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Stochastic Scheduling Problems II*
– Set Strategies –

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Abstract: The paper introduces the finite class of set strategies for stochastic scheduling problems. It is shown that the known *stable* classes of strategies such as ES and MES strategies are of this type, as are *list-scheduling strategies* such as LEPT and SEPT and other, more complicated *priority-type strategies*. Roughly speaking, set strategies are characterized by the fact that the decision as to which jobs should be started at time t depends only on the knowledge of the two sets of jobs finished up to time t and being processed at time t. Contrary to list scheduling strategies, set strategies may involve deliberate idleness of machines, i.e. may not be greedy and can therefore not generally be induced by priority rules. It is demonstrated that set strategies have useful properties. They are e.g. λ^n-almost everywhere continuous and therefore show satisfactory *stability behaviour* w.r.t. weak convergence of the joint distribution of job durations. Furthermore, the optimum w.r.t. *all* strategies is already attained on this class if job durations are independent and exponentially distributed and the performance measure fulfills a certain *shift condition*. This shift property is a quite natural concept and generalizes aspects of the notion of *additivity* in semi-Markov decision theory and stochastic dynamic optimization. Its complete analytical characterization is a major object of this paper. Typical additive cost criteria such as makespan and flowtime are of course covered, which yields simultaneously a first step towards generalization of optimality of LEPT and SEPT rules, as known for special cases. In fact, in view of the obtained optimality result, the question of when deliberate idleness of machines can be avoided, gains considerable interest, as it characterizes stochastic environments in which *priority strategies* are optimal. This provides a major link with current research on the analysis of networks of queues in the context of computer systems.

Key Words: Additive cost criterion, analytic behaviour of strategies, ES strategy, list schedule, MES strategy, priority rule, quasi-stability, regular measure of performance, scheduling problems, set strategy, shift property, stability, stochastic scheduling.

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1. Introduction

In the first paper of this series we have discussed quite general stochastic scheduling problems, allowing for arbitrary joint distributions of activity durations, arbitrary regular measures of performance and arbitrary precedence and resource constraints. The major topics covered were properties of general strategies and certain subclasses thereof, an exploitable monotonicity behaviour, and criteria for the existence of optimal strategies. Less attractive findings have been the possible *non-existence* of optimal strategies and a rather nasty *instability* behaviour that is hardly acceptable for real-life applications. This instability is not only true for the set of all strategies, but also for interesting subclasses such as the *elementary* or the *continuous* strategies. Apart from computational aspects and problems with practical execution, these findings form a strong motivation for restricting oneself to certain well-behaved classes of strategies. One approach has been reported in the first paper of this series. Starting from a quite rigid notion of stability, finite classes of continuous strategies turned out to be of interest. Two combinatorially defined classes of this nature, viz. *ES* and *MES strategies* seem for many reasons to be particularly appealing, above all due to their practical applicability. Here, ES strategies are the earliest start scheduling strategies of feasible partial orders, while MES strategies are defined as the minimum of certain sets of ES strategies. As was reported in the previous paper, it is possible to obtain quite involved equivalent characterizations of these classes by means of *analytical* properties, such as being convex or uniformly continuous, leading to surprising interfaces between discrete and continuous aspects in this field. The key to such results is the notion of *preselectivity*, leading to an uncountable class of strategies which contains *Radermacher* [1984] all (sub-)linear, convex, sub-additive, (super-)additive, concave, uniformly continuous, continuous-elementary and monotonically increasing strategies and which can still essentially (i.e. in the sense of dominance) be identified with the finite class of MES strategies. In fact, in view of the insights available, MES strategies form (in a precise sense) the greatest "reasonable" class of strategies fulfilling the mentioned rigid standard of stability.

In the literature, another special class of strategies is also discussed, viz. strategies induced by (possibly dynamic) *priority rules*, including list-scheduling strategies. Such strategies, which can to some extent be identified with the notion of *greediness*, are not in general stable but are easily implemented. We discuss them in Section 2.2 in the more general framework of priority approaches known for deterministic scheduling problems [cf. e.g. *Elmaghraby; Gewald/Kaspar/Schelle*], and give some hints on their analytical behaviour, including some additional insights into the so-called *Graham anomalies* [*Graham*]. The main interest in the class of priority-type strategies is probably due to the overall optimality of two list-scheduling strategies, viz. LEPT and SEPT, for several different, though quite special, stochastic scheduling problems. Such results are one of the main topics of the introductory volume [*Dempster/Lenstra/Rinnooy Kan*], where surveys on the subject [e.g. *Dempster; Weiss*] can be found, and will also be the issue of the third paper in this series. Some further remarks on this topic are also included in Section 2.3 of this paper, together with hints (Section 2.1) on additional assumptions as to the duration distributions and cost functions needed to obtain such results.

Given this background, and the fact that LEPT and SEPT rules – if optimal-lead to MES strategies [*Möhring/Radermacher/Weiss* (in preparation); *Radermacher*, 1984], the challenge is to find an integration of both approaches into a greater but still finite class of strategies, which should inherit some of the nice features of MES strategies and list-scheduling strategies. The introduction of *set strategies* in Section 3 may be viewed as such an integration. The idea behind the notion of a set strategy is to allow actions taken at any decision point t, to depend only on *sets*, viz. the set of activities finished at time t, and the set of activities still being performed (i.e. started before time t and not completed) at time t. In view of e.g. Section 4.3 of the first paper of this series, this means that though we are more general than in the case of MES or list-scheduling strategies, we still give away important pieces of available information, such as the actual starting or completion times of certain jobs, their (current) duration and t itself.

For set strategies, we will see in Section 3.2 that they indeed inherit some of the nice features of ES and MES strategies, insofar as they are e.g. λ^n-*almost everywhere (uniformly) continuous* and *quasi-stable* (i.e. stable in the case of joint distributions of activity durations having a Lebesgue density). Similarly, some of the nice properties that LEPT and SEPT rules have shown in special cases, are also carried over. In fact, in Section 4.2 we constructively obtain that the *overall optimal value* is attained on this class, regardless of the precedence and resource constraints, provided that the activity durations are independent, exponentially distributed and the cost function has a certain *shift property*. This covers the case of additive cost criteria (such as makespan and flowtime), for which the obtained result is a special version of a well-known theorem in semi-Markov decision theory and stochastic dynamic optimization [*Bertsekas/Shreve*], guaranteeing (under the assumptions made) the existence of an optimal strategy that ist *stationary* and *Markov* (which is in this case equivalent to being a set strategy). The shift property is, however, more general then being additive, cf. Section 4.1; it is, in fact, in a certain sense the *most general condition* under which a result of the given type may be expected. Consequently, characterizing analytically all cost functions with this property, also became a major aim of this paper (cf. Theorem 4.1.3).

Altogether, set strategies turn out to be a reasonable "intermediary" concept between the extremes of general strategies and MES strategies. They are particularly well-suited for a less rigid notion of stability, in which emphasis is placed on distributions with Lebesgue densities rather than on arbitrary duration distributions.

2. Background and Results Used

2.1 Hints on the model

As described in the first paper of this series, we deal with a class of on-line *non-preemptive* stochastic scheduling problems allowing arbitrary *precedence relations* O_0 and *forbidden sets* $\mathbf{N}$ (resource constraints) on the set A of *activities* (jobs). The optimization aim is the *minimization of the expected project cost* w.r.t. all (or maybe special classes of) strategies, arbitrary *joint distributions* P of activitiy durations and arbitrary *regular cost functions* κ (i.e. arbitrary regular measures of performance).

In this paper we will remain essentially within this framework, except for some hints on certain *preemptive* strategies in Remark 4.2.2, which will be of interest in the third paper of this series. However, we will often restrict ourselves to more special distributions P and cost functions κ. For instance, we will frequently require, that P has a *Lebesgue density* f, i.e. that there exists some measurable function f: $\mathbf{R}^n_> \to \mathbf{R}^1_\geqslant$, such that $P(B) = \int_B f\, d\lambda^n$ for all $B \in \mathbf{B}^n_>$. In particular, we will sometimes assume job durations to be *independent and exponentially distributed* with parameters $\lambda_\alpha > 0$, $\alpha \in A$, i.e. f to be of the form $f = \prod_{\alpha \in A} f_\alpha$ with $f_\alpha(x) = \lambda_\alpha e^{-\lambda_\alpha \cdot x}$ for $x > 0$. The expectation of X_α is then $1/\lambda_\alpha$. Interest in the exponential distribution is due to the property that characterizes it, viz. that it is *memory-less* [*Ross*], i.e. fulfills $P(\{X > s + t \mid X > t\}) = P(\{X > s\})$ for all $s, t \in \mathbf{R}^1_\geqslant$, which makes it impossible to gain any additional information on the lifetime of a job (with this distribution) before its completion.

Concerning cost functions, we will often restrict ourselves to functions that are *additive,* [cf. *Weiss/Pinedo*]. This means that there is a set function g from the power set $\mathbf{P}(A)$ of A into $\mathbf{R}^1_>$ with $g(\emptyset) = 0$ such that, given ordered job completion times $t_{i_1} \leqslant t_{i_2} \leqslant \ldots \leqslant t_{i_n}$, for jobs $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}$ κ can be written as $\kappa(t_1, \ldots, t_n) = t_{i_1} \cdot g(A) + (t_{i_2} - t_{i_1}) \cdot g(A_{i_1}) + \ldots + (t_{i_n} - t_{i_{n-1}}) \cdot g(A_{i_{n-1}})$, where $A_{i_0} = A$ and $A_{i_j} = A_{i_{j-1}} \setminus \{\alpha_{i_j}\}$, $j = 1, \ldots, n$. All such additive cost functions are *positively homogeneous* and *uniformly continuous*. Moreover, κ is a regular cost function (i.e. is monotonically increasing w.r.t. the componentwise ordering on $\mathbf{R}^n$) iff g is monotonically increasing on $\mathbf{P}(A)$. The main feature of additive cost criteria is that, given any time $t \in \mathbf{R}^1_>$, the *history* up to t enters into the project cost only through an additive term. This property is a special case of the shift property introduced in Section 4.1.

Note that all the special cost criteria mentioned in [*Möhring/Radermacher/Weiss*, 1984] that are not of the tardiness type are additive. This covers *makespan* (project duration), *(weighted) flowtime, first idleness* and the *sum of k latest jobs* (e.g. take $g(B) = 1$ if $B \neq \phi$ for the makespan and $g(B) = |B|$ for the flowtime), but also some further, interesting performance measures, [*Weiss/Pinedo*].

2.2 Strategies Induced by Priority Rules

There are many concepts of *priority rules* for the treatment of deterministic scheduling problems, [cf. *Elmaghraby; Gewald/Kaspar/Schelle*]. These involve distinctions between *series* and *parallel*, or *static* and *dynamic* rules. Much attention has been given to *list schedules* (constant priority rules), for which (good) *performance bounds* have in some cases been obtained; [cf. *Fisher; Graham*].

For the general case, we mention the unifying result that any priority rule can be transformed into an equivalent (maybe complicated) static and series rule, [cf. *Kaerkes* *et al.*]. In this interpretation, the parallel rules are seen to form a proper subclass of the series rules. W.r.t. the generalization to the stochastic case, the situation is different. In fact, it turns out that, due to their very conceptual structure, series priority rules are unsuited for the on-line scheduling of stochastic problems (and are therefore hardly ever referred to in this context), as they usually necessitate future (i.e. *a posteriori*) knowledge of random data such as activity durations. In contrast, certain parallel rules are applicable to the stochastic case. In a typical situation such a priority rule will, at any decision time t, give a linear ordering $L(t)$ (the *priority ordering*) on the set of jobs not yet started. If $L(t)$ is compatible with $L(0)$ for all decision times, the rule is said to be *static*, if $L(t)$ may vary with t, it is said to be *dynamic*. In a strategy induced by a (parallel) priority rule, actions are only taken at time 0 or at times t when a job is completed, and consist in starting (from the set of jobs available at t) in order of decreasing priority as many jobs as possible in terms of precedence and reources constraints. So, at each decision time t, a *maximal* set of jobs is started that is lexicographically smallest w.r.t. the priority ordering $L(t)$ (greedyness!). Of course, for dynamic rules, the priority orderings $L(t)$ may only vary in a way compatible with the information about x available at time t in order to induce a strategy. Thus, for instance, priority rules based on floats are generally excluded. In fact, among the dynamic rules usually considered [*Gewald/Kaspar/Schelle*], only special ones such as "current earliest start" or "current height" fulfill this property. For these two cases, the induced strategies will in fact even turn out to be *set strategies*, cf. Example 3.1.3.

It follows directly that strategies induced by priority rules are *elementary* and $\leqslant$-*minimal* [c.f. *Möhring/Radermacher/Weiss,* 1984 for these notions]; in fact, they are even pointwise $\leqslant$-minimal for any $x \in \mathbf{R}^n_{>}$. The latter observation reflects the *greediness* of such strategies, i.e. the inherent tendency to minimize idleness of a resource (in accordance with the priority ordering).

The fact, that priority-induced strategies are elementary and $\leqslant$-minimal implies, because of the results mentioned in Section 3.2 in *Möhring/Radermacher/Weiss* [1984], that such strategies can only have strong analytical properties if they coincide with ES strategies or MES strategies [cf. *Radermacher,* 1984]. To be more precise, being sublinear or convex or subadditive is equivalent to being an ES strategy, while being (uniformly) continuous, or monotonically increasing or preselective is equivalent to being an MES strategy. Due to the greediness of priority rules, such a coincidence will be a rare occurrence. In particular, there are problems $[A, O_0, \mathbf{N}]$ such that coincidence of priority-induced strategies with (M)ES strategies is in principle impossible, [cf. *Radermacher*, 1984]. Consequently, any priority-induced strategy in such a case is neither continuous nor monotonically increasing nor convex, and so forth. Such nasty behaviour of priority rules has previously been studied in the deterministic case where it is known as *Graham anomalies* [*Graham*]. Note that in view of Theorem 3.1.2 in *Möhring/Radermacher/Weiss* [1984], this will above all also imply *unstable* behaviour for priority induced strategies in general.

The question remains, what degree of bad analytical behaviour and instability is to be expected. We will see in this paper that for most interesting cases, viz. as long as the induced strategies are set strategies, we can establish e.g. *"piecewise continuity"* and *quasi-stability* (i.e. stability w.r.t. distributions P having a Lebesgue density), cf. Cor.

3.2.3 and Theorem 3.2.4. Even stronger results can be obtained if LEPT and SEPT are optimal (cf. Section 2.3). Apart from some additional insights, this behaviour is due to an observation already known for *elementary* strategies in general (thus in particular for all priority-induced strategies), which states that all these strategies are *piecewise composed of ES strategies;* cf. Section 2.4 and the proof of Theorem 3.2.1. Consequently, the areas of transition from one ES strategy to another are precisely the points where Graham anomalies may occur. Deeper insights into these transitions will be obtained in Section 3.2 of this paper.

2.3 Remarks on the Optimality of LEPT and SEPT Rules

The sometimes poor analytical behaviour and relative instability of priority-induced strategies is partly compensated for by their easy implementability. Also, though only for quite special problem classes, even an overall optimal strategy will be of this type. Subsequently, we give some hints on such cases. Note that the conditions required of P and κ will be compatible with those employed in section 4.2 to show overall optimality of set strategies in a rather general setting. This fact then serves as a first step towards more general results on tractable stochastic scheduling problems, as given in the third paper of this series. Note also that the optimal strategies subsequently discussed are incidentially also *MES strategies* [*Igelmund/Radermacher*, 1982], i.e. have strong analytical properties as well as a completely satisfactory stability behaviour.

The basic type of scheduling problems for which overall optimality of certain list-scheduling strategies has been obtained and which, in fact, motivated the introduction of set strategies, are *m-machine* problems. Such problems do not involve precedence relations, i.e. $(\alpha, \beta) \notin O_0$ for all $\alpha \neq \beta$, and are restricted to special systems $\mathbf{N}$ of the type $\mathbf{N} = \{B \subseteq A \mid |B| = m + 1\}$. These assumptions imply that jobs cannot be distinguished w.r.t. precedence and resource constraints. Furthermore, activity durations are assumed to be stochastically *independent* and *exponentially* distributed and project costs are assumed to be *additive*. Combination of both assumptions yields – for the preemptive case – (cf.Section2.1) that *the states* of the optimization problem are essentially the set of still *uncompleted jobs.* This quite simple state space is the basis for the embedding and comparatively easy handling such problems in the framework of semi-Markov decision processes.

