Duality in Fractional Programming[1])

By K. SWARUP, Delhi[2])

Eingegangen am 19. April 1967

Summary: In this paper, a dual problem to linear fractional functionals programming i. e.

$$\text{Maximise} \quad Z = \frac{c' x}{d' x}$$
$$\text{subject to} \quad A x = b$$
$$x \geq 0$$

is formulated. Certain duality theorems regarding the relationship between primal and dual problems are established.

Zusammenfassung: Es wird zu dem Programm mit gebrochen-linearer Zielfunktion:

$$A x = b, \quad x \geq 0$$
$$Z = \frac{c' x}{d' x} \Rightarrow \text{Max!}$$

ein duales Programm aufgestellt. Gewisse Dualitätssätze über den Zusammenhang zwischen dem Primal- und Dualproblem werden aufgestellt.

Introduction

Linear fractional functionals programming is concerned with maximising (or minimising) a linear fractional functional of n variables subject to linear constraints on the variables. The constraints may be in the form of either equations or inequalities or both. Such mathematical problems arise in a variety of contexts. Under some assumptions linear fractional programming problems, though the objective function is neither convex nor concave, have well been attacked for their solution with the help of simplex method [9, 10, 11]. Conditions of optimality have been derived for a basic feasible solution. However, not much has been carried out in the direction of duality. This paper is a step in this direction.

In duality for any program, we are concerned with two problems called primal problem and dual problem. The duality relationship has both theoretical and computational significance. Duality concepts and relations are well known, now, for convex (concave) programming problems [4, 6, 15].

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[1]) This work was done during my stay at the Indian Institute of Management, Calcutta, as a visiting faculty member.

[2]) K. SWARUP, Faculty of Mathematics, Delhi University, Delhi 7 (India).

In what follows, this concept of duality will be extended further to include optimisation of linear fractional functionals programming. The development, to some extent, parallels to that given in linear programming. Matrix notation will be used throughout the paper. Prime denotes transpose. A vector inequality will apply to each component of the vector i.e. $x \geq 0$ implies that each component of x is non-negative.

A Class of Fractional Programs and their Duals

Consider the following problems:

Primal Problem $(P—P)$

$$\text{Maximise} \quad Z = \frac{c'x}{d'x} \tag{1}$$

$$\text{subject to } Ax = b \tag{2}$$

$$x \geq 0 \tag{3}$$

where

(i) A is $m \cdot n$ matrix.

(ii) x, c, d are $n \cdot 1$ vectors, b is $m \cdot 1$ vector,

(iii) $d'x > 0$ *over a convex polyhedron* "S" of feasible solutions (regular set).

In terms of cartesian notation:

$(P—P)$ i.e.

$$\text{Maximise } Z = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, \ldots, m)$$

$$x_j \geq 0 \quad (j = 1, \ldots, n).$$

Dual Problem $(D—P)$

$$\text{Minimise} \quad W = \frac{b'u}{b'v} \tag{4}$$

$$\text{subject to} \quad b'v(A'u - c) - b'u(A'v - d) \geq 0 \tag{5}$$

$$b'v \geq 0 \tag{6}$$

u, v are unrestricted in sign (7)

and $b'u$ and $b'v$ are not simultaneously zero. (7a)

Theorem 1:

If x is any feasible solution to $(P—P)$ and u, v any feasible solution to $(D—P)$, then

$$\frac{c'x}{d'x} \leq \frac{b'u}{b'v}.$$

Proof:

From $Ax = b$, we find

$$x'A'u = b'u \tag{8}$$

$$x'A'v = b'v. \tag{9}$$

Using (5) and (3), we have

$$b'v(x'A'u - c'x) - b'u(x'A'v - d'x) \geq 0. \tag{10}$$

With (8) and (9), (10) reduces to

$$b'v(b'u - c'x) - b'u(b'v - d'x) \geq 0$$

or

$$-b'v \cdot c'x + b'u \cdot d'x \geq 0. \tag{11}$$

Case (i): If $b'v > 0$, then

$$\frac{c'x}{d'x} \leq \frac{b'u}{b'v}. \tag{12}$$

Case (ii): If $b'v = 0$, then $b'u \neq 0$.

