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What We Know and What We Do not Know about Turán Numbers

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Abstract. The numbers which are traditionally named in honor of Paul Turán were introduced by him as a generalization of a problem he solved in 1941. The general problem of Turán having an *extremely* simple formulation but being *extremely* hard to solve, has become one of the most fascinating *extremal problems* in combinatorics. We describe the present situation and list conjectures which are not so hopeless.

1. The Definition and Equivalent Formulations

A system of r-element subsets (*blocks*) of an n-element set X_n is called a *Turán (n, k, r)-system* if every k-element subset of X_n contains at least one of the blocks. The *Turán number* $T(n, k, r)$ is the minimum size of such a system. The problem of determining $T(n, k, r)$ was posed by Paul Turán [57].

The Turán numbers sometimes appear in different notation. For instance, the *covering number* $C(n, m, p)$ is defined as the minimum number of m-element subsets of X_n needed to cover all p-element subsets ($n \geq m \geq p$). Obviously,

$$C(n, m, p) = T(n, n - p, n - m).$$

Let $U(n, q, r)$ be the minimum number of subsets of X_n whose sizes are at least r and for which the size of any transversal (i.e. a subset intersecting each of them) is at least q. It is easy to see that

$$U(n, q, r) = T(n, n - q + 1, r). \tag{1}$$

It was shown in [45] that $T(n, k, r)$ is equal to the minimal length of a disjunctive normal form with n Boolean variables, which takes value 1 if at least k variables are equal to 1, and takes value 0 if less than r variables are equal to 1.

2. Recursive Inequalities

Schönheim [39], and independently, Katona, Nemetz and Simonovits [19] showed that

$$T(n, k, r) \geq \left\lceil \frac{n}{n - r} T(n - 1, k, r) \right\rceil. \tag{2}$$

Indeed, if we omit one element from a Turán (n,k,r)-system together with all bocks containing this element, we get a Turán $(n-1,k,r)$-system. Omitting an element in n possible ways, we get n such subsystems; each of them has at least $T(n-1,k,r)$ blocks. Every block of the (n,k,r)-system appears in $n-r$ of the considered subsystems. Hence $(n-r)T(n,k,r) \geq nT(n-1,k,r)$. Inequality (2) implies that the ratio $T(n,k,r)\Big/\binom{n}{r}$ is non-decreasing. Thus the limit

$$t(k,r) = \lim_{n\to\infty} \frac{T(n,k,r)}{\binom{n}{r}}$$

exists and

$$T(n,k,r) \leq \binom{n}{r} t(k,r). \tag{3}$$

The values $t(k,r)$ were obtained only for $r = 2$ (except the trivial case $k = r$). Erdös [9] offered a reward for determining $t(k,r)$ for a single pair (k,r) with $k > r > 2$.

A stronger version of Ineq. (2) was given in [43]: if the ratio $T(m,k,r)\Big/\binom{m}{r}$ is constant for $m = n-l, \ldots, n-1$ (which means that the extremal $(n-1,k,r)$-system is an exact l-scheme) and $T(n,k,r)\Big/\binom{n}{r} \neq T(n-1,k,r)\Big/\binom{n-1}{r}$, then

$$T(n,k,r) \geq \left\lceil \frac{nT(n-1,k,r)+l}{n-r} \right\rceil. \tag{4}$$

Given a Turán $(n-1,k,r)$-system $\mathscr{A}$ and a Turán $(n-1,k-1,r)$-system $\mathscr{B}$, a Turán (n,k,r)-system can be produced in a simple way ([41]). We assume that $\mathscr{A}$ and $\mathscr{B}$ share the same element set. By adding a new element v to every block of $\mathscr{B}$ we get a system $\mathscr{B}+v$ whose blocks are r-element subsets. The union of $\mathscr{A}$ and $\mathscr{B}+v$ is a Turán (n,k,r)-system. This gives the inequality

$$T(n,k,r) \leq T(n-1,k,r) + T(n-1,k-1,r-1). \tag{5}$$

Applying (2), we have

$$\begin{aligned}\frac{r}{n}T(n,k,r) &= T(n,k,r) - \frac{n-r}{n}T(n,k,r) \leq T(n,k,r) - T(n-1,k,r)\\ &\leq T(n-1,k-1,r).\end{aligned}$$

Thus

$$\frac{T(n,k,r)}{\binom{n}{r}} \leq \frac{T(n-1,k-1,r-1)}{\binom{n-1}{r-1}}$$

which results in

$$t(k,r) \leq t(k-1, r-1). \tag{6}$$

Obviously, the union of disjoint Turán (n', k', r)- and (n'', k'', r)-systems is a Turán $(n' + n'', k' + k'' - 1, r)$-system. Moreover, the union of l disjoint Turán systems with parameters $(n_i, \alpha_i + 1, r)$ where $i = 1, 2, \ldots, l$, is a Turán system with parameters $(n_1 + n_2 + \cdots + n_l, \alpha_1 + \alpha_2 + \cdots + \alpha_l + 1, r)$. This implies the inequality

$$T\left(\sum_{i=1}^{l} n_i, \sum_{i=1}^{l} \alpha_i + 1, r\right) \leq \sum_{i=1}^{l} T(n_i, \alpha_i + 1, r). \tag{7}$$

3. Lower Bounds

The best known general lower bound is

$$T(n,k,r) \geq \frac{n-k+1}{n-r+1} \binom{n}{r} \Big/ \binom{k-1}{r-1}. \tag{8}$$

A particular case of (8) was proven for $k = 5$, $r = 4$ by Giraud [15], and then generalized for $k = r + 1$ by Sidorenko [42]. In the most general case, Ineq. (8) was proven by de Caen [3].