Typical results obtained in this field show, under the given assumptions, the *optimality* of LEPT (longest expected processing time first) for the expected makespan, the expected first idleness and related objectives, and the optimality of SEPT (shortest expected processing time first) for the expected flow time, the expected weighted flow time (in the case of agreeable weights) and related objectives [*Weiss/Pinedo*]. These results are in a certain sense the hard core of tractable stochastic scheduling problems and allow generalization in various directions, see e.g. the hints in the surveys [*Dempster/Weiss*]. In particular, they yield the interesting observation, discussed in more detail in *Pinedo,* that special stochastic scheduling problems may allow treatment in polynomial, even *linear* time, though the corresponding deterministic counterpart is *NP-complete.* Some caution should however be applied with the notion of solvability, [cf. *Dempster*]. It is the optimal strategy and the associated schedule for any fixed duration vector $x \in \mathbf{R}^n_>$ than can efficiently be computed and implemented, not (in general) the associated optimal cost value. Note that even the iterative analytical description of the optimal value in *Weiss/Pinedo* will not help in this respect.

2.4 Schedule-Induced Partial Orders

A partial order $\Theta = (A, 0)$ is called an *interval order* [cf. *Golumbic*] if a time interval $[t_\alpha, t'_\alpha] \subseteq \mathbf{R}^1_\geqslant$ can be associated with each activity $\alpha \in A$ such that $(\alpha, \beta) \in 0$ for $\alpha \neq \beta$ iff $t'_\alpha \leqslant t_\beta$. Interval orders can equivalently be characterized by the fact that they do not contain a subposet isomorphic to (A', O') with $A' = \{1, 2, 3, 4\}$ and $O' \setminus \{(\alpha, \alpha) \mid \alpha \in A'\} = \{(1, 3), (2, 4)\}$. A third characterization is the existence of a numbering $\alpha_1, \ldots, \alpha_n$ such that $i < j$ implies that either $N_\Theta(\alpha_j) \subset N_\Theta(a_i)$ or $N_\Theta(\alpha_j) = N_\Theta(\alpha_i)$ and $V_\Theta(\alpha_i) \subseteq V_\Theta(\alpha_j)$, where $N_\Theta(\alpha)$ and $V_\Theta(\alpha)$ denote the sets of successors and predecessors of α w.r.t. Θ, respectively. The described *linear* order on the sets of successors (predecessors) is an important feature of interval orders. It allows for a consecutive arrangement $U_1, \ldots, U_r$ of all maximal independent subsets [the so-called *layers in Golumbic*] in such a way that activities $\alpha \in A$ occur consecutively, i.e. $\alpha \in U_i \cap U_j$ with $i < j$ implies that $\alpha \in U_l$ for all l with $i \leqslant l \leqslant j$. This so-called *consecutive-ones property* is in fact a fourth equivalence to Θ being in interval order [*Golumbic*].

The importance of interval orders for scheduling theory becomes apparent as soon as partial orders induced by schedules are studied. To illustrate this, associate with any schedule $T = (T(\alpha_1), \ldots, T(\alpha_n))$ for (A, O, x) a partial order $\Theta(T, x) = (A, O_T)$ defined by $(\alpha, \beta) \in O_T$ for $\alpha \neq \beta$ iff $T(\alpha) + x(\alpha) \leqslant T(\beta)$. Partial orders which can be obtained in this way are called *schedule-induced* [*Radermacher*, 1982] and obviously coincide with interval orders. The poset $\Theta(T, x)$ contains $\Theta = (A, O)$ as a subposet and T is also a schedule for (A, O_T, x). Furthermore, feasibility of T for $[A, O_0, \mathbf{N}, x]$ implies *feasibility* of Θ_T (in the sense of Section 3.2 in *Möhring/Radermacher/Weiss* [1984]) and the inequality $ES_{\Theta(T,x)}(\alpha) \leqslant T(\alpha)$ for each $\alpha \in A$. This observation has the interesting consequence that ES strategies induced by feasible interval orders already determine the overall optimum value in the case of singular distributions P, [cf. *Kaerkes et. al; Radermacher*, 1978], which yields a useful instrument for dealing with deterministic scheduling problems. Furthermore, one obtains a representation of any *elementary* strategy Π as a piecewise composition of ES strategies. To see this, let $Z(\Theta) := \{x \in \mathbf{R}^n_> \mid \Theta(\Pi[x], x) = \Theta\}$ for each feasible (interval) order Θ. Then [cf. *Kaerkes et al.; Radermacher*, 1984]:

$$(1) \bigcup_{\Theta \text{ feasible}} Z(\Theta) = \mathbf{R}^n_> \text{ and } (2)\ \Pi[x] = ES_\Theta[x] \text{ for all } x \in Z(\Theta).$$

This implies that any elementary strategy is composed of a finite number of (well-behaved) ES strategies. This does not, however, imply piecewise nice analytic properties, as the sets $Z(\Theta)$ may be very nasty. However, for the set strategies introduced in Section 3, these sets $Z(\Theta)$ exhibit better behaviour (Theorem 3.2.1), which, in fact, leads to a basic aim of this paper, viz. the intended result on quasi-stability (Theorem 3.2.4) for this particular class.

3. Set Strategies

3.1 Introduction and Elementary Properties of Set Strategies

Definition: Let $[A, O_0, \mathbf{N}, P, \kappa]$ be given.

(1) A strategy Π for this problem is called a *set strategy* if Π is elementary (i.e. jobs start either at time zero or else at the completion times of other jobs) and if the action $B(t)$ taken at any decision point $t \in \mathbf{R}^1_{\geqslant}$ depends only on the sets B^* and $B \setminus B^*$ of jobs already finished or being performed.

(2) The *optimum value* $\rho^{\mathrm{SET}}(\kappa; P)$ is defined as $\rho^{\mathrm{SET}}(\kappa; P) := \min \{\mathbf{E}_P[\kappa(\Pi, \cdot)] \mid \Pi$ is a set strategy$\}$, and a set strategy Π is called *optimal* if $\mathbf{E}_P[\kappa(\Pi, \cdot)] = \rho^{\mathrm{SET}}(\kappa; P)$.

Though set strategies will turn out to be more general than both MES strategies and list-scheduling strategies, they are apparently still rather special, even within the class of all elementary strategies. For if α is a job already finished before t (i.e. $\alpha \in B^*$), it is not permitted to distinguish actions taken at time t according to the observed duration $x(\alpha)$ or the completion time t_α which might be very reasonable in the case of stochastic dependences or special cost functions. Similarly, neither the value t nor the starting times of activities β being presently performed (i.e. $\beta \in B \setminus B^*$) have any influence on the action taken, though this type of information will, due to the respective *conditional* duration distributions, often be of a high value, also in the case of stochastically independent activity durations. To put it short, essential pieces of information that proved to be of crucial importance [cf. Section 4.3 in *Möhring/Radermacher/Weiss* (1984)], are given away, in order to obtain better behaviour than that encountered for general (or continuous or elementary) strategies.

Intuitively speaking, the information that set strategies are permitted to exploit, iteratively consists merely in the sets B^* and $B \setminus B^*$. The *possible actions* are then all sets $B(t) \subseteq A$ whose predecessors are contained in B, i.e. $V_\Theta(\alpha) \subseteq B$ for all $\alpha \in B(t)$, and which fulfil the resource constraints, i.e. $N \nsubseteq (B \setminus B^*) \cup B(t)$ for all $N \in \mathbf{N}$. This clearly implies that there are only *finitely* many such strategies (which in fact all turn out to be *Borel-measurable*) and that an *optimal* set strategy always exists, regardless of the given system $[A, O_0, \mathbf{N}, P, \kappa]$.

Lemma 3.1.1: Each MES strategy for $[A, O_0, \mathbf{N}]$ is a set strategy for $[A, O_0, \mathbf{N}]$.

Proof: Let $\Pi = \min \{\mathrm{ES}_\Theta \mid \Theta \in \mathbf{O}\}$. Let $t > 0$ be any decision time, i.e. the moment of completion of a job. Let B^* and $B \setminus B^*$ be as defined above. According to the definition of Π, the set $B(t)$ of activities started at time t consists of all $\alpha \in A \setminus (B \setminus B^*)$ which are *minimal* in at least one subposet $\Theta \mid (A \setminus B^*)$, $\Theta \in \mathbf{O}$. Therefore, the set $B(t)$ depends only on B^* and $B \setminus B^*$, making Π a set strategy. ∎

Lemma 3.1.1 shows that the class of all set strategies is at least large enough to cover all ES and MES strategies, thus equivalently all *convex (sublinear)* and all *continuous, elementary* strategies [cf. Section 3 in *Möhring/Radermacher/Weiss*, 1984]. Furthermore this also shows that the optimal value $\rho^{\mathrm{SET}}(\kappa; P)$ is as least as good as the objective value associated with any (sub-)linear, convex, subadditive, (super-)additive, concave, uniformly continuous, monotonically increasing or preselective strategy. Finally, Lemma 3.1.1 also straightforwardly implies that $\rho(\kappa;P) \leqslant \rho^{\mathrm{SET}}(\kappa;P) \leqslant \rho^{\mathrm{MES}}(\kappa;P) \leqslant \rho^{\mathrm{ES}}(\kappa;P)$, where all inequalities may be strict. Certainly this adds to the importance of $\rho^{\mathrm{SET}}(\kappa;P)$, e.g. w.r.t. possible use as a *lower bound*.

Call a priority rule *set-type* if, given a decision time t, the priority ordering on the set $A \setminus B$ of jobs available at time t depends only on the sets B^* and $B \setminus B^*$. Since these (possibly dynamic) priority rules use exactly the same information at time t as set strategies, the following lemma is rather obvious.

Lemma 3.1.2: Let Π be a strategy for $[A, O_0, \mathbf{N}]$ which is induced by a (dynamic) set-type priority rule. Then Π is a set strategy for $[A, O_0, \mathbf{N}]$. In particular, all list-scheduling strategies are set strategies.

Proof: Due to the definition of strategies induced by priority rules (cf. Section 2.2), $B(t)$ is uniquely determined as the *lexicographically smallest* maximal set of activities (w.r.t. the priority ordering) that forms a feasible choice. Since the rule is set-type, Π is a set strategy. The case of list-scheduling strategies is then trivial, as constant priority rules are set-type. ∎

There are certainly interesting strategies induced by dynamic priority rules that are not set strategies. However, many of the important ones w.r.t. applications [*Gewald/Kaspar/Schelle*] belong to this class. In the following, we discuss two typical examples of this type. Though the rules are quite different at first sight, it turns out – perhaps surprisingly – that the induced strategies coincide. In fact, they are in both cases essentially determined by the *constant* priority rule given by the numbering of the jobs, which is used (as an additional criterion) to settle ties for the original priority rules.

Example 3.1.3: Let $[A, O_0, \mathbf{N}]$ be given and let $A = \{\alpha_1, \ldots, \alpha_n\}$ be numbered agreeably w.r.t. Θ, i.e. $(\alpha_i, \alpha_j) \in O_0$ for $i \neq j$ implies $i < j$. Then:

(1) The dynamic priority rule *"current earliest start"* (CES) is set-type.

Given a duration vector $x \in \mathbf{R}^n_>$ and a decision time t with corresponding set $A \setminus B$ of still unscheduled jobs, the CES priority ordering on $A \setminus B$ is determined by the earliest start w.r.t. the subposet $\Theta \mid (A \setminus B)$. This means that given $\alpha_i, \alpha_j \in A \setminus B$ with $\alpha_i \neq \alpha_j$, α_i has a *higher priority* than α_j, if $\mathrm{ES}_{\Theta \mid A \setminus B}[x](\alpha_i) \leqslant \mathrm{ES}_{\Theta \mid A \setminus B}[x](\alpha_j)$, where ties are settled by the numbering of jobs, i.e. in the case of equal earliest starts, α_i is preferred to α_j if $i < j$. Obviously the set $B(t)$ is then a uniquely determined subset of the minimal activities in $\Theta \mid A \setminus B$ and thus depends only on B^* and $B \setminus B^*$, making this rule set-type.

(2) The dynamic priority rule *"current height"* (CH) is set-type.

By definition, the main criterion for CH is the largest number of elements in a chain of predecessors of $\alpha_i \in A \setminus B$ (the so-called *height* of α_i in $\Theta \mid (A \setminus B)$), where smaller heights yield higher priorities and ties are again settled by the job numbering. This rule is obviously also set-type. In fact, the induced strategy is the same as that induced by the job numbering being taken as a *(constant)* priority rule. This is similarly true in case (1), and yields that numbering, CES, and CH all induce the same strategy.

Example 3.1.5 below will employ strategies induced by set-type priority rules which are not constant. The following remark characterizes all strategies thus induced within the class of all set strategies.

Remark 3.1.4: A set strategy is induced by a dynamic (set-type) priority rule iff it is *greedy*, i.e. iff it avoids idle resources as long as possible w.r.t. the given priorities.

Proof: The proof of Lemma 3.1.2 already showed that (set-type) priority rules lead to greedy set strategies. Now let Π be a greedy set strategy and B^* and $B \setminus B^*$ be the sets of jobs finished and being performed at a decision time t. Then, either the set $B(t)$ of activities started at t is empty, meaning (because of the greediness) that it is not possible to schedule a new job in addition to $B \setminus B^*$, or $B(t) \neq \phi$. In the latter case, again due to the greediness, no set $B' \supset B(t)$ can be started at t. This behaviour can obviously be simulated by a (set-type) dynamic priority rule which, at any decision time t, assigns the highest priorities to the jobs of $B(t)$. ■

The following example shows that neither MES strategies, nor (set-type) priority-induced strategies will, in general, yield the optimal value $\rho^{\mathrm{SET}}(\kappa; P)$, not even in the case of independent and exponentially distributed job durations and additive (convex) cost functions. In fact, this is even true for the m-machine case (at least if the additive cost function is non-convex), cf. Example 4.2.5. This is different from what might have been expected by the behaviour encountered in cases where LEPT or SEPT are optimal, and demonstrates some of the *difficulties in more general models*, particularly if *non-preemptiveness* is required; (compare Remark 4.2.2 for the different behaviour in the preemptive case).

Example 3.1.5: Let $[A, O_0, \mathbf{N}, P, \kappa]$ be defined by $A = \{1, 2, \ldots, 10\}$, O_0 as given by an arrow diagram in Figure 2, $\mathbf{N}$ be induced by 2 identical machines (i.e. $\mathbf{N} = \{B \subseteq \{1, \ldots, 5\} \mid |B| = 3\} \cup \{\{7, 8, 10\}\}$, $P = \bigotimes_{j=1}^{10} P_j$ where P_j is an exponential distribution with parameter $\lambda_1 = 1/100$, $\lambda_2 = 1/20, \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 1$, $\lambda_9 = 1/20$ and $\lambda_{10} = 5$ and κ be the additive cost function $\kappa(t_1, \ldots, t_{10}) := 100 \cdot t_1 + 100 \cdot t_2 + 100 \cdot \max\{t_2, t_3, t_4\} + 20 \cdot \max\{t_1, t_5\} + \max\{t_1, \ldots, t_{10}\}$.

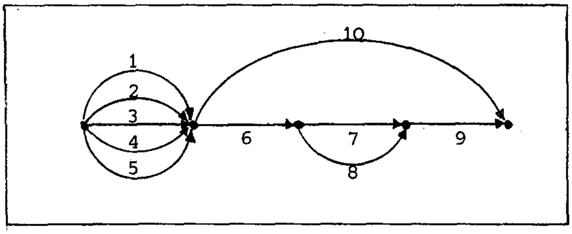

Fig. 2

Theorem 4.2.6 (1) or (3) in *Möhnig/Radermacher/Weiss* [1984] yield existence of an optimal strategy for this problem. In fact, due to Theorem 4.2.1 below, there even be an overall optimal set strategy for this case. We will now indicate how such an optimal set strategy has to behave.

First of all note that one can split up the problem into two parts. The first will consist in optimizing the subproblem belonging to $\{1, \ldots, 5\}$. The optimal solution is accidentially compatible with LEPT and therefore also minimizes the expected makespan for the subproblem. This ecpected makespan enters as an additive term into the second part, which consists in minimizing the expectation of max $\{t_1, \ldots, t_{10}\}$. Because of the special structure of Θ_0, this reduces then to minimizing the expected makespan of the subproblem associated with jobs $\{6, \ldots, 10\}$.

