for all integer i, j, k $(0 \le i,j,k \le d)$, $p_{ij}^k = 0$ (resp. $p_{ij}^k \ne 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two.

Let $Y = (X, \{R_i\}_{0 \le i \le d})$ be a P-polynomial scheme. For convenience, set

$$b_i = p^i_{1,i+1}\ (0 \le i \le d-1), \quad a_i = p^i_{1,i}\ (0 \le i \le d), \quad c_i = p^i_{1,i-1}\ (1 \le i \le d).$$

The P-polynomial property implies

$$b_i > 0\ (0 \le i \le d-1), \quad c_i > 0\ (1 \le i \le d),$$
$$b_0 = b_i + a_i + c_i\ (1 \le i \le d-1). \tag{1.1}$$

By a *kite of length i* in a P-polynomial scheme $Y = (X, \{R_i\}_{0 \le i \le d})$, we mean a 4-tuple $xyzu$ $(x, y, z, u \in X)$ such that

$$(x,y), (x,z), (y,z) \in R_1, \quad (u,x) \in R_i,$$
$$(u,y) \in R_{i-1}, \quad (u,z) \in R_{i-1}.$$

A scheme Y is said to be *Q-polynomial* with respect to the given ordering E_0, $E_1, \ldots, E_d$ of the primitive idempotents, if for all integers i, j, k $(0 \le i, j, k \le d)$, $q^k_{ij} = 0$ (resp. $q^k_{ij} \ne 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two.

Suppose Y is Q-polynomial with respect to $E_0, E_1, \ldots, E_d$. Then the *dual eigenvalues* $\theta^*_i \in \mathbb{R}$ $(0 \le i \le d)$ are defined by

$$E_1 = |X|^{-1} \sum_{i=0}^{d} \theta^*_i A_i.$$

By [3, p384], the dual eigenvalues θ^*_i $(0 \le i \le d)$ are mutually distinct real numbers.

One class of P- and Q-polynomial schemes are the Hamming schemes(see [1, III.2]), defined in the following way. Take S a finite set of cardinality $q \ge 2$. Let X denote the set of all d-tuples of elements taken from S. The ith relation R_i on X is defined as follows:

$$(x,y) \in R_i \Leftrightarrow x, y \text{ differ in precisely } i \text{ coordinates.}$$

Another class of P- and Q-polynomial schemes are the schemes of dual polar spaces(see [1, III.6]), defined in the following way. Let W be a vector space over a finite field equipped with a nondegenerate form F(quadratic, symplectic, or Hermitian). Let X denote the set of all maximal isotropic subspaces of F in W, and let d denote the common dimension of these subspaces. The ith relation R_i on X is defined as follows:

$$(x,y) \in R_i \Leftrightarrow \dim(x \cap y) = d - i.$$

We refer the reader to Bannai and Ito[1, III.6] for more examples of P- and Q-polynomial schemes.

2. The Main Theorem

We divide the main Theorem 2.6 into a few Lemmas. Our work is based on the following theorem of Terwilliger[4, Theorem 3.3(viii)], [5, Theorem 2.11].

Theorem 2.1. *Let $Y = (X, \{R_i\}_{0 \le i \le d})$ denote a P- and Q-polynomial scheme with diameter $d \ge 3$ and dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Then we have the following* (i)–(ii).

(i)

$$\theta_{\eta-2}^* - \theta_{\eta-1}^* = \sigma(\theta_{\eta-3}^* - \theta_\eta^*), \quad (3 \le \eta \le d) \tag{2.1}$$

for appropriate $\sigma \in \mathbb{R}\backslash\{0\}$.

(ii) *Suppose the intersection number $a_1 \ne 0$, and pick any 3-tuple xyz such that $(x, y), (y, z), (x, z) \in R_1$. Set*

$$e_i(xyz) := (p_{ii-1}^1)^{-1} |\{u | u \in X, xyzu \text{ is a kite of length } i\}| \quad (2 \le i \le d).$$

Then

$$e_i(xyz) = \alpha_i e_2(xyz) + \beta \quad (2 \le i \le d), \tag{2.2}$$

where

$$\alpha_i = \frac{(\theta_1^* - \theta_2^*)(\theta_0^* + \theta_1^* - \theta_{i-1}^* - \theta_i^*)}{(\theta_0^* - \theta_2^*)(\theta_{i-1}^* - \theta_i^*)}, \tag{2.3}$$

$$\beta_i = \frac{(\theta_0^* - \theta_1^*)(\theta_2^* - \theta_i^*) - (\theta_1^* - \theta_2^*)(\theta_1^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_2^*)(\theta_{i-1}^* - \theta_i^*)} \tag{2.4}$$

Lemma 2.2. *With the notation of Theorem 2.1(ii), suppose $e_2(xyz) = 0$ and $e_3(xyz) = 0$. Then there exists $b \in \mathbb{R}\backslash\{0, -1\}$ such that*

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad (0 \le i \le d), \tag{2.5}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{2.6}$$

Proof. Set

$$b = \frac{\theta_1^* - \theta_0^*}{\theta_2^* - \theta_1^*}.$$

Then we have

$$\theta_2^* - \theta_0^* = (\theta_1^* - \theta_0^*)\begin{bmatrix} 2 \\ 1 \end{bmatrix} b^{-1}. \tag{2.7}$$

The above b exists since $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are distinct. Observe that $b \ne 0$ and $b \ne -1$.