It is sufficient to use (7) and (8) to get the correct magnitude of $T(k^\gamma, k, r)$ with a fixed γ and $k \to \infty$. Namely, inequality (7) with $l = \left\lfloor \frac{k-1}{r-1} \right\rfloor$, $\alpha_1 = \alpha_2 = \cdots = \alpha_l = r - 1$ yields

$$T(k^\gamma, k, r) \leq l \cdot \frac{1}{r!} \left(\frac{k^\gamma}{l}\right)^r \leq c' k^{(\gamma-1)r+1}.$$

On the other hand, Ineq. (8) gives

$$T(k^\gamma, k, r) \geq (1 + o(1)) \frac{k}{r} (k^{\gamma-1})^r \geq c'' k^{(\gamma-1)r+1}.$$

It also follows from (8) that

$$t(k,r) \geq \frac{1}{\binom{k-1}{r-1}}, \tag{9}$$

$$t(r+1, r) \geq \frac{1}{r}. \tag{10}$$

For odd r, the last inequality was slightly improved by Giraud [17]:

$$t(r+1, r) \geq \frac{2}{r(1 + \sqrt{r/(r+4)})}, \qquad r \equiv 1 \bmod 2. \tag{11}$$

The corresponding upper bound (see the next section) is

$$t(r+1,r) \le \frac{(\frac{1}{2}+o(1))\ln r}{r}.$$

Conjecture 1.

$$\lim_{r\to\infty} r\cdot t(r+1,r) = \infty$$

This was posed by de Caen [5] and he offered a reward for the first proof or disproof.

4. Upper Bounds

According to (3), any upper bound for $t(k,r)$ yields an upper bound for $T(n,k,r)$. Thus we are concentrating on the bounds for $t(k,r)$. We start with $k = r+1$ and then proceed to the general case.

4.1. The Case $k = r+1$

It was found first that $t(r+1,r) = \mathcal{O}(r^{-1/2})$ (see [41]). This fact follows from the inequality

$$t(2s+1,2s) \le \binom{2s}{s}2^{-2s} \tag{12}$$

and Ineq. (6). A major improvement came when Kim and Roush [20] proved

$$t(r+1,r) \le \frac{1+2\ln r}{r}.$$

Their result was consecutively improved by Frankl an Rödl [12]:

$$t(r+1,r) \le \frac{\ln r + \mathcal{O}(1)}{r}, \tag{13}$$

and by Sidorenko ([46]):

$$t(r+1,r) \le \frac{\ln r}{2r}(1+o(1)). \tag{14}$$

For small r, a stronger result was obtained in [7]:

$$t(2s+1,2s) \le \frac{1}{4} + 2^{-2s}. \tag{15}$$

This gives the best known bound for $t(5,4)$ and $t(7,6)$. Starting $s = 4$, inequality (15) can be improved.

Bounds (12), (13), and (15) are based on Constructions 1–3.

Construction 1 [41]. Most of the known constructions of Turán systems have a relatively small number of classes of equivalent elements. In a typical system, the set of elements is partitioned into a fixed number of groups, and whether r elements form a block depends only on the groups they belong to. In contrast, our construction is based on the ordering of the elements. We denote them $1, 2, \ldots, n$. Elements $i_1 < i_2 < \cdots < i_{2s}$ form a block of the system if the Boolean vector

$$((i_1 + 1) \bmod 2, (i_2 + 2) \bmod 2, \ldots, (i_{2s} + 2s) \bmod 2)$$

contains exactly s zeros and s ones. It is easy to prove by induction on s that the resulting system is a Turán $(n, 2s+1, 2s)$-system. Its size is $\left(\binom{2s}{s}2^{-2s}+o(1)\right)\binom{n}{2s}$ which gives (12).

Construction 2 [7]. We identify the n elements with the lines (rows and columns) of an $\left(\left\lfloor \frac{n}{2} \right\rfloor \times \left\lceil \frac{n}{2} \right\rceil\right)$-matrix M whose entries are zeros and ones. We say that a submatrix of M is *even* if the number of its rows, the number of its columns and the sum of its entries are even numbers. Now let $2s$ lines of M form a block of the system if either all of them are rows, or all of them are columns, or they induce an even submatrix. Obviously, the resulting system is a Turán $(n, 2s + 1, 2s)$-system. Its size depends on the matrix M. Let the entries of M be independent random variables taking values 0 and 1 with probability $\frac{1}{2}$ each. Then any $2i$ rows and any $2(s - i)$ columns (with $1 \le i \le s - 1$) induce an even submatrix with probability $\frac{1}{2}$. Thus the expected number of blocks is

$$\begin{aligned}
&\binom{\lfloor n/2 \rfloor}{2s} + \binom{\lceil n/2 \rceil}{2s} + \frac{1}{2}\sum_{i=1}^{s-1}\binom{\lfloor n/2 \rfloor}{2i}\binom{\lceil n/2 \rceil}{2(s-i)} \\
&= \frac{1}{2}\left(\binom{\lfloor n/2 \rfloor}{2s} + \binom{\lceil n/2 \rceil}{2s}\right) + \frac{1}{2}\sum_{i=0}^{s}\binom{\lfloor n/2 \rfloor}{2i}\binom{\lceil n/2 \rceil}{2(s-i)} \\
&= \left(\frac{1}{2^{2s}} + \frac{1}{2}\cdot\frac{1}{2} + \mathcal{O}\left(\frac{1}{n}\right)\right)\binom{n}{2s}
\end{aligned}$$

which yields (15).

Construction 3 [12]. We assume that $n \equiv 0 \bmod l$ and divide the n elements into l equal groups $A_0, A_1, \ldots, A_{l-1}$. For a subset $B \subseteq A_0 \cup A_1 \cup \cdots \cup A_{l-1}$, we denote by $d(B)$ the number of indices $i \in \{0, 1, \ldots, l - 1\}$ satisfying $A_i \cap B = \varnothing$, and set

$$w(B) = \sum_{i=0}^{l-1} i|A_i \cap B|.$$

Let $\mathscr{A}^k$ denote the family of all k-element subsets of $A_0 \cup A_1 \cup \cdots \cup A_{l-1}$. Let a subfamily $\mathscr{B}_j$ consist of those $B \in \mathscr{A}^r$ which satisfy

$$w(B) + j \equiv 0, 1, \ldots \text{ or } d(B) \pmod{l}. \tag{16}$$

We claim that $\mathscr{B}_j$ is a Turán $(n, r+1, r)$-system. Indeed, for any $C \in \mathscr{A}^{r+1}$, there are $n - d(C)$ indices i such that $B \cap A_i \neq \varnothing$ and thus at least one such index can be found among

$$(w(C)+j) \bmod l, \quad (w(C)+j-1) \bmod l, \quad \ldots, \quad (w(C)+j-d(C)) \bmod l.$$

For any $x \in (B \cap A_i)$, the subset $B = C \backslash x$ satisfies (16), since $w(B) = w(C) - i \pmod{l}$ and $d(B) \geq d(C)$.