Starting with the second subproblem, the comparatively long expected duration of job 9 necessitates a procedure that guarantees the earliest possible start of this job, even at the price of postponing the start of job 10 to the completion of either one of jobs 7 or 8. That means *preselectivity* [cf. *Möhring/Radermacher/Weiss*, 1984] with selection 10 on the forbidden set $\{7, 8, 10\}$ and leads to an optimal MES strategy for this subproblem. Note however that *no* priority-induced strategy will ever behave that way. Due to its *greediness*, it will always start activities 6 and 10 after completion of jobs $\{1, \ldots, 5\}$ instead of leaving one machine *idle*, temporarily.

With regard to the subproblem induced by jobs $1, \ldots, 5$, the expected duration and influence on the project cost of jobs 1 and 2 are comparatively so great that these jobs must be started at time 0. Next important are jobs 3, 4, 5 in this order (due to their respective impact on project cost and their identical behaviour otherwise). Job 5 has almost no influence on the cost function, except for the extremly rare case that job 1 ends before job 2. In that rare case, however, job 5 should be started as soon as possible. Altogether this means that the optimal strategy for the first subproblem is induced by the following set-type priority rule:

(i) $1 < 2 < 3 < 4 < 5$ if $x_2 < x_1$ (most probable case)

(ii) $1 < 2 < 5 < 3 < 4$ if $x_1 \leqslant x_2$

Note that the induced strategy is not an MES strategy, as it is not preselective on the forbidden set $\{3, 4, 5\}$. In fact, in case (i) only activity 5 acts as a waiting job, while it is activity 4 otherwise.

Altogether then, neither MES nor priority-induced strategies lead to optimality here; instead, a combination of the given set-type priority rule on jobs $\{1, \ldots, 5\}$ and the given MES strategy on jobs $\{6, \ldots, 10\}$ will suffice.

3.2 Analytical Properties and Quasi-Stability of Set Strategies

Example 3.1.5 demonstrated that MES strategies do not in general determine the optimal value within the class of set strategies. In view of the fact that all strategies in this class are *elementary* and of the representation theorems reported in Section 3 in *Möhring/Radermacher/Weiss* [1984], it becomes clear that optimal set strategies will, in general, be neither *continuous* nor *monotonically increasing*. This behaviour somehow reflects the Graham anomalies for list schedules in the deterministic case [*Graham*] as described in Section 2.2, and has some undesirable side effects. In particular, it means that set strategies do not show the rigid stability behaviour discussed in Section 3 of *Möhring/Radermacher/Weiss* [1984]. However, as was mentioned there already, the unavoidable instabilities will be closely related with *discrete* duration distributions. Certainly, instability w.r.t. a discrete data type is much more easily acceptable for real-life applications than instability w.r.t. "continuous" data such as distributions having a Lebesgue density. This point of view motivates the notion of *quasi-stability*, meaning that the original, rigid standard of stability is only required for the smaller class of "test" distributions P that have a Lebesgue density, [cf. *Möhring/Radermacher/Weiss*, 1984]. We will subsequently show that all set strategies have this property. This will be a consequence of another observation, viz. that set strategies are continuous on the complement of a finite number of hyperplanes, i.e. on the complement of a set of Lebesgue measure zero.

Theorem 3.2.1: Let Π be a set strategy for $[A, O_0, \mathbf{N}]$. Then there exists a partition of $\mathbf{R}^n_>$ into *finitely* many non-empty sets $Z_1, \ldots, Z_m$ such that

(i) each Z_i is a (not necessarily closed) convex polyhedron (even a cone)

(ii) for each Z_i there exists a (unique, feasible) interval order $\Theta_i = (A, O_i)$ such that $\Pi[x] = ES_{\Theta_i}[x]$ for all $x \in Z_i$, $i = 1, \ldots, m$.

Proof: As was described in Section 2.2 in *Möhring/Radermacher/Weiss* [1984], Π induces for each $x \in \mathbf{R}^n_>$ a sequence S of states $(B_0^*, B_0 \setminus B_0^*), (B_1^*, B_1 \setminus B_1^*), \ldots, (B_k^*, B_k \setminus B_k^*)$, where B_j^* and $B_j \setminus B_j^*$ denote the jobs finished and being performed at decision point t_j, $j \in \{0, \ldots, k\}$, respectively. Except for the initial state (ϕ, ϕ) and the final state (A, ϕ), states depend on x, as does k, where of course $k \leqslant n + 1$, due to the elementarity of Π. For any such sequence S, let $Z = Z_S := \{x \in \mathbf{R}^n_> \mid x$ induces the sequence S w.r.t. $\Pi\}$. As there can only be finitely many such sequences, it is sufficient to show that Z fulfills conditions (i) and (ii). This is done below.

(i): The result is shown by induction on $n = |A|$. For $n = 1$, the assertion is trivial. In the inductive step, let S be the sequence of states as given above, $A = \{\alpha_1, \ldots, \alpha_n\}$, $B_1^* = \{\alpha_1, \ldots, \alpha_r\}$ with $r \geqslant 1$ the set of completed jobs at the second decision point t_1 and $B_1 \setminus B_1^* = \{\alpha_{r+1}, \ldots, \alpha_s\}$ the set of jobs still being performed (where $B_1 \setminus B_1^* = \phi$ is possible). Let $B(t_1)$ denote the action taken by Π for x at t_1, i.e. the set of activities started at t_1, where $B(t_1) = \phi$ is possible if $B_1 \setminus B_1^* \neq \phi$. Note, that since Π is a set strategy, $B(t_1)$ is the same for all x that lead to the sequence S.

Consider, on Z, the restriction Π' of Π to the subproblem induced by $A \setminus B_1^*$. Note that in this subproblem, (remaining) activity durations on $B_1 \setminus B_1^*$ will be $x(\alpha) - x(\alpha_1) > 0$, where of course $x(\alpha_1) = \ldots = x(\alpha_r)$, due to the correspondence of Z to the sequence S with start set B_1 and first completed jobs B_1^*. Obviously, Π' will first start activities $B(1) \cup (B_1 \setminus B_1^*)$ and will induce for each $x \in Z$ the unique sequence $(\phi, \phi), (B_2^* \setminus B_1^*, B_2 \setminus B_2^*), \ldots, (B_k^* \setminus B_1^*, B_k \setminus B_k^*)$. By the inductive hypothesis, applied to the subproblem, one obtains that $Z' := \{(y_{r+1}, \ldots, y_s, y_{s+1}, \ldots, y_n) \in \mathbf{R}_>^{n-r} \mid (y_{r+1}, \ldots, y_n) \text{ induces } S\}$ is a convex cone in $\mathbf{R}_>^{n-r}$. Thus there exists a *finite* homogeneous system L' of (possibly strict and non-strict) linear inequalities in the variables $y_{r+1}, \ldots, y_n$, whose feasibility domain is Z'. Now observe that, by construction, the following representation of Z is possible:

$$Z = \{(x_1, \ldots, x_n) \in \mathbf{R}_>^n \mid x_j = x_1; j = 1, \ldots, r; x_j = y_j + x_1, j = r+1, \ldots, s;$$
$$x = y_j, j = s+1, \ldots, n \text{ for some } (y_{r+1}, \ldots, y_n) \in Z'\}.$$

Thus one obtains a description L of Z by a *finite* system L of linear inequalities (which is again homogeneous) by replacing all variables y_j, $j = r+1, \ldots, s$ (in the case of $B_1 \setminus B_1^* \neq \emptyset$) by $y_j + x_1$, and by adding the linear constraints $x_2 = x_1, \ldots, x_r = x_1$. This shows that Z is a convex polyhedron (even a cone).

(ii): As was already mentioned in Sec. 2.4, any elementary strategy has a representation $\Pi[x] = ES_\Theta[x]$, $x \in Z(\Theta)$, $\Theta \in \mathbf{O}$, where $\mathbf{O}$ is a suitable set of feasible interval orders, and where $x \in Z(\Theta)$ iff $\Theta_{\Pi[x]} = \Theta$. Here, $\Theta_{\Pi[x]}$ denotes the unique interval order induced by the feasible schedule $\Pi[x]$ in the sense of Section 3.2 in *Möhring/Radermacher/Weiss* [1984], where the feasibility of Θ immediately follows from the feasibility of $\Pi[x]$. As was also mentioned, the inequality $ES_\Theta[x] \leqslant \Pi[x]$ generally holds. Now for elementary strategies, *equality* is obtained as follows by induction on Θ.

Assume that $ES_\Theta[x](\alpha) < \Pi[x](\alpha)$ for some $\alpha \in A$, while equality holds for all predecessors of α w.r.t. Θ. Note that $(\beta, \alpha) \in 0$ (for $\beta \neq \alpha$) iff $\Pi[x](\beta) + x(\beta) \leqslant \Pi[x](\alpha)$. As Π is elementary, $\Pi[x](\alpha) = \Pi[x](\beta_0) + x(\beta_0)$ for some $\beta_0 \in A$. Obviously, $\beta_0 \in V_\Theta(\alpha)$, and by the inductive hypothesis, $ES_\Theta[x](\alpha) \geqslant ES_\Theta[x](\beta_0) + x(\beta_0) = \Pi[x](\beta_0) + x(\beta_0) = \Pi[x](\alpha) > ES_\Theta[x](\alpha)$, a contradiction. Consequently, Π has a representation as described.

Altogether, it therefore suffices to show that all elements $x \in Z$ in (i) lead to the *same* interval order $\Theta_{\Pi[x]}$. To this end, note that the given sequence S provides the complete information about which maximal sets of activities are processed simultaneously at some time, these sets constituting antichains in an interval representation of $\Theta_{\Pi[x]}$. In chronological order, these special antichains occur as the sets $B(0)$, $B(1) \cup (B_1 \setminus B_1^*), B(2) \cup (B_2 \setminus B_2^*), \ldots, B(k-1) \cup (B_{k-1} \setminus B_{k-1}^*)$. As is well-known, this sequence uniquely defines $\Theta_{\pi[x]}$ [cf. *Golumbic*], concluding the proof. ∎

Remark 3.2.2: Note that the maximal antichains (layers) in the above sequence obviously have the *consecutive ones property* mentioned in Sec. 2.4, which gives another proof of Θ being an interval order. Note also that if all sets in the above sequence are layers, Π is greedy on Z, i.e. Π is (set-type) priority-induced on Z. The converse of this is not in general true (i.e. even if Π is priority-induced, not all sets of the sequence need be layers).

It should, however, be mentioned that for suitably chosen x and for priority-induced Π, a *complete correspondance* between induced sequences and feasible interval orders can be obtained. To this end, let the layers of a feasible interval order be given consecutively as $U_1, \ldots, U_k$. Putting $B_0^* = B_0 = \phi, B(0) := U_1$, and $B_j^* := B_{j-1}^* \cup (U_{j-1} \setminus U_j), B_j \setminus B_j^* = U_j \cap U_{j-1}, B(t_j) = U_j \setminus U_{j-1}$ for $j = 2, \ldots, n-1$, then gives a sequence S of states and actions consisting of the layers of Θ, and Π will generate this sequence for x defined by $x(\alpha) = |\{j \in \{1, \ldots, k\} \mid \alpha \in U_j\}|$, (and consequently then also for any other $x \in Z_S$).

Theorem 3.2.1 means that every set strategy behaves as an ES strategy on each of finitely many *convex cones* forming a partition of $\mathbf{R}^n_>$. Consequently, bad behaviour such as discontinuity or non-monotonicity can only occur on the boundaries of these polyhedra, i.e. obviously only on a finite number of *hyperplanes.* Note that this yields considerable insight into the possible occurrence of *Graham anomalies* [*Graham*] for set-type priority rules, which here turn out to be restricted to these boundaries. Now, taking the union H of all these finitely many hyperplanes obviously yields a *closed set of Lebesgue measure zero*. Altogether, this leads to the following corollary.

Corollary 3.2.3: Let Π be a set strategy for $[A, O_0, \mathbf{N}], Z_1, \ldots, Z_k$ the associated convex cones and H the associated union of boundaries of Lebesgue measure zero. Then:

(1) Π is piecewise well-behaved, i.e. Π is convex (thus subadditive), monotonically increasing, and positively homogeneous on each (open) set $Z_i \setminus H, i = 1, \ldots, k$.

(2) Π is uniformly continuous on the open set $\mathbf{R}^n_> \setminus H$, i.e. in particular, Π is λ^n*-almost everywhere continuous* on $\mathbf{R}^n_>$.

Corollary 3.2.3 is essential for obtaining the intended result on quasi-stability of set strategies. Note that in view of the stability conjecture in *Möhring/Radermacher/Weiss* [1984], quasi-stability should be expected for the optimal value (and for ϵ-optimality) if P has a λ^n-almost everywhere continuous density. The particular feature of set strategies is therefore that *each* individual set strategy is quasi-stable. The proof shows how λ^n-almost-everywhere continuity yields quasi-stability. Note in particular that we allow here arbitrary regular performance measures κ (without *measurability condition*).

Theorem 3.2.4: The class of set strategies is *quasi-stable*, i.e., given $[A, O_0, \mathbf{N}, P, \kappa]$, where P has a Lebesgue-density, the following properties hold for each weakly convergent sequence $(P_j)_{j \in \mathbf{N}} \to P$ of joint distributions for which $\Sigma(x_1, \ldots, x_n) := x_1 + \ldots + x_n$ is uniformly integrable, and for each uniformly convergent sequence

$(\kappa_j)_{j\in\mathbf{N}} \to \kappa$ of cost functions:

1. For any set strategy Π:
   (i) $(P_j)_{\kappa_j(\Pi,\cdot)}$ converges weakly to $P_{\kappa(\Pi,\cdot)}$.
   (ii) $\lim_{j\to\infty} \mathbf{E}_{P_j}[\kappa_j(\Pi,\cdot)] = \mathbf{E}_P[\kappa(\Pi,\cdot)]$.
2. W.r.t. optimality:
   (i) $\lim_{j\to\infty} \rho^{\text{SET}}(\kappa_j, P_j) = \rho^{\text{SET}}(\kappa, P)$.
   (ii) Let Π^j denote, for any $\epsilon > 0$ and any $j \in \mathbf{N}$, a set strategy that is ϵ-optimal in this class w.r.t. (κ_j, P_j). Then there is $j_0 \in \mathbf{N}$ such that for all $j \geqslant j_0$ Π^j is ϵ-optimal in this class w.r.t. $(\kappa.P)$.
   (iii) If Π is an optimal set strategy w.r.t. infinitely many $(\kappa_j, P_j)_{j\in\mathbf{N}}$, then Π is also optimal w.r.t. (κ, P). Furthermore, there is an optimal set strategy for (κ, P) having this property.

Proof: 1(i): The result uses Lemmas 4.2.4 and 4.2.5 in *Möhring/Radermacher/Weiss* [1984] and is a consequence of a quite general theorem on weak convergence, [cf. Theorem 5.5 in *Billingsley*]. Due to this criterion, it is sufficient to show that there is a measurable set N with $P(N) = 0$ which contains the set E of those $x \in \mathbf{R}^n_>$ for which $\kappa_j(\Pi, x_j) \to \kappa(\Pi, x)$ *fails* to hold for some sequence $(x_j)_{j\in\mathbf{N}}$ approaching x.

To apply this criterion, let N_1 denote the (measurable, [cf. p. 225f in *Billingsley*] set of discontinuities of Π, N_2' the (measurable) set of discontinuities of κ and $N_2 := (\Pi + id)^{-1}(N_2')$, where id denotes the identity on $\mathbf{R}^n_>$, i.e. $id(x) = x$. Then $\lambda^n(N_1) = \lambda^n(N_2') = 0$ because of Corollary 3.2.3 and Lemma 4.2.5 in *Möhring/Radermacher/Weiss* [1984], respectively. Taking into account Lemma 4.2.4 in *Möhring/Radermacher/Weiss* [1984] also yields $\lambda^n(N_2) = 0$ for any strategy Π, because otherwise $P_{(\Pi+id)}$ could not have a Lebesgue density, regardless of P. Altogether, putting $N := N_1 \cup N_2$, we obtain $\lambda^n(N) = 0$ as well as $P(N) = 0$, due to the fact that P has a Lebesgue density. It therefore suffices to show that $E \subseteq N$ in order to obtain 1 (i).