From (11) follows $b'u \cdot d'x \geq 0$, which implies $b'u > 0$, as $d'x > 0$ for all feasible x. Thus at this feasible solution u, v to $(D—P)$, we have

$$\frac{b'u}{b'v} \to +\text{infinity}.$$

Hence

$$\frac{c'x}{d'x} \leq \frac{b'u}{b'v}. \tag{13}$$

Theorem 2:

If $\hat{x}$ is a feasible solution to $(P—P)$ and $\hat{u}$, $\hat{v}$ a feasible solution to $(D—P)$ such that

$$\frac{c'\hat{x}}{d'\hat{x}} = \frac{b'\hat{u}}{b'\hat{v}} \tag{14}$$

then $\hat{x}$ is optimal to $(P—P)$ and $\hat{u}$, $\hat{v}$ optimal to $(D—P)$.

Proof:

By assumption there is

$$\frac{c'\hat{x}}{d'\hat{x}}=\frac{b'\hat{u}}{b'\hat{v}}.$$

But for any feasible x to $(P—P)$ we have

$$\frac{c'x}{d'x}\leq\frac{b'\hat{u}}{b'\hat{v}}=\frac{c'\hat{x}}{d'\hat{x}} \tag{15}$$

and therefore $\hat{x}$ is optimal to $(P—P)$.

Similarly for any feasible u, v to $(D—P)$, we have

$$\frac{b'u}{b'v}\geq\frac{c'\hat{x}}{d'\hat{x}}=\frac{b'\hat{u}}{b'\hat{v}} \tag{16}$$

and thus $\hat{u}$, $\hat{v}$ provides an optimal solution to $(D—P)$.

Theorem 3:

If the primal problem has an optimum then the dual is feasible and has an optimum and the two optima are equal.

Proof:

Let $x_B=B^{-1}b$ be an optimal basic feasible solution to the primal problem with optimal basis B. If c_B, d_B are the vectors containing the prices associated with the basic variables in numerator and denominator of (1) [15].

Put

$$Z_{\text{Max}}=\frac{c'_B x_B}{d'_B x_B}=\text{Max}\,\frac{c'x}{d'x}.$$

Now as x_B is optimal to $(P—P)$, we have [9, 10]

$$z^{(2)}(z_j^{(1)}-c_j)-z^{(1)}(z_j^{(2)}-d_j)\geq 0 \quad \text{for all } j \tag{17}$$

where

$$z^{(2)}=d'_B x_B,\quad z^{(1)}=c'_B x_B$$

$$z_j^{(2)}=d'_B B^{-1}a_j,\quad z_j^{(1)}=c'_B B^{-1}a_j \quad (a_j \text{ is a column of } A).$$

i.e. $$d'_B x_B(c'_B B^{-1}a_j-c_j)-c'_B x_B(d'_B B^{-1}a_j-d_j)\geq 0 \quad \text{for all } j\ (j=1,2,\dots,n).$$

Therefore

$$d'_B x_B (c'_B B^{-1} A - c') - c'_B x_B (d'_B B^{-1} A - d') \geq 0 \tag{18}$$

or

$$d'_B B^{-1} b (c'_B B^{-1} A - c') - c'_B B^{-1} b (d'_B B^{-1} A - d') \geq 0 . \tag{19}$$

Consider

$$\begin{cases} c'_B B^{-1} = u' & (20) \\ d'_B B^{-1} = v' . & (21) \end{cases}$$

Using (20) and (21) in (19), we get

$$v' b (u' A - c') - u' b (v' A - d') \geq 0$$

or

$$b' v (A' u - c) - b' u (A' v - d) \geq 0 . \tag{22}$$

Also

$$d'_B B^{-1} b > 0 \tag{23}$$

i.e.

$$v' b > 0, \qquad \text{or} \;\; b' v > 0 . \tag{24}$$

(23) also implies $d'_B B^{-1} \neq 0$. Hence u, v given by (20) and (21) represent a feasible solution to $(D—P)$.

Next we show that

$$u' = c'_B B^{-1}, \; v' = d'_B B^{-1}$$

is an optimal solution to $(D—P)$. Now

$$W = \frac{b' u}{b' v} = \frac{c'_B B^{-1} b}{d'_B B^{-1} b} = \frac{c'_B x_B}{d'_B x_B} = Z_{\text{Max}} . \tag{25}$$

Thus from theorem 2, we conclude that u, v represent an optimal solution to $(D—P)$.