We denote $\mathscr{A}'_i = \{B \in \mathscr{A}^r : B \cap A_i = \varnothing\}$. Now, to estimate $\min\{|\mathscr{B}_0|, |\mathscr{B}_1|, \ldots, |\mathscr{B}_{l-1}|\}$, we use the fact that any $B \in \mathscr{A}^r$ belongs to exactly $d(B)+1$ families among $\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_{l-1}$. Thus

$$\sum_{j=0}^{l-1} |\mathscr{B}_j| = \sum_{B \in \mathscr{A}^r} (d(B)+1) = |\mathscr{A}^r| + |\mathscr{A}'_0| + |\mathscr{A}'_1| + \cdots + |\mathscr{A}'_{l-1}|$$

$$= \binom{n}{r} + l \cdot \binom{n(l-1)/l}{r} \leq \left(1 + l \cdot \left(1 - \frac{1}{l}\right)^r\right)\binom{n}{r} \tag{17}$$

and

$$\min\{|\mathscr{B}_0|, |\mathscr{B}_1|, \ldots, |\mathscr{B}_{l-1}|\} \leq \frac{1}{l} \sum_{j=0}^{l-1} |\mathscr{B}_j| \leq \left(\frac{1}{l} + \left(1 - \frac{1}{l}\right)^r\right)\binom{n}{r}.$$

It gives

$$t(r+1, r) \leq \frac{1}{l} + \left(1 - \frac{1}{l}\right)^r.$$

The substitution $l = \frac{r}{\ln r}(1 + o(1))$ produces (13).

To obtain (14), we combine the main ideas of Constructions 1–3. Consider a system of r-element blocks. Its automorphism is a permutation of the elements which preserves the set of blocks. The automorphism group generates an equivalence relation on the elements as well as on the blocks. We denote the classes of equivalent elements by $A_1, A_2, \ldots, A_l$. Then any equivalence class $\mathscr{B}$ of blocks corresponds to an integer partition $r = b_1 + b_2 + \cdots + b_l$ such that

$$\mathscr{B} = \mathscr{B}(b_1, b_2, \ldots, b_l) = \{B : |B \cap A_1| = b_1, |B \cap A_2| = b_2, \ldots, |B \cap A_l| = b_l\}.$$

Suppose that the considered system is a Turán $(n, r+1, r)$-system. We claim that one may omit at least half of the blocks from every present class $\mathscr{B}(b_1, b_2, \ldots, b_l)$ whenever $b_1 \geq 2, b_2 \geq 2, \ldots, b_l \geq 2$. As a result, we will have a Turán $(n, r+1, r)$-system of a smaller size. Let us first suppose that $r = 2l$ and $b_1 = b_2 = \cdots = b_l = 2$. We pick up an l-dimensional matrix M whose ith dimension corresponds to the equivalence class A_i. A block $B \in \mathscr{B}(2, 2, \ldots, 2)$ corresponds to a $(2 \times 2 \times \cdots \times 2)$-submatrix of M. We say that B is *even* or *odd* if the sum of the entries of the corresponding submatrix is such. Similarly to Construction 2, we may omit all odd blocks from $\mathscr{B}(2, 2, \ldots, 2)$. Now let $r \geq 2l$, and $b_1, b_2, \ldots, b_l$ satisfiy $b_1 + b_2 + \cdots + b_l = r$ and $b_i \geq 2$ with $i = 1, 2, \ldots l$. Similarly to Construction 1, we linearly order the elements within each class A_i. For a block $B \in \mathscr{B}(b_1, b_2, \ldots, b_l)$, we name its *2-projection* the $(2l)$-element subset which includes the two maximal elements from each intersection $B \cap A_i$. We omit all blocks $B \in \mathscr{B}(b_1, b_2, \ldots, b_l)$ which have odd 2-projections (with respect to the l-dimensional matrix M). If the entries of M are independent random variables equal to zero or one with probability $\frac{1}{2}$, the expected number of omitted blocks is $\frac{1}{2}|\mathscr{B}(b_1, b_2, \ldots, b_l)|$. Applying this to Construction 3 and choosing $l = r/(\ln r + \ln\ln r)$, we get (14) (see [46] for details).

If we are able to strengthen one of the intermediate assertions, we would improve inequality (14). Namely, having n elements divided into l equal groups, we need to find a family of (ml)-element subsets, when (a) every subset contains m elements from each group, and (b) any $(ml + 1)$-element subset, that contains at least m elements from each group, has to contain at least one of the chosen (ml)-element subsets. If we find such a family of size

$$(C_m + o(1))\binom{n/l}{m}^l \qquad \text{as } n, l \to \infty,$$

we will get by similar arguments the upper bound

$$t(r + 1, r) \leq (C_m + o(1))\frac{\ln r}{r}.$$

To obtain (14), we used $m = 2$ and $C_2 = \frac{1}{2}$.

4.2. The General Case

The simplest general upper bound

$$t(k, r) \leq \left(\frac{r-1}{k-1}\right)^{r-1} \tag{18}$$

was obtained in [41]. The construction is based on the elementary fact:

Lemma. *Let $b_0, b_1, \ldots, b_{l-1}$ be (cyclically ordered) reals, and $b = (b_0 + b_1 + \cdots + b_{l-1})/l$. Then there exists an index m such that*

$$\forall s = 1, 2, \ldots, l: \qquad b_m + b_{m+1} + \cdots + b_{m+s-1} \geq sb$$

(we denote $b_l = b_0$, $b_{l+1} = b_1$ and so on).

Proof. Choose $m, q \in \{0, 1, \ldots, l-1\}$ to minimize the value $b_{m-q} + b_{m-q+1} + \cdots + b_{m-1} - qb$. Then

$$\begin{aligned} &b_{m-q} + b_{m-q+1} + \cdots + b_{m-1} + b_m + \cdots + b_{m+s-1} - (q + s)b \\ &\quad \geq b_{m-q} + b_{m-q+1} + \cdots + b_{m-1} - qb \end{aligned}$$

which implies the statement of the Lemma.