Let $x \notin N$. Then x is a continuity point of Π and $\Pi[x] + x$ is a continuity point of κ. Therefore, if $(x_j)_{j\in\mathbf{N}}$ is any sequence approaching x and if $y_j := \Pi[x_j] + x_j$, then y_j approaches $y := \Pi[x] + x$ and $\kappa(y_j)$ approaches $\kappa(y)$. This yields

$$\kappa(y) = \lim_{j\to\infty} \kappa_j(y) \overset{(*)}{=} \lim_{j\to\infty} \kappa_j(y_j),$$

where (*) is a direct consequence of uniform convergence (of sequences of functions)) as formulated e.g. in Theorem 7.11 in *Rudin*, concluding this part of the proof.
1 (ii): Note first that the uniform convergence $\kappa_j \to \kappa$ yields, for any $\epsilon > 0$, the existence of some $j(\epsilon) \in \mathbf{N}$ such that $|\kappa_j(z) - \kappa(z)| \leqslant \epsilon$ for all $z \in \mathbf{R}^n_>$ and all $j \geqslant j(\epsilon)$, implying $|\mathbf{E}_Q[\kappa_j(\Pi,\cdot)] - \mathbf{E}_Q[\kappa(\Pi,\cdot)]| \leqslant \epsilon$ for any such $j \geqslant j(\epsilon)$ and

any probability distribution Q on $(\mathbf{R}^n_>, \mathbf{B}^n_>)$. We can therefore straightforwardly restrict ourselves to the special case $\kappa_j \equiv \kappa$ for all $j \in \mathbf{N}$.

Because of the quite general Theorem 5.4 in *Billingsley* and 1 (i), it is then sufficient to show the *uniform integrability* of $\kappa(\Pi, \cdot)$ w.r.t. $(P_j)_{j\in\mathbf{N}}$. To see this, note that the assumed uniform integrability of Σ w.r.t. $(P_j)_{j\in\mathbf{N}}$, i.e. the validity of $\lim_{r\to\infty} \sup_{j\in\mathbf{N}} \int_{\{\Sigma \geqslant r\}} |\Sigma|\, dP_j = 0$, implies uniform integrability for each function $f: \mathbf{R}^n_> \to \mathbf{R}^n_\geqslant$, fulfilling $f \leqslant n_0 \cdot \Sigma + b_0$; $n_0, b_0 \in \mathbf{N}$ arbitrary but fixed. Since κ is linearly bounded in our model, i.e. $\kappa(x) \leqslant m_0 \cdot \Sigma(x) + c_0$ for some $m_0, c_0 \in \mathbf{N}$, and since $\Pi[x](\alpha) + x(\alpha) \leqslant x_1 + \ldots + x_n = \Sigma(x)$ for all $x \in \mathbf{R}^n_\geqslant$ and all $\alpha \in A$ because of (Sto 4), we obtain that $\kappa(\Pi, x) \leqslant \kappa(\Sigma(x), \ldots, \Sigma(x)) \leqslant m_0 \cdot n \cdot \Sigma(x) + c_0$, i.e. uniform integrability of $\kappa(\Pi, \cdot)$ w.r.t. $(P_j)_{j\in\mathbf{N}}$ for arbitrary κ and Π, concluding this part.

2 (i): This is an immediate consequence of 1 (ii) and the fact that there exists only a finite number of set strategies, reducing the proof to the possible interchange of the operations "limit" and "minimum".

2 (ii): If the statement were not correct, there would be an infinite subsequence $(\Pi^j)_{j\in I}$, $I \subseteq \mathbf{N}$ such that $\mathbf{E}_{P_j}[\kappa_j(\Pi^j, \cdot)] \leqslant \rho^{\text{SET}}(\kappa_j, P_j) + \epsilon$ but $\mathbf{E}_P[\kappa(\Pi^j, \cdot)] > \rho^{\text{SET}}(\kappa, P) + \epsilon$ for all such $j \in I$. Due to the fact that the number of set strategies is finite, we can w.l.o.g. assume that all $(\Pi^j)_{j\in I}$ equal one particular Π. This implies: $\rho^{\text{SET}}(\kappa, P) + \epsilon < \mathbf{E}_P[\kappa(\Pi, \cdot)] \overset{1(\text{ii})}{=} \lim_{j\in I} \mathbf{E}_P[\kappa_j(\Pi^j, \cdot)] \leqslant \lim_{j\to\infty} \rho^{\text{SET}}(\kappa_j, P_j) + \epsilon$
$j = \rho^{\text{SET}}(\kappa, P) + \epsilon$, a contradiction.

2 (iii): This statement follows immediately from 1 (ii). In particular, since the class of set strategies is *finite*, there is some set strategy which is optimal for infinitely many $(\kappa_j, P_j)_{j\in\mathbf{N}}$ and consequently also for (κ, P), concluding the proof. ■

Note that Theorem 3.2.4′shows *quasi stability* for a large class of discontinuous strategies for stochastic scheduling problems, considerably relaxing the requirements formulated in Theorem 3.1.2 in *Möhring/Radermacher/Weiss* [1984]. Of particular practical importance is again part 2 (ii), which tells us that "almost" optimal scheduling w.r.t. a "good" approximation $(\kappa_j, P_j)_{j\in\mathbf{N}}$ of the "unknown" data (κ, P) already yields "almost" optimal scheduling, also w.r.t. the correct data (κ, P).

With a view to the applicability of Theorem 3.2.4, it should again be noted that the required uniform integrability of Σ is not very restrictive. It holds, for example, if P has a bounded support, or if the approximating $(P_j)_{j\in\mathbf{N}}$ are all members of one of the usually encountered classes of distributions with Lebesgue-densities such as uniform, triangle, Beta, truncated normal or Erlang distributions. Particularly, the cases dealt with in Sec. 4 and in *Dempster/Lenstra/Rinnooy Kan*, i.e. those falling nicely into the Markov-decision framwork, are of this type, since e.g. all additive cost functions are continuous and products of exponential distributions have a Lebesgue-density. This last observation is all the more interesting because in this case there is even an overall optimal set strategy (Theorem 4.2.1), i.e. in this case we obtain in addition to Theorem 3.1.1 in *Möhring/Radermacher/Weiss* [1984], a stronger version of quasi-stability for the *overall* optimum value, too.

4. Overall Optimality of Set Strategies and the Shift Property

In the following we demonstrate that there is an overall optimal set strategy for an interesting class of stochastic scheduling problems. This overall optimality of one out of a finite number of rather simple strategies is related to the optimality of priority-induced strategies such as LEPT and SEPT in special problem classes, [cf. *Dempster; Weiss/Pinedo*]. In fact, the intention to fully exploit the nice features of priority-induced strategies in special cases was one of our aims, and led to the introduction of set strategies. In a feedback, this result on the optimality of set strategies initiated more general results on the optimality of e.g. LEPT and SEPT rules as will be given in the third paper of this series.

If we ask for appropriate conditions for optimality of set strategies, we will have to make sure that the sets of completed jobs and of the jobs currently being performed *"essentially"* describe the state of the problem at some time t. Set strategies will then just be the *stationary Markov strategies* and optimality will follow from standard results in stochastic dynamic optimization and semi-Markov decision theory [*Bertsekas/Shreve; Hinderer*, 1970; *Strauch*].

In view of Section 4.3 in *Möhring/Radermacher/Weiss* [1984], such a restriction means that at any decision time t, the optimal decision does not depend on usually important information such as the duration of completed jobs, the current duration of jobs being performed, the already known completion times and the present time t.

The first two types of information are inessential if we restrict the joint duration distributions to products of exponential distributions (for job durations restricted to $\mathbf{N}$, geometric distributions would do as well). In order to render the last two types of information inessential, one needs cost functions for which different future developments from time t onwards (induced by different strategies) depend on the history up to t only in such a way that the ordering among the different strategies w.r.t. expected future costs remains the same for all possible histories and futures (*strategy-equivalent* cost functions).

One well-known class of such cost functions are the *additive cost functions*, for which the history enters into the total cost only as an additive term $b = b\,(t_{i_1}, \ldots, t_{i_k}, t)$ depending on the completion times $t_{i_1} \leqslant \ldots \leqslant t_{i_k} \leqslant t$ and t. Preservation of the ordering w.r.t. expected cost follows for additive cost functions from the fact that, with $T(f) = f + b$, $b \in \mathbf{R}^1$,

$$\int f\, dP \leqslant \int g\, dP \text{ iff } \int T(f)\, dP \leqslant \int T(g)\, dP$$

for all probability measures P on $(\mathbf{R}^n_>, \mathbf{B}^n_>)$ and all measurable functions f, g: $\mathbf{R}^n_> \to \mathbf{R}^1_\geqslant$.

Now the most general transformations T with this property are exactly the *affine transformations* $T(f) = af + b$ $(a, b \in \mathbf{R}^1, a > 0)$, cf. Theorem 8.2 in *Fishburn*. So the most general cost functions, for which a general optimality result for set strategies can be expected, are cost functions for which the t-history enters the future cost development only via a (positive-) affine transformation (so-called *shift property*). We indeed obtain such an optimality result, which is related to the existence of optimal *stationary* strategies for stochastic dynamic optimization problems with additive cost criterion [*Bertsekas/Shreve; Hinderer,* 1970] (which is a much more general notion of additivity than the one used in this paper).

Before proving this theorem, we first give, – as a central aim of this paper –, the complete and quite involved characterization of the class of cost functions with the *shift property* and discuss their relationship with the additive cost functions.

4.1 The Shift Property

Definition: Let $\kappa: \mathbf{R}^n_\geqslant \to \mathbf{R}^1_\geqslant$ be a regular cost function. κ is said to have the *shift property* if for every vector $(t_1, \dots, t_n) \in \mathbf{R}^n_\geqslant$ of completion times, for every set $B \subset \{1, \dots, n\}$, and $t \in \mathbf{R}^1_\geqslant$ fulfilling $\max \{t_j \mid j \in B\} \leqslant t \leqslant \min \{t_j \mid j \notin B\}$ where $\max_\phi := 0)$

$$\kappa(t_1, \dots, t_n) = a \cdot \kappa((t_1 - t)^+, \dots, (t_n - t)^+) + b,$$

where $(t_j - t)^+ = t_j - t$ if $t_j \geqslant t$ and 0 otherwise, and $a, b \in \mathbf{R}^1_\geqslant$ depend only on the history up to t, i.e. $a = a_B(t, t_j \mid j \in B)$ and $b = b_B(t, t_j \mid j \in B)$.

Note that B may be viewed as the set of jobs already completed by time $\max \{t_j \mid j \in B\}$, with no further completion occurring in the time up to t. The shift property then states that the future problem, starting at the present time t, can w.r.t. cost aspects essentially – i.e. up to an affine transformation – be handled as if $t = 0$ and as if all jobs $\alpha_j, j \in B$ were eliminated. Elimination here means taking $t_\alpha = 0$. To obtain this representation is also the reason why in this context κ must be defined on $\mathbf{R}^n_\geqslant$ and not, as above, on $\mathbf{R}^n_>$ only.

The following lemma gives an initial insight into which cost functions have the shift property. Note that again costs involving *tardiness* will generally not have this property.

Lemma 4.1.1:

(1) Let κ have the shift property and $c \in \mathbf{R}^1_\geqslant$ be any constant. Then $\kappa + c$ also has the shift property. In particular, any cost function κ' with the shift property is of the type $\kappa + c$, where $\kappa(0, \dots, 0) = 0$ and $c \in \mathbf{R}^1_\geqslant$. Note that κ is uniquely determined by κ' and that the respective multiplicative terms in the shift representation are the same.

(2) Let $\kappa: \mathbf{R}^n_{\geqslant} \to \mathbf{R}^1_{\geqslant}$ have the shift property, and let $\phi \subset D \subset A := \{1, \ldots, n\}$. Then the projection κ^D of κ onto the components of D, i.e. the function $\kappa^D: \mathbf{R}^{|D|}_{\geqslant} \to \mathbf{R}^1_{\geqslant}$ with $\kappa^D (x_D) = \kappa (x_D, O_{A \setminus D})$, has the shift property, and $\bar{\kappa}: \mathbf{R}^1_{\geqslant} \to \mathbf{R}^1_{\geqslant}$ with $\bar{\kappa} (t) := \kappa (t, \ldots, t)$ has the shift property.

(3) Let $\kappa: \mathbf{R}^n_{\geqslant} \to \mathbf{R}^1_{\geqslant}$ be a regular cost function with the shift property and let $\kappa (0, \ldots, 0) = 0$. If $\kappa (x) = \kappa (y)$ for $x, y \in \mathbf{R}^n_{\geqslant}$ with $x_i < y_i$, $i = 1, \ldots, n$, then κ is constant on $\mathbf{R}^n_{\geqslant}$. [Note, however, that in this case the projections κ^D of κ may have different values or may even be strictly monotonically increasing].

(4) Each additive regular cost function κ fulfils $\kappa (0, \ldots, 0) = 0$ and has the shift property with $a_B (t, t_j \mid j \in B) = 1$ for all B and $t_j, j \in B$.

(5) Let κ be any additive cost function with $\kappa (0, \ldots, 0) = 0$ and $\kappa' (t_1, \ldots, t_n) := e^{\kappa(t_1, \ldots, t_n)} - 1$, then $\kappa' (0, \ldots, 0) = 0$, κ' is regular and κ' is not additive but has the shift property.

(6) Let $c_B \in \mathbf{R}^1_{\geqslant}$ for all $\phi \neq B \subseteq A$ and put

$$\kappa (t_1, \ldots, t_n) := \sum_{\phi \neq B \subseteq A} c_B \cdot (e^{\lambda \cdot \max_{\alpha \in B} t_\alpha} - 1), \lambda > 0,$$

then $\kappa (0, \ldots, 0) = 0$, κ is regular and κ has the shift property (but is not additive).

(7) Let $A = \{1, 2\}$ and put

$$\kappa (t_1, t_2) := \begin{cases} a (e^{\lambda t_1} - 1) + e^{\lambda t_1} a_1 (e^{\lambda_1 (t_2 - t_1)} - 1) & t_1 \leqslant t_2 \\ a (e^{\lambda t_2} - 1) + e^{\lambda t_2} a_2 (e^{\lambda_2 (t_1 - t_2)} - 1) & t_1 \geqslant t_2 \end{cases},$$

where $\lambda, \lambda_1, \lambda_2, a, a_1, a_2 \in \mathbf{R}^1_{\geqslant}$ with $\lambda \geqslant \lambda_1, \lambda_2$ and $a_1, a_2 > 0$. Then $\kappa (0, \ldots, 0) = 0$, κ is regular and κ has the shift property (but is not additive).

(8) Let $g, f: \mathbf{P}(A) \to \mathbf{R}^1$ with $g (\phi) = f (\phi) = 0$ be set functions and define, given ordered completion times $t_{i_1} \leqslant t_{i_2} \leqslant \ldots \leqslant t_{i_n}$, for jobs $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}$,

$$\kappa (t_1, \ldots, t_n) = g (A) \cdot (e^{f(A) t_{i_1}} - 1) + e^{f(A) t_{i_1}} g (A_{i_1}) (e^{f(A_{i_1})(t_{i_2} - t_{i_1})} - 1)$$

$$+ \ldots + e^{f(A) t_{i_1} + f(A_{i_1})(t_{i_2} - t_{i_1}) + \ldots + f(A_{i_{n-1}})(t_{i_{n-1}} - t_{i_n})}$$

$$\cdot g (A_{i_{n-1}}) \cdot (e^{f(A_{i_{n-1}})(t_{i_n} - t_{i_{n-1}})} - 1),$$

where $A = \{i_1, \ldots, i_n\}$ and $A_{i_k} = A \setminus \{i_1, \ldots, i_k\}$, $k = 1, \ldots, n-1$.

Then $\kappa (0, \ldots, 0) = 0$, and κ has the shift property (but is not additive). Moreover, κ is regular iff f and g are monotonically increasing.

Proof: (We only give hints on the non-trivial parts)

(1): It is sufficient to show the following: If κ' has the shift property, then

$\kappa(t_1, \ldots, t_n) := \kappa'(t_1, \ldots, t_n) - \kappa'(0, \ldots, 0)$ fulfills $\kappa \geqslant 0$, $\kappa(0, \ldots, 0) = 0$, κ is regular and κ has the shift property. In fact, κ fulfills the shift property with the same a_B as κ' and a modified b_B.

(2) is obvious, since the functions a_B^D and b_B^D of κ^D are just the D-projections of the functions a_B and b_B for κ. For $\bar{\kappa}$, we obtain $\bar{a}(t) = a_\phi(t, \ldots, t)$ and $\bar{b}(t) = b_\phi(t, \ldots, t)$.