Numerical Example

$$(P-P) \quad \begin{cases} \text{Maximise} \;\; Z = \dfrac{2x_1 + 3x_2}{x_1} \\ \text{subject to} \\ \qquad x_1 + x_2 + x_3 = 1 \\ \qquad 4x_1 + x_2 - x_4 = 2 \\ \qquad x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

$$(D-P)\quad \begin{cases} \text{Minimise} & W=\dfrac{u_1+2u_2}{v_1+2v_2} \\ \text{subject to} & (v_1+2v_2)(u_1+4u_2-2)-(u_1+2u_2)(v_1+4v_2-1)\geq 0 \\ & (v_1+2v_2)(u_1+u_2-3)-(u_1+2u_2)(v_1+v_2)\geq 0 \\ & (v_1+2v_2)u_1-(u_1+2u_2)v_1\geq 0 \\ & -(v_1+2v_2)u_2+(u_1+2u_2)v_2\geq 0 \\ \text{and} & (v_1+2v_2)\geq 0 \\ & u_1+2u_2, \quad v_1+2v_2 \\ \text{are not zero simultaneously.} & \end{cases}$$

The optimal solution to the primal problem [9] is

$$x_1=\frac{1}{3}, \quad x_2=\frac{2}{3}, \quad x_3=0,\ x_4=0$$

with

$$B=\begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix} \quad c_B=\begin{pmatrix}2\\ 3\end{pmatrix} \quad d_B=\begin{pmatrix}1\\ 0\end{pmatrix}$$

thus

$$B^{-1}=\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{4}{3} & -\frac{1}{3}\end{pmatrix}$$

The optimal solution to $(D-P)$ will be given as

$$(u_1,u_2)=(2,3)\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{4}{3} & -\frac{1}{3}\end{pmatrix}=\left(\frac{10}{3},\quad -\frac{1}{3}\right),$$

i.e.

$$u_1=\frac{10}{3},\quad u_2=-\frac{1}{3}.$$

$$(v_1,v_2)=(1,0)\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{4}{3} & -\frac{1}{3}\end{pmatrix}=\left(-\frac{1}{3},\quad \frac{1}{3}\right),$$

i.e.

$$v_1=-\frac{1}{3} \quad v_2=\frac{1}{3}.$$

Also $Z_{\text{Max}}=8=W_{\text{Min}}$.

Remark:

If there is an optimal solution to $(P—P)$, we can construct the optimal solution to $(D—P)$.

Conclusion:

Here, it has not been possible to provide a converse of the duality theorem. However, some work, in this direction has already been done by the author in the paper "Some Aspects of Duality in Fractional Functionals Programming" [13].

Acknowledgement

I am extremely grateful to Professor R. S. VARMA, Dean, Faculty of Mathematics, Delhi University, for continued inspiration throughout this work.

References

[1] CHARNES, A. and W. W. COOPER: Programming with linear fractional functionals. — Nav. Res. Log. Quart. Vol. IX, 1962, pp. 181—186.

[2] DINKELBACH, W.: Die Maximierung eines Quotienten zweier linearer Funktionen unter linearen Nebenbedingungen. — Z. Wahrscheinlichkeitstheorie, Vol. 1, 1962, pp. 141—145.

[3] DORN, W. S.: Linear fractional programming — I. B. M. Research Report, 1962.

[4] —: A duality theorem for convex programs. — I. B. M. Res. Jour. 4, 1960.

[5] HADLEY, G.: Linear Programming. — Reading Mass: Addison Wesley, 1964.

[6] HANSON, M. A.: Duality theorem in non-linear programming with non-linear constraints. — Australian Journal of Statistics, Vol. 3, 1961, pp. 64—71.

[7] KUHN, H. W. and A. W. TUCKER: Non-Linear programming. — Proc. Second Berkeley Symposium on Math. Stat. and Prob. 1951, pp. 481—492.

[8] MARTES, B.: Hyperbolic programming. — Nav. Res. Log. Quart., Vol. 11, 1964, pp. 135—155.

[9] SWARUP, K.: Linear fractional functionals programming. — Operations Res. vol. 13, No. 6, 1965, pp. 1029—1036.

[10] —: Some aspects of linear fractional functionals programming. — Australian Journal of Statistics, Vol. 7, No. 3, 1965.

[11] —: Transportation technique in linear fractional functionals programming. — J. R. N. S. S. Vol. 21, No. 5, 1966, pp. 256—260.

[12] —: Fractional programming with non-linear constraints. — ZAMM, Vol. 46, No. 7, 1966.

[13] —: Some aspects of duality in fractional programming. — To appear in ZAMM, Vol. 47, 1967.

[14] —: A duality theorem for non-linear programming. — ZAMM, Vol. 45, No. 5, 1965, pp. 311—314.

[15] WOLFE, P.: Duality theorem for nonlinear programming. — Quart. Applied Maths., Oct. 1961, pp. 239—244.

Vorgel. v.: H. P. KÜNZI.