Construction 4 [41]. We select integers l and d such that $r = \lceil dk/l \rceil$ and divide the n elements into l approximately equal groups $A_0, A_1, \ldots, A_{l-1}$. An r-element block B is included into the system if the sizes of intersections $b_i = |B \cap A_i|$ $(i = 0, 1, \ldots, l-1)$ satisfy the condition: there exists an index m such that

$$\forall s = 1, 2, \ldots, d: \qquad \sum_{i=1}^{s} b_{m+i-1} \geq s\frac{k}{l}$$

(we again denote $b_l = b_0$, $b_{l+1} = b_l$ and so on). It follows from Lemma that this is a Turán (n,k,r)-system. Counting the blocks, we get

$$T(n,k,r) \leq \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)\left(\frac{n}{l}\right)^r l \sum \frac{1}{a_1!\,a_2!\ldots a_d!} \tag{19}$$

where the sum is taken over all sequences of non-negative integers $(a_1, a_2, \ldots, a_d)$ satisfying $a_1 + a_2 + \cdots + a_d = r$ and $a_1 + a_2 + \cdots a_s \geq s\frac{k}{l}$ with $s = 1, 2, \ldots, d$. If we choose $l = k - 1$ and $d = r - 1$, the value of the sum is equal to $\frac{1}{r!}(r-1)^{r-1}$ which gives (18). For some k and r, there is a better choice. In particular, (19) with $k = 7, r = 5, l = 5, d = 3$ yields $t(7,5) \leq \frac{121}{625}$.

Frankl an Rödl [12] used probabilistic arguments to prove the bound

$$t(r+a,r) \leq \frac{a(a+4+o(1))\ln r}{\binom{r}{a}} \qquad \text{as } r \to \infty, \qquad a = const. \tag{20}$$

Their arguments were extended by Sidorenko [46] to prove the inequalities

$$t(k,r) \leq \frac{(1+o(1))(k-r+1)\ln\binom{k}{r}}{\binom{k}{r}} \qquad \text{as } r \to \infty, \qquad k \geq r + r/\log_2 r, \tag{21}$$

and

$$t(k,r) \leq \frac{(c_\gamma + o(1))r^2}{\binom{k}{r}} \qquad \text{as } r \to \infty, \qquad k = (\gamma + o(1))r \tag{22}$$

with $c_\gamma = (\gamma - 1)[\gamma \ln \gamma - (\gamma - 1)\ln(\gamma - 1) + o(1)]$, $\gamma > 1$. It is interesting to compare (21) or (22) with the lower bound (9).

If $r = 2$ (see Section 6), the equality in (18) is attained. The conjectured values of the Turán numbers with $r = 3$ (see Section 7) also satisfy the equality in (18). Turán [58] conjectured that the equality is attained whenever $k - 1$ is a multiple of $r - 1$. A counterexample $k = 13$, $r = 4$ was found in [41]. In general, by using (7) with $l = 2s - 1$, $\alpha_1 = \alpha_2 = \cdots \alpha_l = 2s$ and applying (12), we get the counterexample $k = 4s^2 - 2s + 1$, $r = 2s$, $\frac{k-1}{r-1} = 2s$ for any $s \geq 2$. Inequality (22) shows that the conjecture fails for any ratio $\frac{k-1}{r-1}$ when r is sufficiently large.

5. The Case of Small $\frac{n}{k-1}$

In the case $qr \le n$, one may easily prove by induction that the system of q disjoint r-element subsets is the optimal Turán $(n, n-q+1, r)$-system. Thus

$$T(n, \alpha+1, r) = n - \alpha \qquad \text{with } 1 \le \frac{n}{\alpha} \le \frac{r}{r-1}. \tag{23}$$

The next zone was found in [43] (the proof was also published in [45]):

$$T(n, \alpha+1, r) = \begin{cases} \left\lceil \dfrac{3r-2}{r} n \right\rceil - 3\alpha & \text{if } r \equiv 0 \bmod 2, \dfrac{r}{r-1} \le \dfrac{n}{\alpha} \le \dfrac{3r}{3r-4}; \\ 3n - \left\lfloor \dfrac{3r-1}{r-1}\alpha \right\rfloor & \text{if } r \equiv 1 \bmod 2, \dfrac{r}{r-1} \le \dfrac{n}{\alpha} \le \dfrac{3r+1}{3r-3}. \end{cases} \tag{24}$$

The right hand side of (24) gives a lower bound for any larger n.

6. The Case $r = 2$

Mantel [26] in 1907 determined $T(n, 3, 2)$, and Turán [56] in 1941 determined $T(n, k, 2)$ for any k. We write the result of Turán in the following form:

$$T(n, \alpha+1, 2) = mn - \frac{m(m+1)}{2}\alpha \qquad \text{if } m \le \frac{n}{\alpha} \le m+1. \tag{25}$$

Turán also proved that the extremal system is unique. Namely, the n elements are divided into $\alpha = k-1$ nearly equal groups; a pair of elements $\{x, y\}$ is a block of the system if x and y belong to the same group.

7. The Case $r = 3$

Two old conjectures of Turán concern the numbers $T(n, 4, 3)$ and $T(n, 5, 3)$. In order to get a Turán $(n, 4, 3)$-system, we divide the n elements into 3 almost equal groups A_0, A_1, A_2, and take all triples $\{x, y, z\}$ with $x,\ y \in A_i$, $z \in (A_i \cup A_{i+1})$, $i = 0, 1, 2$, where $A_3 = A_0$. This gives

$$T(n, 4, 3) \le \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m; \\ m^2(2m-1) & \text{if } n = 3m+1; \\ m^2(2m+1) & \text{if } n = 3m+1. \end{cases} \tag{26}$$

One of the famous conjectures of Turán is

Conjecture 2. *The equality in* (26) *holds.*

Katona, Nemetz and Simonovits [19] established the equality for $n \le 9$, Stanton and Bate [47] for $n = 10$, Radziszowski and Zou [37] for $n = 11, 12, 13$.

In contrast to the case $r = 2$, this construction, if optimal, is not unique. Other variants of $(n, 4, 3)$-systems with the same number of blocks were found by Brown [2]. Later Kostochka [21] found 2^{m-2} non-isomorphic constructions of Turán $(3m, 4, 3)$-systems of size $m(m-1)(2m-1)$. A nice representation of those constructions in terms of directed graphs was given by Fon-Der-Flaass [10]. The Kostochka's systems are indexed by ordered partitions $m = l_1 + l_2 + \cdots + l_s$ where $l_1 \geq 1, \ldots, l_{s-1} \geq 1, l_s \geq 2$. There are 2^{m-2} such partitions. Given a partition, we divide $3m$ elements into $3s$ subsets A_{ij} such that $i \in \mathbf{Z}_3$, $j \in \{1, 2, \ldots s\}$, and $|A_{ij}| = l_j$. Now we take triples T which satisfy

1. $T \subseteq \bigcup_{j=1}^{s} A_{ij}$ with some $i \in \mathbf{Z}_3$, or
2. $T = \{x, y, z\}$ where $x \in A_{ij}$ and $y, z \in (\bigcup_{t=1}^{s+1-j} A_{i+1,t}) \cup (\bigcup_{t=s+2-j}^{s} A_{i-1,t})$.