(3): We first show the assertion for $n = 1$. Assume first that $\kappa(x) = 0$ for some $x > 0$. Let $x_0 := \sup\{x \geqslant 0 \mid \kappa(x) = 0\}$. If $x_0 < \infty$, let $0 < t < x_0$ and $x > x_0$ such that $0 < x - t < x_0$. Then the shift property yields

$$0 < \kappa(x) = a_\phi(t)\kappa(x - t) + b_\phi(t) = b_\phi(t) \text{ and}$$

$$0 = \kappa(t) = a_\phi(t)\kappa(0) + b_\phi(t) \qquad = b_\phi(t),$$

a contradiction. Hence $\kappa \equiv 0$.

Assume now that $\kappa(x) > 0$ and that there are $x_1 \neq x_2$ with $\kappa(x_1) = \kappa(x_2) = c > 0$. Then κ', defined by $\kappa'(x) := \kappa(x_1 + x) - c$ is again a regular function with the shift property and $\kappa'(0) = 0$. Since $\kappa'(x_2 - x_1) = 0$, we obtain by the first part of the proof that $\kappa' = 0$. Hence $\kappa(x) = c$ for all $x \geqslant x_1$.

Now let $x_1^* = \inf\{x \geqslant 0 \mid \kappa(x) = c\}$. We must show that $x_1^* = 0$. Note first that $b_\phi(t) = \kappa(t)$ for all $t \geqslant 0$, since $\kappa(t) = a_\phi(t) \cdot \kappa(0) + b_\phi(t)$ and $\kappa(0) = 0$. If $x_1 > x_1^*$, we obtain for all $t \geqslant 0$ that $c = \kappa(x_1 + t) = a(t)\kappa(x_1) + \kappa(t) = c \cdot a(t) + \kappa(t)$, i.e. $a(t) = (c - \kappa(t))/c$. Choose $x < x_1^*$ with $2x > x_1^*$. Then $c = \kappa(2x) = a(x)\kappa(x) + \kappa(x) = \dfrac{c - \kappa(x)}{c}\kappa(x) + \kappa(x)$ yields the quadratic equation $\kappa(x)^2 - 2c\kappa(x) + c^2 = 0$ which has the unique solution $\kappa(x) = c$. Since $x < x_1^*$, we obtain that $x_1^* = 0$.

In the n-dimensional case, assume that $\kappa(x^0) = \kappa(y^0)$ for $x^0, y^0 \in \mathbf{R}^n_\geqslant$ with $x_i^0 < y_i^0$, $i = 1, \ldots, n$. If the interval $[x^0, y^0] \subseteq \mathbf{R}^n_\geqslant$ has a non-trivial intersection I (i.e. $|I| > 1$) with the diagonal $D := \{x \in \mathbf{R}^n_> \mid x_1 = \ldots = x_n\}$, one obtains that $\bar{\kappa}$ is constant on $\{t \in \mathbf{R}^1 \mid (t, \ldots, t) \in I\}$. Hence $\bar{\kappa}$ is constant on $\mathbf{R}^1_>$, implying that κ is constant on $\mathbf{R}^n_>$. If $|I| \leqslant 1$, consider κ^* defined by $\kappa^*(x) := \kappa(x + x^0) - \kappa(x^0)$. κ^* has again the shift-property and $\kappa^*(0) = \kappa^*(y^0 - x^0)$. The above arguments then yield $\bar{\kappa}^* \equiv 0$. Hence $\kappa^* \equiv 0$ on $\mathbf{R}^n_\geqslant$.

Then one obtains $\kappa(y^*) = \kappa^*(y^* - x^0) = \kappa^*(0) = \kappa(x^0)$ for each $y^* \geqslant x^0$. Choosing y^* large enough yields $|[x^0, y^*] \cap D| > 1$, and the above argument can be applied to κ. Hence $\kappa \equiv c$ on $\mathbf{R}^n_>$ for some constant c.

However, the projections κ^D of κ may be different from c. (As an example, define $\kappa: \mathbf{R}^2_{\geqslant} \to \mathbf{R}^1$ by

$$\kappa(x_1, x_2) := \begin{Bmatrix} 1 & x_1 > 0,\ x_2 > 0 \\ 1 - e^{\lambda x_1} & x_1 \geqslant 0,\ x_2 = 0 \\ 1/2 & x_1 = 0,\ x_2 > 0 \end{Bmatrix}. \quad)$$

(4): As described in Section 2.1, additive cost functions are, given ordered completion times $t_{i_1} \leqslant t_{i_2} \leqslant \ldots \leqslant t_{i_n}$, for jobs $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}$, defined by

$$\kappa(t_1, \ldots, t_n) = t_{i_1} \cdot g(A) + (t_{i_2} - t_{i_1}) \cdot g(A_{i_1}) + \ldots + (t_{i_n} - t_{i_{n-1}}) \cdot g(A_{i_{n-1}}),$$

where $A_{i_r} := A \setminus \{i_1, i_2, \ldots, i_r\}$ for $r = 1, \ldots, n-1$, and $g: \mathbf{P}(A) \to \mathbf{R}^1_{\geqslant}$ is monotonically increasing in order to guarantee regularity.

Now let $|B| = r < n$, where we can restrict ourselves w.l.o.g. to the case $B \neq \phi$. Obviously then $B = \{i_1, \ldots, i_r\}$, i.e. B consists of the r jobs completed first. We consequently have $t_{i_r} \leqslant t \leqslant t_{i_{r+1}}$ and the cost occurring in the time intervals $[t_{i_r}, t]$ and $[t, t_{i_{r+1}}]$ are obviously $(t - t_{i_r}) \cdot g(A_{i_r})$ and $(t_{i_{r+1}} - t) \cdot g(A_{i_r})$, respectively. Therefore we obtain

$$\begin{aligned} \kappa(t_1, \ldots, t_n) &= t_{i_1} \cdot g(A) + \ldots + (t_{i_r} - t_{i_{r-1}}) \cdot g(A_{i_{r-1}}) + (t - t_{i_r}) \cdot g(A_{i_r}) \\ &\quad + (t_{i_{r+1}} - t) \cdot g(A_{i_r}) + \ldots + (t_{i_n} - t_{i_{n-1}}) \cdot g(A_{i_{n-1}}) \\ &= [t_{i_1} \cdot (g(A) - g(A_{i_1})) + \ldots + t_{i_r} \cdot (g(A_{i_{r-1}}) - g(A_{i_r})) + t \cdot g(A_{i_r})] \\ &\quad + [(t_{i_{r+1}} - t) \cdot g(A_{i_r}) + \ldots + (t_{i_n} - t_{i_{n-1}}) \cdot g(A_{i_{n-1}})]. \end{aligned}$$

Obviously, the expression in the first (square) brackets is an additive term b depending only on t and the values $t_j, j \in B$. Also, the expression in the second brackets is easily identified as $\kappa(t'_1, \ldots, t'_n)$. Thus choosing $a \equiv 1$ yields the intended representation.

(5): By (3), $a \equiv 1$ for additive cost functions. This implies the shift property for κ', as the functions a'_B for κ' can be chosen as e^{b_B}, while the functions b'_B for κ' are of the form $(e^{b_B} - 1)$, b_B denoting the respective additive terms for κ.

(6): Let $\phi \neq B' \subset A$ be arbitrary and $\max\{t_j \mid j \in B'\} \leqslant t \leqslant \min\{t_j \mid j \notin B'\}$. Then

$$\begin{aligned} \kappa(t_1, \ldots, t_n) &= \sum_{\substack{\phi \neq B \subseteq A \\ B \nsubseteq B'}} c_B \cdot (e^{\lambda \max_{\alpha \in B} t_\alpha} - 1) + \sum_{\phi \neq B \subseteq B'} c_B \cdot (e^{\lambda \max_{\alpha \in B} t_\alpha} - 1) \\ &= e^{\lambda t} \cdot \Big(\sum_{\substack{\phi \neq B \subseteq A \\ B \nsubseteq B'}} c_B \cdot (e^{\lambda \max_{\alpha \in B} (t_\alpha - t)^+} - 1) \end{aligned}$$

$$+ \Big[\sum_{\substack{\phi \neq B \subseteq A \\ B \nsubseteq B'}} c_B \cdot (e^{\lambda t} - 1) + \sum_{\phi \neq B \subseteq B'} c_B \cdot (e^{\lambda \max_{\alpha \in B} t_\alpha} - 1) \Big]$$

Obviously, this is the representation required.

(7): Put $a = a_\phi(t) = e^{\lambda t}$ and $b = b_\phi(t) = a(e^{\lambda t} - 1)$ as well as $a = a_{\{i_1\}}(t_{i_1}, t) \cdot e^{\lambda \cdot t_{i_1}} \cdot e^{\lambda_{i_1} \cdot (t - t_{i_1})}$ and $b = b_{\{i_1\}}(t_{i_1}, t) = a_1 \cdot e^{\lambda t_{i_1}} \cdot e^{\lambda_{i_1}(t - t_{i_1})} - a_1 \cdot e^{\lambda t_{i_1}} + a(e^{\lambda t_{i_1}} - 1)$ where $t_{i_1} \leqslant t \leqslant t_{i_2}$ and $i_1 \in \{1, 2\}$, respectively.

(8) generalizes (7) by induction. ■

Remark 4.1.2: Lemma 4.1.1 exhibits a variety of cost functions having the shift property. By part (1) the study of such functions may essentially be restricted to the case that $\kappa(0, \ldots, 0) = 0$. Part (2) shows that projections and $\bar{\kappa}$ again have the shift property. This fact will be used in the inductive proof of Theorem 4.1.3. Part (3) shows that regular cost functions with the shift property are usually *strictly monotonically increasing* as long as all activities are still being processed. After a completion, however, the corresponding subfunction may be constant, cf. the characterization in Theorem 4.1.3. The only exceptions to this behaviour (within the class of monotonically increasing functions) are functions that are constant on $\mathbf{R}^n_>$. In that case, the projections may be chosen arbitrarily as long as the shift property and the regularity are preserved. This also shows that (regular) functions with the shift property may still be *discontinuous*, necessitating the more involved argumentation occuring in the proof of the complete characterization in Theorem 4.1.3. Note that the mentioned cases will in fact turn out to be the only discontinuous (regular) functions occuring. Thus, in particular, all functions having the shift property are continuous on $\mathbf{R}^n_>$, i.e. the rare occurance of discontinuities is restricted to cases that involve certain job durations to be zero. Part (4) shows that the additive cost functions are a subclass fulfilling $a_B \equiv 1$. We will see below that this property characterizes the additive case (as do uniform continuity in the unbounded case and positive homogeneity).

Part (5) shows that the shift property also holds for all *discounted* variants of the additive case, while (6) and also (7) show that these do still not give all functions with the shift property. Note that here a multiplicative constant depending on the set B but *not* on the order in which the activities of B are completed) is involved.

Part (7) then shows that the multiplicative term a_B in general depends on the order of completions in B. Finally, note that (8) includes all cases (4) – (7). In fact, it will turn out below that (8) gives *all* functions having the shift property with $a_B \neq 1$ for all projections of κ.

Using ideas from the theory of functional equations [cf. *Aczél*], we next give a characterization of all cost functions having the shift property. This characterization is a basic aim of this paper and needs a quite sophisticated argumentation. Note that a major problems arises from the fact that we do not know that κ is continuous, let alone *differentiable.* To overcome this difficulty, we will essentially rely on regularity, in particular on known properties of monotonic functions $f\colon \mathbf{R}^1 \to \mathbf{R}^1$, [cf. e.g. *Royden; Rudin*].

Theorem 4.1.3: Let κ be any regular cost function with the shift property such that $\kappa(0, \ldots, 0) = 0$ and κ is not constant on $\mathbf{R}^n_>$. Then:

(1) κ is continuous and has the representation

$$\begin{aligned}\kappa(t_1, \ldots, t_n) = {} & [g(A)\, t_{i_1} + h(A)(\exp[f(A)\, t_{i_1}] - 1)] \\ & + \exp[f(A)\, t_{i_1}]\, [g(A_{i_1})(t_{i_2} - t_{i_1}) + h(A_{i_1})(\exp[f(A_{i_1}) \\ & \quad (t_{i_2} - t_{i_1})] - 1)] \\ & + \ldots + \exp[f(A)\, t_{i_1} + \ldots + f(A_{i_{k-1}})(t_{i_k} - t_{i_{k-1}})] \\ & \cdot [g(A_{i_k})(t_{i_{k+1}} - t_{i_k}) + h(A_{i_k})(\exp[f(A_{i_k})(t_{i_{k+1}} - t_{i_k})] - 1)] \\ & + \ldots + \exp[f(A)\, t_{i_1} + \ldots + f(A_{i_{n-2}})(t_{i_{n-1}} - t_{i_{n-2}})] \\ & \cdot [g(A_{i_{n-1}})(t_{i_n} - t_{i_{n-1}}) + h(A_{i_{n-1}})(\exp[f(A_{i_{n-1}})(t_{i_n} - t_{i_{n-1}})] \\ & - 1)]\end{aligned}$$

where $t_{i_1} \leqslant t_{i_2} \leqslant \ldots \leqslant t_{i_n}$ are the ordered completion times, $A_{i_n} := A \setminus \{i_1, \ldots, i_k\}$, $f, g, h\colon \mathbf{P}(A) \to \mathbf{R}^1$ are set functions which fulfil the orthogonality condition $g(B) \cdot f(B) = 0, B \in \mathbf{P}(A)$, and $\exp x = e^x$.

(2) The following statements are equivalent:

- (i) κ is additive,
- (ii) $a_B \equiv 1$ in the shift equation for all $\phi \subset B \subset A$,
- (iii) κ is unbounded and uniformly continuous.
- (iv) κ is positively homogeneous (i.e. $\kappa(\lambda x) = \lambda \cdot \kappa(x)$ for all $x \in \mathbf{R}^n_\geqslant$ and all $\lambda \geqslant 0$).

Proof: Let κ be any cost function as considered. Then Lemma 4.1.1 implies that κ and thus also $\bar{\kappa}$ (with $\bar{\kappa}$ defined as in Lemma 4.1.1) are *strictly* monotonically increasing on $\mathbf{R}^n_>$ and $\mathbf{R}^1_>$, respectively. Hence $\bar{\kappa}$ is λ^1-almost everywhere differentiable, [cf. *Royden*].

Applying the shift property for $t_1 = t_2 = \ldots = t$ yields $b_\phi(t) = \bar{\kappa}(t)$, i.e. for $t \leqslant \min_{\alpha \in A} t_\alpha$, $\kappa(t_1, \ldots, t_n) = a(t) \cdot \kappa((t_1 - t), \ldots, (t_n - t)) + \bar{\kappa}(t)$. This has two immediate consequences:

First, $a := a_\phi$ is a measurable function of t. To see this, let $T > 0$. Then $\bar{\kappa}(T) > 0$ and the shift property yields $a(t) = \frac{1}{\bar{\kappa}(T)}[\bar{\kappa}(T+t) - \bar{\kappa}(t)]$, $t \in \mathbf{R}^1_{\geqslant}$. Since $\bar{\kappa}(T+t)$ and $\bar{\kappa}(t)$ are monotone functions of t, $a(t)$ is measurable [cf. *Royden*].

Second, we obtain the *functional equation* $a(s+t) = a(s) \cdot a(t)$ for all $s, t \in \mathbf{R}^1_{\geqslant}$, which together with the shown measurability of $a(t)$ implies $a(t) = e^{\lambda t}$, [cf. *Ross*]. To see this, let $s + t \leqslant \min_{\alpha \in A} t_\alpha$. Then

$$\begin{aligned}\bar{\kappa}(x+s+t) &= a(s) \cdot \bar{\kappa}(x+t) + \bar{\kappa}(s)\\ &= a(s) \cdot a(t) \cdot \bar{\kappa}(x) + a(s) \cdot \bar{\kappa}(t) + \bar{\kappa}(s)\\ &= a(s) \cdot a(t) \cdot \bar{\kappa}(x) + \bar{\kappa}(s+t) \qquad \text{as well as}\\ \bar{\kappa}(x+s+t) &= a(s+t) \cdot \bar{\kappa}(x) + \bar{\kappa}(s+t),\end{aligned}$$

where x is chosen such that $\bar{\kappa}(x) > 0$. Altogether, we therefore obtain for $t \leqslant \min_{\alpha \in A} t_\alpha$:

$$\kappa(t_1, \ldots, t_n) = e^{\lambda t}(\kappa(t_1 - t, \ldots, t_n - t)) + \bar{\kappa}(t).$$

Case 1: Assume $\lambda = 0$. We then have the *functional equation* $\bar{\kappa}(s+t) = \bar{\kappa}(s) + \bar{\kappa}(t)$, implying $\bar{\kappa} = c \cdot t$ with $c \in \mathbf{R}^1_{\geqslant}$ [cf. *Ross*] (and the fact that $\bar{\kappa}(0) = 0$). It is straightforwardly obtained by induction that assuming $\lambda = 0$ also for all projections of κ (cf. Lemma 4.1.1) yields all additive cost functions. Obviously, these have all the properties stated in (2).