Setting $i = 3$ in (2.2) we have $\beta_3 = 0$, so setting $i = 3$ in (2.4) we find

$$(\theta_0^* - \theta_1^*)(\theta_2^* - \theta_3^*) = (\theta_1^* - \theta_2^*)^2. \tag{2.8}$$

Evaluating (2.8) using (2.1) with $\eta = 3$, we get

$$(\theta_0^* - \theta_1^*)(\sigma^{-1}(\theta_1^* - \theta_2^*) - (\theta_0^* - \theta_2^*)) = (\theta_1^* - \theta_2^*)^2,$$

or equivalently we have

$$(\theta_2^* - \theta_0^*)^2 - (1 + \sigma^{-1})(\theta_1^* - \theta_0^*)(\theta_2^* - \theta_0^*) + (1 + \sigma^{-1})(\theta_1^* - \theta_0^*)^2 = 0. \quad (2.9)$$

Combining (2.7), (2.9) we have

$$1 + b + b^2 = \sigma^{-1} b,$$

so $1 + b + b^2 \neq 0$ and

$$\sigma = \frac{b}{b^2 + b + 1}. \quad (2.10)$$

Now we prove (2.5) by induction on i. The cases $i = 0$, 1 are trivial and the case $i = 2$ is from (2.7). Now suppose $i \geq 3$. Then (2.1) implies

$$\theta_i^* = \sigma^{-1}(\theta_{i-1}^* - \theta_{i-2}^*) + \theta_{i-3}^*. \quad (2.11)$$

Evaluate (2.11) using (2.10) and the induction hypothesis, we find $\theta_i^* - \theta_0^*$ is as in (2.5).

Definition 2.3. A P-polynomial scheme Y is said to have *classical parameters* (d, b, α, β) whenever the diameter of Y is d, and the intersection numbers of Y satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq d), \quad (2.12)$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq d), \quad (2.13)$$

where [ ] as in (2.6).

Lemma 2.4. *Let Y denote a P- and Q-polynomial scheme with diameter $d \geq 3$ and dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Suppose that the intersection number $a_1 \neq 0$, and further suppose Y has no kites of length 2 or 3. Then Y has classical parameters (d, b, α, β), for some $b \in \mathbb{R}\backslash\{0, -1\}$, and some $\alpha, \beta \in \mathbb{R}$.*

Proof. In view of Terwilliger[4, Theorem 4.2(iii)], it suffices to prove that there exists $b \in \mathbb{R}\backslash\{0, -1\}$ such that

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad (0 \leq i \leq d),$$

where [ ] as in (2.6).
But this is immediate from Lemma 2.2.
The following lemma comes from a simple observation.

Lemma 2.5. *Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ denote a P-polynomial scheme, where $d \geq 2$. Suppose Y has no kites of length 2. Then $a_2 - a_1 c_2 \geq 0$.*

Proof. Fix $x, y \in Y$ with $(x, y) \in R_2$. For $u, z \in X$ with (x, z), (x, u), (u, z), $(z, y) \in R_1$, we have $(u, y) \in R_2$, otherwise $xzuy$ is a kite of length 2. For any $z' \in X$ with (x, z'),

$(u, z'), (z', y) \in R_1$, we have $z = z'$ by a similar argument. Now

$$\begin{aligned} a_1 c_2 &= |\{u \in X: (x,u), (z,u), (x,z), (z,y) \in R_1 \text{ for some } z \in X\}| \\ &\leq |\{u \in X: (x,u) \in R_1, (u,y) \in R_2\}| \\ &= a_2. \end{aligned}$$

Theorem 2.6. *Let Y be a P- and Q-polynomial association scheme. Suppose Y has diameter $d \geq 3$, and suppose Y has intersection number $a_1 \neq 0$. Then the following* (i)–(iii) *are equivalent.*

(i) *Y has classical parameters (d, b, α, β), and either $b < -1$, or Y is a dual polar scheme or a Hamming scheme.*
(ii) *Y has no kites of length* 2 *and no kites of length* 3.
(iii) *Y has no kites of any length i $(2 \leq i \leq d)$.*

Proof. (ii) $\rightarrow$ (i). Suppose (ii) is true. Then by Lemma 2.4, Y has classical parameters (d, b, α, β). First suppose $\alpha = 0$. Then by [2, Theorem 9.4.4], Y is a dual polar scheme or a Hamming scheme. Now suppose $\alpha \neq 0$. From (1.1), (2.12), (2.13), and Lemma 2.5, we have

$$\begin{aligned} (-\alpha)(1 + b)(b + a_1 + 1) &= a_2 - a_1 c_2 \\ &\geq 0. \end{aligned} \tag{2.14}$$

By direct calculation from (2.12), and by (1.1) we get

$$\begin{aligned} (c_2 - b)(b^2 + b + 1) &= c_3 \\ &> 0. \end{aligned} \tag{2.15}$$

Since b is an integer [2, p195], we have

$$b^2 + b + 1 > 0.$$

Then from (2.15), we get

$$c_2 > b. \tag{2.16}$$

Using (2.12), (2.16) we get

$$\begin{aligned} \alpha(1 + b) &= c_2 - b - 1 \\ &\geq 0. \end{aligned}$$

But $\alpha \neq 0$, $b \neq -1$, so

$$\alpha(1 + b) > 0.$$

Applying this to (2.14), we find

$$b + a_1 + 1 \leq 0.$$

Therefore we have $b \leq -(a_1 + 1) < -1$, since $a_1 \neq 0$.

(i) → (iii). For $b < -1$, Y has no kites of any length i ($2 \leq i \leq d$) by [5, Theorem 2.12]. It is well known that the Hamming schemes and the dual polar schemes have no kites. See [2, Theorem 9.2.1, Theorem 9.4.3] for details.

(iii) → (ii). Clear.

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