They form a Turán $(3m, 4, 3)$-system of the required size.

For $m \leq 4$, all $(3m, 4, 3)$-systems of size $m(m-1)(2m-1)$ are Kostochka's systems (it was shown in [44] for $m = 3$ and in [37] for $m = 4$).

We say that a subset of elements of a system is a *clique* if any r-tuple within this subset is a block of the system. In each of the Kostochka's systems, the elements are divided into 3 cliques of size m.

Conjecture 3. *The elements of any Turán* $(3m, 4, 3)$*-system of size* $m(m-1)(2m-1)$ *are divided into* 3 *cliques of size m.*

By dividing n elements into two almost equal groups and taking all triples within each group, we get a Turán $(n, 5, 3)$-system. Hence,

$$T(n, 5, 3) \leq \binom{\lceil n/2 \rceil}{3} + \binom{\lfloor n/2 \rfloor}{3}. \tag{27}$$

Turán conjectured (see, for instance, [58]) that this construction is the best possible and the equality in (27) holds. However, it was disproved for all odd $n \geq 9$ by Sidorenko and Kostochka [44] (for $n = 9$ it was done earlier by Surányi [49]).

Construction 5. We divide the n elements into 9 groups $A_1, A_2, \ldots, A_9$ which correspond to the points $a_1, a_2, \ldots, a_9$ of the finite affine plane of order 3. The 12 lines of the plane are denoted as follows:

$$\begin{aligned}
L_1 &= \{a_1, a_6, a_8\}, & L_9 &= \{a_2, a_4, a_9\}, & M_1 &= M_9 = \{a_1, a_5, a_9\}, \\
L_2 &= \{a_1, a_2, a_3\}, & L_8 &= \{a_7, a_8, a_9\}, & M_2 &= M_8 = \{a_2, a_5, a_8\}, \\
L_3 &= \{a_3, a_4, a_8\}, & L_7 &= \{a_2, a_6, a_7\}, & M_3 &= M_7 = \{a_3, a_5, a_7\}, \\
L_4 &= \{a_1, a_4, a_7\}, & L_6 &= \{a_3, a_6, a_9\}, & M_4 &= M_6 = \{a_4, a_5, a_6\}.
\end{aligned}$$

We note that $a_i \in L_i$ and a_i, $a_5 \in M_i$ for $i \neq 5$. Now we take triples $x \in A_i$, $y \in A_j$, $z \in A_k$ that satisfy

1. $i = j = k$, or
2. i, j, k are pairwise distinct, and a_i, a_j, a_k are collinear, or
3. $i = j$, $k \neq i$ and $a_k \in L_i \cup M_i$.

If we also require $|A_5| = 1$, then the resulting triples form a Turán $(n, 5, 3)$-system. Making the sizes of the remaining 8 groups nearly equal,

$$|A_1| \geq |A_3| \geq |A_9| \geq |A_7| \geq |A_2| \geq |A_6| \geq |A_8| \geq |A_4| \geq |A_1| - 1,$$

we get

$$T(2m+1, 5, 3) \leq \binom{m+1}{3} + \binom{m}{3} - f(m) \tag{28}$$

where

$$f(m) = \begin{cases} \frac{m}{2} & \text{if } m \equiv 0 \bmod 4; \\ \frac{m-1}{4} & \text{if } m \equiv 1 \bmod 4; \\ \frac{m-4}{2} & \text{if } m \equiv 2 \bmod 4; \\ \frac{m-3}{4} & \text{if } m \equiv 3 \bmod 4. \end{cases}$$

for $m = 7$, a better bound $T(15, 5, 3) \leq 89$ is known [44, 1].

The values $T(n, 5, 3)$ and all the optimal systems for small values of n were determined in [49, 47, 1]:

$$T(6,5,3) = 2, \qquad T(7,5,3) = 5, \qquad T(8,5,3) = 8, \qquad T(9,5,3) = 12,$$

$$T(10,5,3) = 20, \qquad T(11,5,3) = 29, \qquad T(12,5,3) = 40, \qquad T(13,5,3) = 52,$$

The unique $(13, 5, 3)$-system of size 52 is formed by the collinear triples of points of the projective plane of order 3. For even values of n, no counterexample is known to the Turán's conjecture on the equality in (27).

In contrast to $(n, 4, 3)$-systems, one may find $(n, 5, 3)$-systems with $2\binom{n/2}{3}$ blocks where the size of the largest clique is only $\sqrt{n}$.

Construction 6. Let p be a permutation of order m. We construct a system whose elements are divided into $2m$ groups $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$. A triple $\{x, y, z\}$ is a block of the system if

1. $x \in A_i$, $y \in A_j$, $z \in A_k$ where i, j, k are distinct, or
2. $x \in B_i$, $y \in B_j$, $z \in B_k$ where i, j, k are distinct, or
3. $x, y \in A_i$, $z \in B_j$ where $j \neq i$, or
4. $x, y \in B_j$, $z \in A_i$ where $i \neq p(j)$, or
5. $x, y, z \in A_i$, or
6. $x, y, z \in B_j$.

It is easy to check that any 5-element subset contains at least one block of the system. We denote $a_i = |A_i|$, $b_i = |B_i|$, $n = a_1 + a_2 + \cdots + a_m$, and set the additional condition:

7. $a_i = b_i = a_{p(i)}$ for $i = 1, 2, \ldots, m$.