Case 2: Assume $\lambda \neq 0$. We first consider the one-dimensional case, i.e. the behaviour of $\bar{\kappa}$.

We first show that $\bar{\kappa}$ is differentiable. To this end, let t_1 be an (existing) point, where the derivative of $\bar{\kappa}$ exists. Then, by the shift property,

$$\bar{\kappa}'(t_1)_+ = \lim_{t \downarrow 0} \frac{\bar{\kappa}(t_1 + t) - \bar{\kappa}(t_1)}{t} = \left(\lim_{t \downarrow 0} \frac{e^{\lambda t} - 1}{t}\right) \cdot \bar{\kappa}(t_1) + \lim_{t \downarrow 0} \frac{\bar{\kappa}(t) - \bar{\kappa}(0)}{t},$$

implying that the right derivative of $\bar{\kappa}$ for $t = 0$ exists. This yields the existence of the right derivative for each $t_2 \in \mathbf{R}^1_{\geqslant}$, since

$$\bar{\kappa}'(t_2)_+ = \lim_{t \downarrow t_2} \frac{\bar{\kappa}(t_2 + t) - \bar{\kappa}(t_2)}{t} = \lambda \cdot \bar{\kappa}(t_2) + \bar{\kappa}'(0).$$

We therefore only need to show the same behaviour for left derivatives. To this end, assume first that $\bar{\kappa}$ is continuous from the left. Then

$$\bar{\kappa}'(t_2)_- = \lim_{t \downarrow 0} \frac{\bar{\kappa}(t_2) - \bar{\kappa}(t_2 - t)}{t} = \lim_{t \downarrow 0} \left(\frac{e^{\lambda t} - 1}{t} \cdot \bar{\kappa}(t_2 - t) + \kappa'(0) \right)$$

$$= \lambda \cdot \bar{\kappa}(t_2) + \bar{\kappa}'(0),$$

i.e. we obtain the stated behaviour. To see that $\bar{\kappa}$ is continuous from the left just note that a discontinuity from the left at a particular point $t_0 \neq 0$ would yield that $\bar{\kappa}$ is nowhere continuous from the left because of the shift property. This contradicts $\bar{\kappa}$ being monotonical.

Thus, altogether, $\bar{\kappa}'$ exists everywhere on $\mathbf{R}^1_{\geqslant}$. This then implies that $\bar{\kappa}$ has to fulfil the differential equation $\bar{\kappa}'(t) - \lambda \cdot \bar{\kappa}(t) = \bar{\kappa}'(0)$, for which *all* solutions are given by $\bar{\kappa}(x) = c \cdot (e^{\lambda x} - 1), x \geqslant 0, c \in \mathbf{R}^1$. The cases $c > 0, \lambda > 0$ and $c < 0, \lambda < 0$ then yield all strictly monotonically increasing one-dimensional cost functions with the shift property for this case.

In the n-dimensional case, we obtain first that all functions with the representation given in (1) have the shift property. In fact, given $t \leqslant t_{i_1} \leqslant \ldots \leqslant t_{i_n}$ (i.e. $B = \phi$) or $t_{i_1} \leqslant \ldots \leqslant t_{i_k} \leqslant t \leqslant t_{i_{k+1}} \leqslant \ldots \leqslant t_{i_n}$ (i.e. $B = \{i_1, \ldots, i_k\}$), it follows that

$$a_\phi(t) = \exp[f(A)\, t],$$

$$b_\phi(t) = \exp[f(A)\, t]\, [g(A)\, t + h(A)(\exp[f(A)\, t] - 1)],$$

$$a_{\{i_1, \ldots, i_k\}}(t_{i_1}, \ldots, t_{i_k}, t) = \exp[f(A)\, t_{i_1} + \ldots + f(A_{i_{k-1}})(t_{i_k} - t_{i_{k-1}})$$
$$+ f(A_{i_k})(t - t_{i_k})], \text{ and}$$

$$b_{\{i_1, \ldots, i_k\}}(t_{i_1}, \ldots, t_{i_k}, t) = h(A_{i_k})(a_{\{i_1, \ldots, i_k\}}(t_{i_1}, \ldots, t_{i_k}, t) - 1)$$
$$+ [g(A)\, t_{i_1} + h(A)(\exp[f(A)\, t_{i_1}] - 1)] + \ldots$$
$$+ \exp[f(A)\, t_{i_1} + \ldots + f(A_{i_{k-2}})(t_{i_{k-1}} - t_{i_{k-2}})]$$
$$\cdot [g(A_{i_{k-1}})(t_{i_k} - t_{i_{k-1}}) + h(A) \exp[f(A_{i_{k-1}})(t_{i_k} - t_{i_{k-1}})] - 1]$$
$$+ \exp[f(A)\, t_{i_1} + \ldots + f(A_{i_{k-1}})(t_{i_k} - t_{i_{k-1}})]$$
$$\cdot [g(A_{i_k})(t - t_{i_k}) + h(A)(\exp[f(A_{i_k})(t - t_{i_k})] - 1)].$$

We now show by induction that each κ has a representation in the sense of (1). Recall that $\kappa(t_1, \ldots, t_n) = e^{\lambda t} \cdot \kappa(t_1 - t, \ldots, t_n - t) + \bar{\kappa}(t)$ for all $t, t_1, \ldots, t_n$ with $t \leqslant \min\{t_1, \ldots, t_n\}$. From the above characterization of $\bar{\kappa}$, we obtain that $\bar{\kappa}(t) = g_1 t + h_1 [\exp(f_1\, t) - 1]$ for suitable constants g_1, h_1, f_1 with $g_1 \cdot f_1 = 0$.

Putting $t_1 = \ldots = t_n = 2t$ then gives

$$\kappa(2t) = e^{\lambda t} \cdot \kappa(t) + \kappa(t),$$

i.e.

$$2g_1 t + h_1 [\exp(2f_1 t) - 1] = e^{\lambda t} \cdot [g_1 t + h_1 [\exp(f_1 t) - 1]] +$$
$$[g_1 t + h_1 [\exp(f_1 t) - 1]].$$

The case $f_1 = 0$ yields $2g_1 t = (e^{\lambda t} + 1) g_1 t$, i.e. $\lambda = f_1 = 0$.

If $f_1 \neq 0$, then $g_1 = 0$ and we obtain

$$h_1 [\exp(2f_1 t) - 1] = (e^{\lambda t} + 1) \cdot h_1 [\exp(f_1 t) - 1],$$

where w.l.o.g. $h_1 \neq 0$. Choosing $t = 1/|f_1|$ then yields again that $\lambda = f_1$. So we have the recursive representation

$$\kappa(t_1, \ldots, t_n) = \exp(f_1 t) \cdot \kappa(t_1 - t, \ldots, t_n - t) + [g_1 t + h_1 [\exp(f_1 t) - 1]]$$

of κ for all $t, t_1, \ldots, t_n$ with $t \leqslant \min\{t_1, \ldots, t_n\}$.

Now let $t_{i_1} \leqslant t_{i_2} \leqslant \ldots \leqslant t_{i_n}$ and $t = t_{i_1}$. Then $\kappa(t_1 - t, \ldots, t_n - t)$ is the projection $\kappa^{A_{i_1}}(t_j - t \mid j \in A_{i_1})$ of κ and thus fulfils the shift property (cf. Lemma 4.1.1). By the inductive hypothesis, this projection has the representation

$$\begin{aligned}\kappa(t_1 - t, \ldots, t_n - t) &= [g_{i_1}(A_{i_1})(t_{i_2} - t_{i_1}) + h_{i_1}(A_{i_1})(\exp[f_{i_1}(A_{i_1})(t_{i_2} - t_{i_1})] \\ &\quad - 1)] \\ &\quad + \ldots + \exp[f_{i_1}(A_{i_1}) + \ldots + f_{i_1}(A_{i_{n-2}})(t_{i_{n-1}} - t_{i_{n-2}})] \\ &\quad \cdot [g_{i_1}(A_{i_{n-1}})(t_{i_n} - t_{i_{n-1}}) \\ &\quad + h_{i_1}(A_{i_{n-1}})(\exp[f_{i_1}(A_{i_{n-1}})(t_{i_n} - t_{i_{n-1}})] - 1)]\end{aligned}$$

where $g_{i_1}, h_{i_1}, f_{i_1}$ are set functions on $\mathbf{P}(A_{i_1})$ (which may depend on i_1, i.e. on which activity ends first) with $f_{i_1}(B) \cdot g_{i_1}(B) = 0$ for all $B \subset A_{i_1}$. Consideration of all other first completions $k_1 \neq i_1$ on the set $X_{k_1 i_1} = \{(t_1, \ldots, t_n) \in \mathbf{R}^n_{\geqslant} \mid t = t_{k_1} = t_{i_1}$, $t_j = 2t$ for all $j \neq k_1, i_1\}$ yields that g_{i_1}, h_{i_1} and f_{i_1} are compatible with g_{k_1}, h_{k_1} and f_{k_1} in the sense that $g_{i_1}(A_{i_l}) = g_{i_k}(A_{i_l})$, $h_{i_1}(A_{i_l}) = h_{i_k}(A_{i_l})$ and $f_{i_1}(A_{i_l}) = f_{i_k}(A_{i_l})$ for all $l \geqslant 2$. Therefore $g(A) := g_1$, $g(A_{i_l}) = g_{i_1}(A_{i_l})$, $l \geqslant 1$

(and similarly for h and f) defines set functions f, g, h on $\mathbf{P}(A)$ that lead to the stated representation of κ. Obviously, all functions thus obtained are continuous, concluding the proof of (1).

Part (2) is an immediate consequence of (1), if one notes that uniform continuity yields $f(A_{i_k}) \leqslant 0$ and unboundedness yields $f(A_{i_k}) \geqslant 0$. ∎

An *equivalent* (not too difficult to obtain) and sometimes more convenient representation of cost functions with the shift property is as follows:

Let $U(t)$ denote the set of unfinished jobs at time $t \in \mathbf{R}^1_{\geqslant}$, and let f^* and g^* be set functions on $\mathbf{P}(A)$. Then

$$\kappa(t_1, \ldots, t_n) = \int_0^\infty e^{\int_0^t f^*(U(\tau))d\tau} g^*(U(t))\, dt$$

In this "continuous" representation, $g^*(U(t))$ may be interpreted as the *cost rate* at time t, while $e^{\int_0^t f^*(U(\tau))d\tau}$ is a distortion factor depending on the history up to time t.

The connection with the "discrete" representation in the previous theorem is then given by the equations

$$f^*(B) = f(B) \text{ and } g^*(B) = \begin{cases} h(B) \cdot f(B) & \text{if } f(B) \neq 0 \\ g(B) & \text{if } f(B) = 0 \end{cases}$$

for all $B \subseteq A$.

Both representations show that *all* (cost) functions with the shift property are obtained additively (in the usual sense in stochastic dynamic programming) over the decision periods, where e.g. the cost incurred in the current period t_{i_k} to $t \leqslant t_{i_{k+1}}$ is given by $[g(A_{i_k})(t - t_{i_k}) + h(A_{i_k})(\exp f(A_{i_k})(t - t_{i_k}) - 1)]$ distorted by the history up to t_{i_k} by the factor $\exp[f(A)\, t_{i_1} + \ldots + f(A_{i_{k-1}})(t_{i_k} - t_{i_{k-1}})]$. Due to the orthogonality of g and f, this distortion factor is only caused by those periods $[t_{i_l}, t_{i_{l+1}}]$ in which the period costs are non-additive, i.e. $f(A_{i_l}) \neq 0$. In this context, of course, the additive cost functions behave particularly nicely by never causing a distortion.

Since the proof of Theorem 4.1.3 uses only the monotonicity of $\bar{\kappa}$, we have in fact obtained a characterization of *all* functions $\kappa: \mathbf{R}^n_{\geqslant} \to \mathbf{R}^1_{\geqslant}$ with the shift property that can be iteratively constructed from the characterized 1-dimensional functions. Of course, the *regular cost functions* form a proper subclass of these, where the monotonicity condition imposes further conditions on the set functions g, h, f or g^*, f^*.

Expressed in terms of g^* and f^*, κ is monotonically increasing iff the same is true for g^* and f^* and, in addition, $g^* \geqslant 0$, i.e. iff

$$g^* \geqslant 0 \text{ and } (B \subseteq C \Rightarrow g^*(B) \leqslant g^*(C) \text{ and } f^*(B) \leqslant f^*(C)).$$

In particular, the condition on f^* means that *exponentially unbounded periods* (i.e. $f(A_{i_l}) > 0$) must precede *additive periods* (i.e. $f(A_{i_l}) = 0$), and these in turn must precede *exponentially bounded periods* (i.e. $f(A_{i_l}) < 0$). Thus f determines how many (if any) of these different periods will occur for a fixed order of completion times.

Also other properties of κ such as *convexity* or *concavity* – which might be of great importance in the context of the *non idleness problem* discussed in connection with Example 4.2.5 – can be stated equivalently in terms of g, h, f or g^*, f^*. In these characterizations, the notions of *submodularity* and *supermodularity* of set functions play a key role.

A set function f is called *submodular* if $f(B \cup C) + f(B \cap C) \leqslant f(B) + f(C)$ for all $B, C \subseteq A$, or, equivalently, if all "marginal surplus functions" f_α: $\mathbf{P}(A \setminus \{\alpha\}) \to \mathbf{R}^1$ with $f_\alpha(B) := f(B \cup \{\alpha\}) - f(B)$ are non-increasing. It is called *supermodular*, if $-f$ is submodular. An overview on submodular functions and the relationship of this notion with convexity can be found in *Lovasz*.

One of these relationships can immediately be interpreted in terms of additive cost functions and yields the desired characterization of convex and concave additive (cost)functions: An additive (cost)function κ is convex [concave] if the associated set function g is submodular [supermodular].

For arbitrary – i.e. non-additive – (cost)functions with the shift property, the characterization of convexity [concavity] is much more tedious and leads to more complicated conditions, which can best be represented by using both the "discrete" representation by h, g, f and the "continuous" representation by g^* and f^*. The result then is as follows:

A function with the shift property is convex [concave] iff the following three conditions hold:

(i) g^* is non-negative and submodular [supermodular]
(ii) $f = f^*$ is non-decreasing [non-increasing] and submodular [supermodular]
(iii) h is non-negative [non-positive] and non-decreasing [non-increasing].

Certainly the shift property is a much more rigid requirement then the additivity notion in stochastic dynamic optimization, as it implies a certain internal homogenity between the additive terms caused by transition into different states. Therefore, it allows stronger results, i.e. w.r.t. optimization, not just a restriction to stationary Markov strategies but in certain cases to a finite subclass thereof, viz. the set strategies. A result of this type now follows.

4.2 Overall Optimality of Set Strategies

We derive here the intended result on the overall optimality of set strategies for stochastic scheduling problems for which job durations are realized according to a product of exponential distributions and for which the cost function has the shift property introduced in Section 4.1. [Note that these cost functions will in general not be linearly bounded as required in the general model assumptions. However, the model can be extend straightforwardly to this case by suitable assumptions on P, e.g. if $\mathbf{E}_P [\exp (\lambda \cdot \Sigma x_i)] < \infty$ for some appropriate $\lambda > 0$.]

Note that the cases covered are still quite special. This meant not so much w.r.t κ (cf. the comments in Section 2.1. and the characterization obtained in Theorem 4.1.3),but rather w.r.t. P, as the occurrence of exponential distributions in practical applications of scheduling theory is quite limited. Interest in this class is thus particularly motivated by its well-behavedness, a familiar phenomenon with many other stochastic models. Still, we cover a variety of problems discussed in literature, in fact the hard core of models previously discussed, [cf. e.g. *Dempster/Lenstra/Rinnooy Kan*]. Moreover, these cases give additional insight into the principal behaviour of stochastic scheduling problems and also provide *bounds* for more complicated models, cf. the discussion of the monotonicity behaviour in Section 5 in *Möhring/Radermacher/Weiss* [1984]. (Note that the optimality result is the very basis for these bounds, since in general, ρ^{SET} has no nice monotonicity properties; cf. Example 5.1.7 in *Möhring/ Radermacher/Weiss* [1984]). Altogether, these are good reasons for working with the special cases, also from an application point of view.

Since our assumptions imply that P has a Lebesgue-density and that κ is continuous on $\mathbf{R}^n_{>}$ (cf. Theorem 4.1.3), the existence of optimal strategies – though not necessarily set-type – is clear, cf. Theorem 4.2.6 (3) or (1) in *Möhring/Radermacher/ Weiss* [1984]. We give two proofs of the intended result. The first is based on standard optimality results in stochastic dynamic optimization and semi-Markov decision theory [*Bertsekas/Shreve*]. It exploits the fact that the shift property can be seen as a special case of additivity in stochastic dynamic optimization (which is much more general than the additivity of the cost functions discussed in this paper). The second proof is a typical approach within stochastic scheduling. It has the advantage that it is direct and *constructive* in the sense that it shows how to replace any strategy by a set strategy that is at least as good. As the complete proof is quite lengthy (due to technical necessities), we restrict ourselves to an outline of the main ideas.

Theorem 4.2.1: Let $[A, O_0, \mathsf{N}, P, \kappa]$ be such that P is a product of exponential distributions and κ has the shift property. Then there is an overall optimal set strategy, i.e. $\rho(\kappa, P) = \rho^{\mathrm{SET}}(\kappa, P)$.

Proof 1: Recall the stochastic dynamic optimization type description of stochastic scheduling problems given in Section 2.2 in *Möhring/Radermacher/Weiss* [1984]. If we specialize this approach to the assumptions of the theorem, it follows that we can describe the states by the sets B^* and $B \setminus B^*$ of completed and currently being performed jobs, the observed (ordered) completion times t_α, $\alpha \in B^*$, and the current time t. Once these data are fixed, the influence of the completion times t_α and of t on the future evolution is reduced to a multiplicative term because of the required shift property of κ. [Moreover, in the case of additive cost functions, we can even restrict the state space to the sets B^*, $B \setminus B^*$ only.] This multiplicative term is, given B^* and $B \setminus B^*$, either zero for all possible t_α, $\alpha \in B^*$ and t, or different from zero for all $t_\alpha \neq 0$, $\alpha \in B$ and $t \neq 0$, cf. Theorem 4.1.3. Furthermore, the cost functions considered are all additive in the broader sense of stochastic dynamic optimization, i.e. they can be interpreted as a sum of cost terms associated with the different decision periods, where each term depends on the respective history, decision and transition into the next state, [cf. e.g. *Bertsekas/Shreve; Hinderer*, 1970]. In view of the standard assumptions on κ and P in our model, general results [*Bertsekas/Shreve; Hinderer*, 1970] guarantee the existence of a non-randomized, stationary (even Markov) optimal strategy Π for this problem. In the additive case, in which the states are essentially given by the sets B^* and $B \setminus B^*$, Π is already a set strategy.

In general, however, Π may not be a set strategy, as it may, given sets B^*, $B \setminus B^*$, choose different actions according to the values of t_α, $\alpha \in B^*$ and t, which enter into the state via a distortion factor. If we could exclude this dependence, we would have achieved our aim. For then the states would essentially be given by the sets of completed or currently being performed jobs, and the stationary strategies would then just be the set strategies.

It is not difficult to obtain this additional property by an inductive argument, using the iterative decomposition of strategies described in Section 2.3 in *Möhring/Radermacher/Weiss* [1984], together with the fact that if a particular substrategy is optimal for any sequence t_α, $\alpha \in B^*$ and t, then it is optimal for every such sequence. The desired set strategy is then composed of such optimal sub-(set-)strategies, where only (universal) measurability has to be guaranteed. We omit these steps here, as they will occur analogously in the second proof that is given next. ■

Proof 2: We give a stronger version of Theorem 4.2.1 by considering strategies (general and set strategies) that contain certain prescribed jobs in their starting set. The proof is as follows:

Assume κ and P to be given and let $\bar{\Pi}$ be any (general) strategy for this problem. Let B_0 be the set of jobs started by $\bar{\Pi}$ at time zero and assume the prescribed starting jobs to belong to B_0. We will show *constructively* how to replace $\bar{\Pi}$ by a *set strategy* Π^*, such that the prescribed starting jobs (in fact all jobs from B_0) are started at time zero also by Π^*, and such that $\mathbf{E}_P\,[\kappa\,(\Pi^*, \cdot\,)] \leqslant \mathbf{E}_P\,[\kappa\,(\bar{\Pi}, \cdot\,)]$. Obviously, this will prove the theorem.

The construction of Π^* is inductive and consists basically of three steps:

(1) The standard decomposition of $\bar{\Pi}$ yields strategies $\bar{\Pi}^0, \bar{\Pi}^1, \ldots, \bar{\Pi}^l$. From these we choose one with the least expected project cost; let this be $\bar{\Pi}^r$.

(2) We modify $\bar{\Pi}^r$ in a way that does not increase the objective, and that leads to a strategy Π^r for which the first decision is of the type (B_0^r, ∞) with $B_0 \subseteq B_0^r$.

(3) We apply the inductive hypothesis to (P-almost) all subproblems occurring in the sense of Section 2.3 in *Möhring/Radermacher/Weiss* [1984], and use special set strategies for these subproblems to modify Π^r into a *set strategy* Π^* such that the objective value associated with Π^* is at least as good as that associated with Π^r.

Step 1: We decompose $\bar{\Pi}$ in the sense of Section 2.3 in *Möhring/Radermacher/Weiss* [1984]. Let $(B_0, t_1), \ldots, B_l, t_{l+1})$ with $0 < t_1 < t_2 < \ldots < t_l < t_{l+1} = \infty$ be the sequence of decision times associated with $\bar{\Pi}$. W.l.o.g. we can assume $t_1 < \infty$ (otherwise put $\Pi^r := \bar{\Pi}$). Remember that the measurable set $Z_r := \{x \in \mathbf{R}^n_{>} \mid t_r < t_j + x(\alpha)$ for all $\alpha \in B_j, j = 0, \ldots, r-1$ and $t_j + x(\alpha) \leqslant t_{r+1}$ for some $\alpha \in B_j$, some $j = 0, \ldots, r\}$ gives all duration vectors for which the first job completion occurs in $]t_r, t_{r+1}]$, $r = 1, \ldots, l$. Analogously to Section 2.3 in *Möhring/Radermacher/Weiss* [1984], we can define strategies $\bar{\Pi}^1, \bar{\Pi}^2, \ldots, \bar{\Pi}^l$ which reflect the behaviour of $\bar{\Pi} := \bar{\Pi}^0$ on the sets $A^1 := \mathbf{R}^n_{>} \setminus Z_0, A^2 := \mathbf{R}^n_{>} \setminus (Z_0 \cup Z_1), \ldots, A^l := \mathbf{R}^n_{>} \setminus (Z_1 \cup Z_2 \cup \ldots \cup Z_{l-1})$ by putting

$$\bar{\Pi}^r[x](\alpha) := \begin{cases} \bar{\Pi}[x^r](\alpha) - (t_r - t_j) & \alpha \in B_j, j = 0, \ldots, r \\ \bar{\Pi}[x^r](\alpha) - t_r & \text{otherwise} \end{cases}$$

where

$$x^r(\alpha) := \begin{cases} x(\alpha) + (t_r - t_j) & \text{if} \quad \alpha \in B_j,\ j = 0, 1, \ldots, r-1 \\ x(\alpha) & \text{otherwise} \end{cases}$$

We chose from $\bar{\Pi}^0, \bar{\Pi}^1, \ldots, \bar{\Pi}^l$ a strategy with the least objective value; let this be $\bar{\Pi}^r$.

Step 2: Consider the standard decomposition and the induced sequence of strategies associated with $\bar{\Pi}^r$ in the sense of Step 1. Obviously, this decomposition is given by $(B^r_0, t_{r+1}), (B_{r+1}, t_{r+2}), \ldots, (B_l, t_{l+1})$, where $B^r_0 := B_0 \cup B_1 \cup \ldots \cup B_r$, i.e. in particular $B_0 \subseteq B^r_0$. Of course, the behaviour of $\bar{\Pi}^r$ on the sets $A^{r+1}, \ldots, A^l$ is analogously given by the strategies $\bar{\Pi}^{r+1}, \ldots, \bar{\Pi}^l$ already introduced. By construction, $\mathbf{E}_P[\kappa(\bar{\Pi}^r, \cdot)] \leqslant \mathbf{E}_P[\kappa(\bar{\Pi}^{r+1}, \cdot)]$. Thus if we replace $\bar{\Pi}^{r+1}$ on A^{r+1} by $\bar{\Pi}^r$, i.e. "restart" $\bar{\Pi}^r$ if there is no completion before time t_{r+1}, this will not increase the objective value, because of the shift property whose use is crucial at this point. If we repeat this argument at times $2 \cdot t_{r+1}, 3 \cdot t_{r+1}, \ldots$, we obtain a sequence of strategies which converges pointwise. The limit Π^r of this sequence is then again a strategy, cf. Section 2.2 in *Möhring/Radermacher/Weiss* [1984]. In fact, Π^r is the strategy claimed in Step 2. First of all, each strategy of the above sequence has an objective value at least as good as $\bar{\Pi}^r$. Since the sequences $\bar{\Pi}^r[x]$ become stationary for all x by construction, we have $\mathbf{E}_P[\Pi^r(\kappa,\cdot)] \leqslant \mathbf{E}_P[\bar{\Pi}^r(\kappa,\cdot)]$. (Another argument for this inequality is here given by the continuity of κ on $\mathbf{R}^n_{\gtrless}$ (cf. Theorem 4.1.3) and Lemma 4.2.1 in *Möhring/Radermacher/Weiss* [1984]). Furthermore, the standard decomposition for Π^r is (B^r_0, ∞) by construction, yielding all desired properties since $B_0 \subseteq B^r_0$.

Step 3: We use an inductive argument. Note that the inductive base is trivial and that the assumptions concerning κ and P carry over to the subproblems induced by a job completion in the sense of Section 2.3 in *Möhring/Radermacher/Weiss* [1984]. We can therefore assume the stronger version of Theorem 4.2.1 (i.e. with prescribed starting jobs) to hold for these subproblems. Since P has a Lebesgue-density, no two jobs from B^r_0 will P-almost sure end at the same time. We can therefore restrict ourselves to (notationally simpler) subproblems of the form $[A, O_0, \mathbf{N}, P, \kappa] \mid x(\alpha)$, meaning that the first job completion occurred at time $x(\alpha)$, $\alpha \in B^r_0$, with all other completions occurring later. By the inductive assumption, there exists for each of these subproblems a set strategy $\Pi'_{x(\alpha)}$ which has the jobs from $B^r_0 \setminus \{\alpha\}$ in the starting set (thus later yielding *non-preemptiveness*) and is optimal among *all* strategies with this property for the subproblem. As the history at the completion of α (i.e. at the transition into the associated subproblem) is given by $\{\alpha\}, B^r_0 \setminus \{\alpha\}, t = t_\alpha = x(\alpha)$, the shift property tells us that each particular $\Pi'_{x(\alpha)}$ is optimal for all subproblems induced by the completion of α (i.e. regardless of $x(\alpha)$). Choose a fixed such Π'_α for each $\alpha \in B^r_0$, and use them to compose a strategy Π^* with standard decomposition (B^r_0, ∞) and respective substrategies Π'_α, $\alpha \in B^r_0$ arbitrary. (If several jobs end at the same time, we can proceed analogously.)

In view of the remarks in Section 2.3 in *Möhring/Radermacher/Weiss* [1984], Π^* is a (*non-preemptive*) strategy provided that Π^* is (universally) measurable. This is obvious here, as Π^* is obtained by composing a *finite* number of measurable strategies defined on Borel sets. Moreover, "being a set strategy" fulfils a (restricted) conservation property in the sense that if the sub set-strategy is the same for each $B^r_0 \backslash B^*$, then the composed strategy (here Π^*) is again a set strategy, provided that it is elementary on B^r_0. So it only remains to be shown that $\mathbf{E}_P[\kappa(\Pi^*,\cdot)] \leqslant \mathbf{E}_P[\kappa(\Pi^r,\cdot)]$. According to the comments in Section 2.3 in *Möhring/Radermacher/Weiss* [1984] it suffices to show that $\mathbf{E}_{P(\cdot \mid Z_{0,\alpha})}[\kappa(\Pi^*,\cdot)] \leqslant \mathbf{E}_{P(\cdot \mid Z_{0,\alpha})}[\kappa(\Pi^r,\cdot)]$ where $Z_{0,\alpha} := \{x \in \mathbf{R}^n_{\gtrless} \mid x(\alpha) < x(\beta)$ for all $\beta \in B^r_0 \setminus \{\alpha\}\}$, $\alpha \in B^r_0$ arbitrary and $P(\cdot \mid Z_{0,\alpha})$ denotes the conditional distribution w.r.t. the set $Z_{0,\alpha}$. This inequality is proved by conditioning on the value $x(\alpha)$. By construction, the sub-strategy associated with Π is optimal for every value $x(\alpha)$, i.e. the associated conditional expected value is, as a function of $x(\alpha)$, everywhere less than or equal to the conditional expected value associated with Π^r. As is well-known *Bauer; Hinderer* [1972], this implies the intended inequality, concluding the proof. ∎

Remark 4.2.2: With a view to the third paper in this series, we include a few remarks on a different type of scheduling problem, in which *preemptions* are allowed at times when a job ends. This means that, at a job completion, a complete rescheduling of all unfinished jobs is possible. This is a special version of more general preemptive models (where jobs may be rescheduled any time [*Dempster/Lenstra/Rinnooy Kan; Weiss*]). The present case is particularly nice to handle. Deliberate idleness can now only pay off for non-elementary strategies. Hence elementary strategies are greedy, i.e. induced by (dynamic) priority rules. The optimal value will be at least as good as in the non-preemptive case, but may of course be better (e.g. in Example 3.1.5).

The analogoues to *set strategies* for this preemptive model are elementary strategies for which decisions only depend on the *set* B^* of completed jobs, i.e. strategies which are induced by dynamic priority rules that only depend on B^*. We will indicate below to what extent the results on set strategies have preemptive analogues. Such preemptive results can be useful for the non-preemptive case, too, as an optimal preemptive strategy (which can often be obtained more easily) may incidentially turn out to be non-preemptive, i.e. may make no use of the possibility of rescheduling. In fact that is what happens in many of the tractable cases in which the rules LEPT and SEPT are optimal, [cf. *Weiss; Weiss/Pinedo*].

In fact, the results on *analytic behaviour* (Theorem 3.2.1 and Corollary 3.2.3), on *quasi-stability* (Theorem 3.2.4) and, in particular, on *overall optimality* (Theorem 4.2.1) all straightforwardly carry over to the preemptive case. There is only one problem, which consists in an appropriate adaptation of the notion of *continuity* of preemptive strategies. Here it means continuity for the starting times, preemption times (if preemption occurs) and completion times of all activities, whereas for non-preemptive strategies Π, it (only) means continuity for the starting times $\Pi[x](\alpha)$ (which already implies continuity for the completion times $\Pi[x](\alpha) + x(\alpha)$). With this definition, Theorem 3.2.1 yields continuity of preemptive set strategies piecewise on finitely many convex polyhedra.

We finally discuss, with a view to Section 4.3 in *Möhring/Radermacher/Weiss* [1984], how essential the shift property is for obtaining the nice behaviour formulated in Theorem 4.2.1. Note that since P is a product of exponential distributions, the part of the t-history that may be employed for decision-making is restricted to the sets of completed or currently being performed jobs together with the observed completion times and the present time t. This fact was used in the first proof of Theorem 4.2.1. The shift property is designed specially to make the last two types of information superfluous. Without such an assumption on κ, a characterization of a nice class of special strategies that determine the optimum is still missing and may be hard to obtain. For instance, assuming only products of exponential distributions will not yield a restriction to *elementary* strategies.