Now any element belongs to exactly $\binom{n-1}{2}$ blocks. Therefore, we have a Turán $(2n, 5, 3)$-system of size $2\binom{n}{3}$. If indices $i_1, i_2, \ldots, i_k$ form a cycle C of the permutation p, all numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_k}, b_{i_1}, b_{i_2}, \ldots, b_{i_k}$ must be equal to the same value $d = d(C)$. We denote the cycles of the permutation by $C_1, C_2, \ldots, C_t$. Let c_i be the length of C_i, and $d_i = d(C_i)$. Then

$$\begin{aligned} c_1 + c_2 + \cdots + c_t &= m, \\ c_1 \cdot d_1 + c_2 \cdot d_2 + \cdots + c_t \cdot d_t &= n. \end{aligned} \tag{29}$$

We call a set of parameters $(c_1, c_2, \ldots, c_t, d_1, d_2, \ldots, d_t)$ that satisfies (29) a *representation.* Every representation *produces* a $(2n, 5, 3)$-system of size $2\binom{n}{3}$. Sometimes, different representations produce isomorphic systems. We name a representation $(c_1, c_2, \ldots, c_t, d_1, d_2, \ldots, d_t)$ *canonical* if one satisfies the following conditions:

(i) $d_1 \geq d_2 \geq \cdots \geq d_t$;
(ii) $d_i = d_{i+1}$ implies $c_i \geq c_{i+1}$;
(iii) If $t = 1$ then $d_1 > 1$;
(iv) If $t > 1$ then $d_{t-1} > 1$;
(v) If $t = 2$ then $c_1 + c_2 > 2$;
(vi) If $t = 2$, $c_1 = 1$, $d_2 = 1$ then $d_1 \neq c_2$.

Different canonical representations produce non-isomorphic Turán systems.

Construction 7. Consider a pair of representations

$$c'_1 \cdot d'_1 + c'_2 \cdot d'_2 + \cdots + c'_{t'} \cdot d'_{t'} = n', \tag{30}$$

$$c''_1 \cdot d''_1 + c''_2 \cdot d''_2 + \cdots + c''_{t''} \cdot d''_{t''} = n''. \tag{31}$$

Let $A'_1, A'_2, \ldots, B'_1, B'_2, \ldots$ and $A''_1, A''_2, \ldots, B''_1, B''_2, \ldots$ denote the equivalence classes of elements in the systms produced by these representations. Take all blocks of both systems and add blocks $\{x, y, z\}$ which satisfy

8. $x, y \in A'_i$, $z \in B''_j$, or
9. $x, y \in A''_i$, $z \in A'_j$, or
10. $x, y \in B'_i$, $z \in A''_j$, or
11. $x, y \in B''_i$, $z \in B'_j$, or
12. $x \in A'_i$, $y \in A'_j$, $i \neq j$, $z \in A''_k$, or
13. $x \in A''_i$, $y \in A''_j$, $i \neq j$, $z \in B'_k$, or
14. $x \in B'_i$, $y \in B'_j$, $i \neq j$, $z \in B''_k$, or
15. $x \in B''_i$, $y \in B''_j$, $i \neq j$, $z \in A'_k$.

It gives a Turán $(2(n' + n''), 5, 3)$-system of size $2\binom{n' + n''}{3}$. We call it the *system produced by the pair of representations* (30), (31). Different pairs of canonical representations with $t' > 1$ and $t'' > 1$ produce non-isomorphic system, and these systems are distinct from the systems produced by single representatons.

The systems in Constructions 6 and 7 do not have to contain large cliques. In particular, the representation $2k \cdot k = 2k^2$, as well as the pair of representations $k \cdot k = k^2$, $k \cdot k = k^2$, produces a Turán $(4k^2, 5, 3)$-system with $2\binom{2k^2}{3}$ blocks, where the size of the largest clique is $2k$.

The best known $(n, k, 3)$-systems with $k \geq 6$ can be constructed as unions of $(n', 5, 3)$-, $(n'', 4, 3)$- and $(n''', 3, 3)$-systems. In particular, using (7), (26) and (27), we get

$$T(n, \alpha + 1, 3) \leq \frac{1}{2} \sum_{i=0}^{\alpha-1} \binom{\lfloor (2n + i)/\alpha \rfloor}{3}, \tag{32}$$

or equivalently,

$$T(n, \alpha + 1, 3) \leq \binom{m}{2} n - \binom{m+1}{3} \alpha \qquad \text{for } m \leq \frac{2n}{\alpha} \leq m + 1.$$

Applying (28), one may improve (32) for those values of n where $\frac{2n}{\alpha}$ is not integral. In the case when $\frac{2n}{\alpha}$ is an integer, we expect that the old conjecture of Turán [58] is correct:

Conjecture 4.

$$T\left(\frac{\alpha m}{2}, \alpha + 1, 3\right) = \frac{\alpha}{2} \binom{m}{3}.$$

Thus we believe that the asymptotic behavior of the numbers $T(n, k, 3)$ is what Turán and other researchers (e.g. see [4]) conjectured:

Conjecture 5.

$$t(\alpha + 1, 3) = \left(\frac{2}{\alpha}\right)^2.$$

The best known lower bound for $t(4, 3)$ was found by Giraud [17] (see formula (11) in Section 3):

$$t(4, 3) \geq \frac{7 - \sqrt{21}}{6} = 0.4029\ldots.$$

For small ratios $\frac{n}{\alpha}$, the values $T(n, \alpha + 1, 3)$ were found in [43, 44]:

$$T(n, \alpha + 1, 3) = \begin{cases} n - \alpha & \text{if } 1 \leq \frac{n}{\alpha} \leq \frac{3}{2}; \\ 3n - 4\alpha & \text{if } \frac{3}{2} \leq \frac{n}{\alpha} \leq 2; \\ 4n - 6\alpha & \text{if } 2 \leq \frac{n}{\alpha} \leq \frac{9}{4}, \quad 4n - 9\alpha \neq -1; \\ 4n - 6\alpha + 2 & \text{if } 4n - 9\alpha = -1, 1, 2. \end{cases} \tag{33}$$

We expect the next zone to be as follows.

Conjecture 6.

$$T(n, \alpha + 1, 3) = 8n - 2\left\lfloor \frac{15}{2}\alpha \right\rfloor \qquad \text{if } \frac{9}{4} \leq \frac{n}{\alpha} \leq \frac{5}{2}.$$

The corresponding upper bound follows from formulae (7), (28) and (32).

8. The Case $r = 4$

Formulae (23) and (24) imply

$$T(n, \alpha + 1, 4) = \begin{cases} n - \alpha & \text{if } \alpha \leq n \leq \frac{4}{3}\alpha; \\ \left\lceil \frac{5}{2}n \right\rceil - 3\alpha & \text{if } \frac{4}{3}\alpha \leq n \leq \frac{3}{2}\alpha. \end{cases} \tag{34}$$

It was proven in [43] that

$$T(6s + 1, 4s + 1, 4) = 3s + 4,$$
$$T(6s + 2, 4s + 2, 4) = 3s + 3.$$

Feng-Chu Lai and Gerard J. Chang [23] proved $U(n, q, 4) + n \geq \frac{9}{2}q$, and this can be casted as an upper bound for Turán numbers (see Section 1, formula (1)) with $n \geq \frac{3}{2}\alpha$:

$$T(n, \alpha + 1, 4) \geq \frac{7}{2}n - \frac{9}{2}\alpha.$$

Note that Turán numbers with $r = 2$ form convex sequence:

$$T(n, k, 2) - T(n - 1, k, 2) \leq T(n + 1, k, 2) - T(n, k, 2).$$

The same is true for all determined values of Turán numbers with $r = 3$. As (24) and (34) show, there is no convexity starting $r = 4$. For instance, if $\frac{2}{3}\alpha \leq m \leq \frac{3}{4}\alpha$, then

$$T(2m, \alpha + 1, 4) = 5m - 3\alpha, \qquad T(2m + 1, \alpha + 1, 4) = 5m - 3\alpha + 3,$$

$$T(2m + 2, \alpha + 1, 4) = 5m - 3\alpha + 5,$$

$$T(2m + 1, \alpha + 1, 4) - T(2m, \alpha + 1, 4) > T(2m + 2, \alpha + 1, 4) - T(2m + 1, \alpha + 1, 4).$$

The only case with $r > 3$, when there exists a plausible conjecture on the asymptotic behavior of Turán numbers, is $k = 5$, $r = 4$. In this case, a beautiful construction was found by Giraud [16] and then generalized by de Caen, Kreher and Wiseman [7]. We have already described it in Section 4 (see Construction 2). Let $h(l, m)$ denote the minimal number of even (2×2)-submatrices in an $(l \times m)$-matrix. We also set $h(2l) = h(l, l)$ and $h(2l + 1) = h(l, l + 1)$. Construction 2 yields the upper bound

$$T(n, 5, 4) \le \binom{\lfloor n/2 \rfloor}{4} + \binom{\lfloor (n+1)/2 \rfloor}{4} + h(n). \tag{35}$$

Minimization of the number of even (2×2)-submatrices is equivalent to the minimization of the sum of square entries of AA^* where A is an $(l \times m)$-matrix with entries equal to 1 or -1. Naturally, this problem is related to the existence of Hadamard matrices. We are going to show that

$$h(n) = \begin{cases} \binom{\lfloor n/2 \rfloor}{2}\left(\binom{\lfloor (n+2)/4 \rfloor}{2} + \binom{\lfloor (n+3)/4 \rfloor}{2}\right) & \text{if } 8k - 3 \le n \le 8k + 1, \\ \binom{\lfloor n/2 \rfloor}{2}\left(\binom{\lfloor (n+2)/4 \rfloor}{2} + \binom{\lfloor (n+3)/4 \rfloor}{2}\right) & \\ \quad + \binom{\lfloor n/4 \rfloor}{2} + \binom{\lfloor (n+2)/4 \rfloor}{2} & \text{if } 8k - 5 \le n \le 8k - 4, \end{cases} \tag{36}$$

provided that there exists an Hadamard matrix of order $4k$. Indeed, the lower bound in (36) for $8k - 3 \le n \le 8k + 1$ follows from the inequalities

$$h(2, m) \ge \binom{\lfloor m/2 \rfloor}{2} + \binom{\lfloor (m+1)/2 \rfloor}{2}$$

and

$$h(l, m) \ge \binom{l}{2} h(2, m).$$

In any $(3 \times (4k - 2))$-matrix, there are two rows which induce more than

In any $(3 \times (4k-2))$-matrix, there are two rows which induce more than $2\binom{2k-1}{2}$ even (2×2)-submatrices. Thus

$$h(l, 4k-2) \geq \binom{l}{2} \cdot 2\binom{2k-1}{2} + T(l, 3, 2)$$

which is the lower bound in (36) for $n = 8k-4, 8k-5$. Now assume that there exists an Hadamard matrix of order $4k$. We replace its negative entries with zeros. The resulting matrix and its submatrices of size $\left(\left\lfloor \frac{n}{2} \right\rfloor \times \left\lfloor \frac{n+1}{2} \right\rfloor\right)$ yield the upper bound in (36) for $8k-5 \leq n \leq 8k$. To get the upper bound for $n = 8k+1$, we extend the matrix by an arbitrary column.

The identity matrix of size 5 yields $h(10) = 40$. The case $n = 8k+2$, $k > 1$ remains unresolved.

The exact values of $T(n, 5, 4)$ were determined for $n \leq 10$ in [40] and [6]. In the former work, the uniqueness of the Turán $(10, 5, 4)$-system of size 50 was also proven. It is interesting that the equality in (35) is attained (and thus the described construction is optimal) for every $n \leq 10$. We expect that this construction is asymptotically optimal.

Conjecture 7.

$$t(5, 4) = \frac{5}{16}.$$

For $k = 6, 7$, we describe the best asymptotical constructions that we know.

Construction 8. We take two matrices with the same number of rows: an $(m \times m')$-matrix M' and an $(m \times m'')$-matrix M''; their elements are zeros and ones. The elements of the system are divided into 3 groups of sizes m, m' and m''; they correspond to the set of rows (which is common for both matrices), the set of columns of M', and the set of columns of M''. We take all quadruples within each group as well as those quadruples which correspond to even (2×2)-submatrices in either matrix. We also take every quadruple which corresponds to a pair of columns of M' and a pair of columns of M''. It gives us a Turán $(m + m' + m'', 6, 4)$-system. Therefore,

$$T(m + m' + m'', 6, 4) \leq \binom{m}{4} + \binom{m'}{4} + \binom{m''}{4} + h(m, m') + h(m, m'') + \binom{m'}{2}\binom{m''}{2}. \tag{37}$$

If $m' = m'' = \lfloor np \rfloor$, $m = n - m' - m''$, inequality (37) and the simple probabilistic bound $h(l, m) \leq \frac{1}{2}\binom{l}{2}\binom{m}{2}$ yield

$$T(n, 6, 4) \leq (8p^4 + 6p^2(1-2p)^2 + (1-2p)^4)\binom{n}{4} + \mathcal{O}(n^3).$$

The polynomial in the right hand side attains the minimum value 0.176614234... at $p = 0.29419\ldots$. Thus we get $t(6,4) < 0.17662$.