This and other aspects are demonstrated by the following two examples, which use cost functions involving *tardiness costs*. Note that since P has a Lebesgue density, there is an optimal strategy because Theorem 4.2.6 (3) in *Möhring/Radermacher/Weiss* [1984]. Furthermore, any λ^n-almost everywhere continuous strategy will have the stability behaviour formulated in Theorem 3.2.4 (1). We present optimal strategies with this property, making the use of *simulation* in dealing with these examples acceptable. Note that in the first example, each optimal strategy must employ deliberate idleness of machines and a first decision of the type $(B,t), t<\infty$, while in the second example, the optimal strategy is elementary but bases the decision of which job to start next on wether some jobs end before a certain time t. So in the first example, the actions still are based on the sets of completed and presently being performed jobs, i.e., compared with set strategies only the elementarity is lost, whereas in the second example it is just the other way round. In both cases the given numbers are only an approximation of the real t, based on 3000 simulation runs.

Example 4.2.3: Let $[A, O_0, \mathbf{N}, P, \kappa]$ be given by $A = \{1, 2, 3, 4\}$, O_0 by its arrow diagram in Figure 3, $\mathbf{N} = \{B \subseteq A \mid |B| = 3\}$ (i.e. all jobs compete for two identical machines), $P = \bigotimes_{j=1}^{4} P_j$ where each P_j is an exponential distribution with (identical) parameter $\lambda_j = 1/2$, and κ is a weighted unit penalty cost function with (identical) due dates $d_j = 3$ for $j = 1, 2, 3, 4$ and $w_1 = 0, w_2 = 2, w_3 = 10$ and $w_4 = 10$. This means that the project cost is the sum of individual cost terms for each job, which are zero if the job ends before its due date $d_j = 3$, and w_j otherwise.

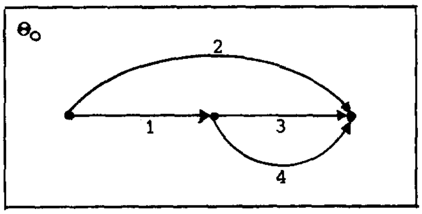

Fig. 3

Certainly, job 1 should be started at time zero to allow job 3 and 4, which may cause the largest penalties w_j, to start early. For the same reason, one should not start job 2 immediately, but wait until a certain time t and put jobs 3 and 4 on the two machines if job 1 is completed before t. If, however, job 1 is not completed by time t, then for t succifiently close to 3, it will become reasonable to start job 2, because then there is little hope for completing jobs 3 and 4 before their due date.

Altogether, one must consider a family of strategies Π^t, $t \in [0, 3]$, with associated first decision $(\{1\}, t)$ which acts as described above. Each such strategy is λ^n-almost everywhere continuous. Moreover, the associated objective value $\mathbf{E}_P\,[\kappa\,(\Pi^t, \cdot\,)]$ is a *continuous* function of t, as $t_j \to t$ implies $\Pi^{t_j} \to \Pi$ (pointwise for $x_1 \neq t$), and this in turn implies $\mathbf{E}_P\,[\kappa\,(\Pi^{t_j}, \cdot\,)] \to \mathbf{E}_P\,[\kappa\,(\Pi^t, \cdot\,)]$, analogously to the proof of Theorem 4.2.6 (3) in *Möhring/Radermacher/Weiss* [1984].

The values $\mathbf{E}_P\,[\kappa\,(\Pi^t, \cdot\,)]$ were (simultaneously) determined for $t = 0.0, 0.1, 0.2, \ldots, 2.8, 2.9$ and 3.0 by simulation as well as by numerical evaluation. The resulting function of t is given in Figure 4. The (approximate) optimal value 12.522 was attained for $t = 2.216$, i.e. $(\{1\}, 2.216)$ should be taken as the first decision.

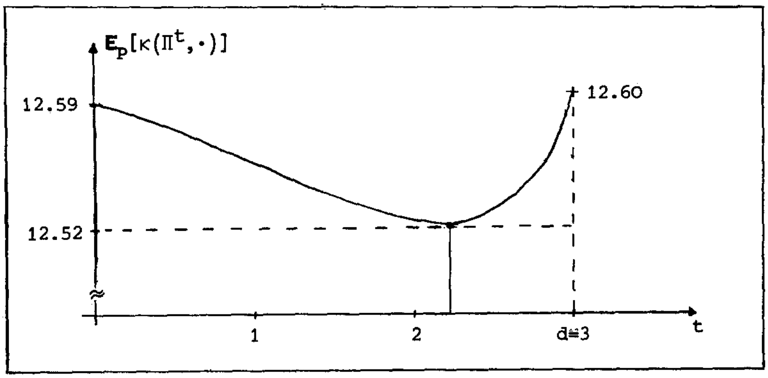

Fig. 4: Expected project cost $\mathsf{E}_P[\kappa(\Pi^t, \cdot)]$ as function of $t \in [0, 3]$

Example 4.2.4: Let $[A, O_0, \mathbf{N}, P, \kappa]$ denote a 2-machine problem with four independent, exponentially distributed jobs $1, \ldots, 4$ with parameters $\lambda_1 = \lambda_2 = 1/12$ $\lambda_3 = 1/6$ and $\lambda_4 = 1/2$. As in Example 4.2.3, κ is a weighted unit penalty cost function with due dates $d_1 = d_2 = 12, d_3 = 10, d_4 = 12$ and weights $w_1 = w_2 = 20$, $w_3 = 3$ and $w_4 = 1$.

Certainly, jobs 1 and 2 should be started at time zero because of their large weight. If the first of them ends "early", i.e. before a certain time t (which still gives job 3 a good chance to end before its due date), job 3 is started. Otherwise, i.e. if the first of jobs 1 and 2 ends after t, then job 4 is started.

This procedure defines a family of strategies Π^t, $t \in [0, 12]$ for the given problem. The associated values $\mathsf{E}_P[\kappa(\Pi^t, \cdot)]$, which are again a continuous function of t, were determined by simulation (which may here be applied for the same reasons as in Example 4.2.3). The (approximate) optimal value was attained for $t_0 = 8.6$. Altogether, this means that the optimal strategy is elementary, but not a set strategy.

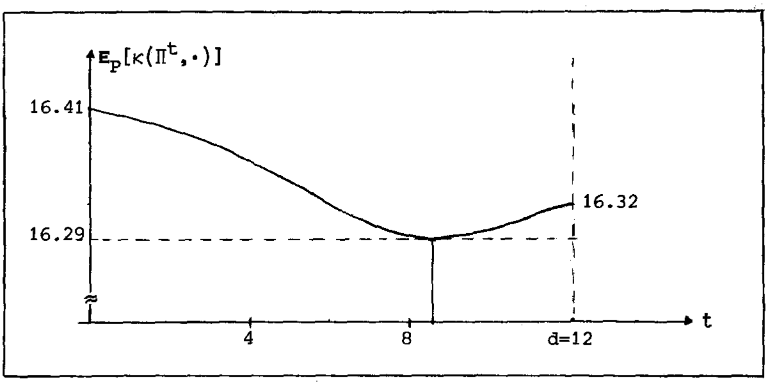

Fig. 5: Expected project cost $\mathsf{E}_P[\kappa(\Pi^t, \cdot)]$ as function of t

It should be noted that by modifying the values d_j and w_j accordingly, a large variety of quite different functions was observed in both examples. Altogether, these results show that the natural and the most general stochastic environment to study and to hope for possible optimality of simple-structured strategies such as LEPT or SEPT is given by independent, exponentially distributed job durations and cost functions with the shift property, thus leading to the investigation of set strategies. Now a very decisive but yet open step in this direction would be to obtain additional conditions which guarantee the optimality of a *priority-type* set strategy (thus a greedy set strategy), i.e. exclude *deliberate idleness* of resources. In this respect, the following example is crucial, as it demonstrates that deliberate idleness may even occur in a very restrictive model, viz. for *m-machine problems with additive cost functions.* It thus demonstrates the full extent of the limitations under which optimality of priority rules can be expected.

Example 4.2.5: Let $[A, O_0, \mathbf{N}, P, \kappa]$ denote a 3-maschine problem with six independent, exponentially distributed jobs $1, \ldots, 6$ with parameters $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1$. The additive cost function κ is given by its associated set function g, where

$$g(B) := \begin{cases} a \gg 1 & \text{if } 1 \in B \text{ and } 2 \in B \\ 1 & \text{if } 1 \notin B \text{ and } \{2,5\} \subseteq B \text{ or } \{2,6\} \subseteq B \\ 1 & \text{if } 2 \notin B \text{ and } \{1,3\} \subseteq B \text{ or } \{1,4\} \subseteq B \\ 0 & \text{otherwise} \end{cases}$$

Due to symmetry reasons and the fact that $a \gg 1$, only the following three strategies need to be considered:

Π_1: Start 1, 2 at time $t = 0$ and leave the third machine *idle*. If 1 ends first, proceed with 2, 5, 6 according to priorities $5 < 6 < 3 < 4$. If 2 ends first, proceed with 1, 3, 4 according to priorities $3 < 4 < 5 < 6$.

Π_2: Start 1, 2, 3 at time $t = 0$. Proceed according to priorities

$$\begin{cases} 5 < 6 < 4 & \text{if} \quad 1 \text{ ends first} \\ 4 < 5 < 6 & \text{if} \quad 2 \text{ ends first} \\ 4 < 5 < 6 & \text{if} \quad 3 \text{ ends first} \end{cases}$$

Π_3: Start 1, 2, 3, at time $t = 0$. Proceed according to priorities

$$\begin{cases} 5 < 6 < 4 & \text{if} \quad 1 \text{ ends first} \\ 4 < 5 < 6 & \text{if} \quad 2 \text{ ends first} \\ 5 < 4 < 6 & \text{if} \quad 3 \text{ ends first} \end{cases}$$

The associated expected costs then are as follows:

$$E[\kappa(\Pi_1, \cdot)] = \frac{a}{2\lambda} + \frac{1}{\lambda+2} + \frac{1}{(\lambda+2)(\lambda+1)}$$

$$E[\kappa(\Pi_2, \cdot)] = \frac{a}{2\lambda} + \frac{\lambda}{2\lambda+1}\left[\frac{2}{\lambda+2} + \frac{3}{(\lambda+2)(\lambda+1)} + \frac{1}{(\lambda+2)^2} + \frac{2}{(\lambda+2)^2(\lambda+1)}\right.$$

$$+ \frac{1}{(2\lambda+1)(\lambda+1)} + \frac{1}{(2\lambda+1)(\lambda+2)} + \frac{1}{(2\lambda+1)(\lambda+2)^2}$$

$$+ \frac{1}{(2\lambda+1)(\lambda+2)^2(\lambda+1)}$$

$$+ \frac{1}{(2\lambda+1)(\lambda+2)(\lambda+1)} + \frac{1}{(2\lambda+1)^2(\lambda+2)}$$

$$\left.+ \frac{1}{(2\lambda+1)^2(\lambda+2)(\lambda+1)} + \frac{1}{(2\lambda+1)^3(\lambda+1)}\right].$$

Hence $E[\kappa(\Pi_2, \cdot)] - E[\kappa(\Pi_1, \cdot)] = \dfrac{2\lambda^4 - 8\lambda^3 - 23\lambda^2 - 19\lambda - 6}{(\lambda+1)(\lambda+2)^2(2\lambda+1)^4}$

$$> 0 \Leftrightarrow \lambda > 6.1385$$

Similarly, $E[\kappa(\Pi_3, \cdot)] - E[\kappa(\Pi_2, \cdot)] = \dfrac{2\lambda^3 + 3\lambda^2 + 3\lambda + 1}{(\lambda+1)(\lambda+2)^2(2\lambda+1)^4}$

$$> 0 \text{ for all } \lambda > 0.$$

Thus Π_1 is optimal for all $\lambda > 6.14$. Since Π_1 leaves a machine *idle* until the first job completion, Π_1 is not priority-induced.

This example shows that optimality of priority-induced set strategies such as LEPT or SEPT can, even in exponential m-machine models, only be expected under additional assumptions on the (additive!) cost function. It seems that convexity or concavity (or, equivalently, submodularity or supermodularity of the associated set function; cf. Section 4) might be conditions that permit *greediness*, i.e. that will never necessitate deliberate idleness. In this respect, the weighted flowtime $\kappa(t_1, \ldots, t_n) = \Sigma w_i t_i$, which is both convex and concave, may play a key role in establishing such an optimality result. In any case, gaining more insight into this *Non–Idleness-Problem* seems to be a very involved task and certainly constitutes at present one of the most important and most challenging open problems in stochastic scheduling.

5. Concluding Remarks

This paper deals with an intermediary concept between general strategies and known special classes of strategies. The requirements for such an intermediary class are formulated at the end of the first paper in this series, and the set strategies introduced here fulfill these requirements.

The notion of a set strategy arises in a rather natural way. In fact one only has to note what ES and MES strategies and list-scheduling strategies such as LEPT and SEPT (or certain more involved strategies induced by certain dynamic priority rules) have in common. The main common feature is that these strategies depend on the observed past only via the sets of completed or currently being performed jobs. This property then leads to a finite class of elementary strategies which are all λ^n-almost everywhere continuous, thus implying quasi-stability of this class. This is a variant of the original rigid notion of stability that places emphasis on distributions with Lebesgue densities rather than general duration distributions.

The restricted use of the observed history made by set strategies makes them particularly suited for a restricted class of stochastic scheduling problems, the hard core of which is given by duration distributions that are products of exponential distributions and cost functions having the shift property. The first assumption makes the observance of the duration of completed jobs and of the current duration of jobs being performed superfluous, while the second one, which is a stronger version of the notion of additivity in stochastic dynamic optimization, ensures that the observed job completion times and the present time t enter into the total cost only via an affine transformation. This essentially means that though the past influences the objective value, it does not influence the future ranking of the objectives of different strategies (strategy equivalence). Given this particular influence of the past on the future, it is not surprising that set strategies lead to overall optimality in these cases.

It remains to be seen to what extent this observation can be of use for the treatment of stochastic scheduling problems with more complicated job duration distribution, e.g. in delivering bounds, though such possibilities are certainly restricted. An immediate use, which is in the direction of further specialisation, however, will become apparent in the third paper of this series. The main emphasis there is on tractable cases, the hard core of which presently deals with the optimality of LEPT and SEPT rules and thus necessarily belongs in the framework in which set strategies are overall optimal. Presently available results of that type do usually not allow any distinction, however marginal, of jobs w.r.t. precedence and resource constraints, thereby considerably reducing the range of possible practical applications. In the third paper of this series, we will discuss to what extent this restriction may be lifted. It will eventually transpire that optimality of LEPT can, under certain agreeability conditions, be obtained if the precedence constraints are given by a (strict) interval order. Analogous results yield the optimality of LEPT or SEPT rules under certain regularity conditions concerning the forbidden sets, i.e. the resource constraints.

The cases covered are still quite special, but it is a nice feature that they naturally occur in the course of an iterative treatment of set strategies by means of branch- and bound-methods based on simulation. This is a straightforward way to cope with such computationally difficult stochastic versions of optimization problems, whose deterministic counterparts are all NP-complete. The approach falls into the framework of implicit enumeration and follows the lines previously employed for dealing with ES and MES strategies [*Igelmund/Radermacher; Kaerkes et al.; Pinedo*]. In this context, optimality results as intended allow certain subtrees to be handled fast and may considerably improve algorithmic treatment. Work on the implementation of such algorithmic methods is presently being performed at the Universities of Aachen, Hildesheim and Passau, also with a view to better treatment of the MES case. There is some hope of also employing these tools for delivering bounds for certain general stochastic scheduling problems.

Finally, in view of Remark 3.1.4 and Theorem 4.2.1, the next and challenging step towards optimality results for priority-induced set strategies is to find conditions under which deliberate idleness of machines does not pay off. Example 4.2.5 gives some indication what type of condition may be needed, e.g. additivity in combination with convexity or concavity for the cost function κ (or, equivalently, submodularity or supermodularity of the associated set function). Much work is presently going into the establishment of such results and we would like to see them included in the third paper of this series. We are certain that such results will also give some additional justification for the regular use of priority rules in stochastic scheduling problems and queuing theory, particularly with respect to applications in computer systems, what makes their establishment even more urgent.

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1. page numbers 207 and 208 have to be interchanged.
2. page numbers 211 and 212 have to be interchanged.
3. page 222, first line: the correct term is $1/2 \cdot 57$ instead of $\frac{1}{2} \cdot 57/2$.
4. the following acknowledgement is missing on the first page: "The work of the first two authors was supported by the Minister für Wissenschaft und Forschung des Landes Nordrhein-Westfalen, while the work of the last author was supported by the DAAD."