The exact values of $T(n,6,4)$ were determined in [8] for $n \leq 10$.

Construction 9. In order to construct a Turán $(n,7,4)$-system, we divide the n elements into 8 groups A_{ijk} where $i,j,k \in \{0,1\}$ and

$$|A_{000}| \leq |A_{001}| \leq |A_{010}| \leq |A_{011}| \leq |A_{100}| \leq |A_{101}| \leq |A_{110}| \leq |A_{111}| \leq |A_{000}| + 1.$$

The system includes quadruples Q which satisfy

1. $Q \subseteq A_{ijk}$, $i,j,k \in \{0,1\}$, or
2. $|Q \cap A_{ijk}| = 3$, $|Q \cap A_{l,1-j,k}| = 1$, $i,j,k,l \in \{0,1\}$, or
3. $|Q \cap A_{ijk}| = |Q \cap A_{lmk}| = 2$, $i,j,l,m \in \{0,1\}$, $(i,j) \neq (l,m)$, or
4. $|Q \cap A_{ijk}| = 2$, $|Q \cap A_{0,m,1-k}| = |Q \cap A_{1,m,1-k}| = 1$, $i,j,k,m \in \{0,1\}$, or
5. $|Q \cap A_{00k}| = |Q \cap A_{01k}| = |Q \cap A_{10k}| = |Q \cap A_{11k}| = 1$, $k \in \{0,1\}$.

This construction yields the asymptotical upper bound $t(7,4) \leq \frac{3}{32}$ and the best known bounds for small n: $T(11,7,4) \leq 17$, $T(12,8,4) \leq 26$, $T(13,7,4) \leq 40$, $T(14,7,4) \leq 58$, $T(15,7,4) \leq 81$, $T(16,7,4) \leq 108$. The construction is optimal for $n \leq 10$: $T(8,7,4) = 2$, $T(9,7,4) = 4$, $T(10,7,4) = 10$.

9. The case of Small $n - k$ (Covering Numbers)

While the behavior of the Turán numbers $T(n,k,r)$ is studied with a fixed r (or fixed r and k), the covering numbers $C(n,m,p) = T(n, n-p, n-m)$ are usually investigated with fixed m and p. The titles "Turán problem" and "covering problem" represent two different viewpoints in the 3-dimensional parameter space: they state along which subspaces the problem will be primarily studied. A detailed survey of results on the covering numbers can be found in [34]. Tables of the best known upper bounds for $C(n,m,p)$ with $n \leq 32$, $m \leq 16$, $p \leq 8$ are published in [18].

Inequality (2) yields the Schönheim bound:

$$C(n,m,p) \geq L(n,m,p),$$

where

$$L(n,m,p) = \left\lceil \frac{n}{m} \left\lceil \frac{n-1}{m-1} \left\lceil \cdots \left\lceil \frac{n-m+1}{1} \right\rceil \cdots \right\rceil \right\rceil \right\rceil.$$

Rödl [38] proved that

$$C(n,m,p) = (1 + o(1))L(n,m,p) \tag{38}$$

where m, p are fixed and $n \to \infty$. Kuzyurin [22] showed that (38) can be extended to the case when p is fixed and $\frac{n}{m} \to \infty$.

The case $m = 3$, $p = 2$ was solved by Fort and Hedlund [11]:

$$C(n,3,2) = L(n,3,2),$$

and the case $m = 4$, $p = 2$ by Mills [27, 28]:

$$C(n,4,2) = \begin{cases} L(n,4,2) + 2 & \text{if } n = 19; \\ L(n,4,2) + 1 & \text{if } n = 7, 9, 10; \\ L(n,4,2) & \text{if } n \neq 7, 9, 10, 19. \end{cases}$$

The case $m = 5$, $p = 2$ was not resolved completely [14, 31, 24, 35, 36, 33, 48]. For instance, the subcases $n \equiv 0 \bmod 4$ and $n \equiv 13 \bmod 20$ are still open.

For $m = 4$, $p = 3$, it was proven in [29, 32] that $C(n,4,3) = L(n,4,3)$ for any n except a finite number of values $n \equiv 7 \bmod 12$ in the range $n \leq 54211$.

If a system of m-element blocks covers all p-element subsets of an n-element set, one may replace each element by l distinct elements and get a system of (lm)-element blocks covering all p-element subsets of an (ln)-element set. Thus

$$C(ln, lm, p) \leq C(n, m, p). \tag{39}$$

Because of (39), one may hope to classify all values n, m such that $C(n,m,p) = t$ (with fixed p, t) in terms of $\frac{n}{m}$. In the cases $p = 2$ and $p = 3$ such a classification was obtained for $t \leq 13$ [30, 53] and $t \leq 8$ [50, 54], respectively.

Todorov [51] proved

$$C\left(\sum_{i=0}^{l} a_i p^{l-i}, \sum_{i=0}^{l-1} a_i p^{l-1-i}, l\right) = \frac{p^{l+1} - 1}{p - 1}$$

where p is a prime power and $a_0 \geq a_1 \geq \cdots \geq a_l > 0$. The construction is based on finite projective geometries, and the lower bound follows from the Schönheim bound. Some related results are given in [13, 25, 52, 55].

In most of the cited results, $C(n,m,p)$ is equal or very close to $L(n,m,p)$. On the other hand, formulae (23) and (24) imply

$$C(n,m,p) = p + 1 \quad \text{if } n \leq \frac{(p+1)m}{p}, \tag{40}$$

$$C(n,m,p) = \begin{cases} 3(p+1) - \left\lfloor \dfrac{2n}{n-m} \right\rfloor & \text{if } n - m \equiv 0 \bmod 2, \\ & \dfrac{(p+1)m}{p} \leq n \leq \dfrac{3(p+1)m}{3p-1}, \\ 3(p+1) - \left\lfloor \dfrac{2(n-p-1)}{n-m-1} \right\rfloor & \text{if } n - m \equiv 1 \bmod 2, \\ & \dfrac{(p+1)m}{p} \leq n \leq \dfrac{(p+1)(3m-1)}{3p-1}. \end{cases} \tag{41}$$

The right hand side of (40) is equal to the Schönheim bound, $L(n, m, p)$, but the right hand side of (41) is far from $L(n, m, p)$.

Inequalities (4) and (8) sometimes are better than the Schönheim bound.

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