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SUBGROUPS OF SEMIFREE GROUPS

By
A. BAUDISCH (Berlin)

1. Introduction

We call a group semifree iff it has a presentation with defining relations $[a, b]=1$ between the defining generators only. In [1] it is proved:

THEOREM 1.1. *Every Abelian subgroup of a semifree group is free-Abelian.*

Our main result in this paper is

THEOREM 1.2. *Let u and v be elements of a semifree group. Then $[u, v]\neq 1$ implies that $\{u, v\}$ is a basis of a free group.*

To prove Theorem 1.2 cancellation arguments as firstly developed by Nielsen are used.

COROLLARY 1.3. *Every subgroup of a semifree group generated by two elements is semifree.*

Furthermore, it will be shown that Corollary 1.3 is not true for subgroups generated by three elements. The counterexample group that is a subgroup of a semifree group has the property that every Abelian subgroup is free-Abelian.

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2. Basic facts on semifree groups

The results stated in this section are proved in [1].

Let A be a set. A *word* over A is a finite sequence $a_1^{\alpha_1}a_2^{\alpha_2}\dots a_n^{\alpha_n}$, where $a_i\in A$ and $\alpha_i\neq 0$ are integers. Let e be the *empty word*. $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ is *reduced* iff $a_i\neq a_{i+1}$.

If R is a set of words, we use (A, R) to denote the group F/N, where F is the free group with basis A and N is the normal closure of R in F. Then a group is *semifree* iff it has a presentation (A, R) where $R\subseteq\{[a, b]: a, b\in A\}$.

We use elements of A (letters) and words over A to denote elements of (A, R). Obviously, for every $u\in(A, R)$ there is some reduced word $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ in the coset of u. Two words u and v represent the same element of the semifree group (A, R) iff we can carry u into v applying a finite number of the following transformations:

(R1) Replace $a_1^{\alpha_1}\dots a_i^{\alpha_i}\, a_{i+1}^{\alpha_{i+1}}\dots a_k^{\alpha_k}$ by $a_1^{\alpha_1}\dots a_i^{\alpha_i}aa^{-1}\, a_{i+1}^{\alpha_{i+1}}\dots a_k^{\alpha_k}$ for some $a\in A$.

(R2) If $a_i=a=a_{i+1}$, replace $a_i^{\alpha_i}a_{i+1}^{\alpha_{i+1}}$ in $a_1^{\alpha_1}\dots a_k^{\alpha_k}$ by $a^{\alpha_i+\alpha_{i+1}}$ and delete it if $\alpha_i+\alpha_{i+1}=0$.

(R3) Replace $a_1^{\alpha_1}\dots a_i^{\alpha_i}a_{i+1}^{\alpha_{i+1}}\dots a_k^{\alpha_k}$ by $a_1^{\alpha_1}\dots a_{i+1}^{\alpha_{i+1}}\, a_i^{\alpha_i}\dots a_k^{\alpha_k}$ if $[a_i, a_{i+1}]\in R$.

$u=a_1^{\alpha_1}\ldots a_n^{\alpha_n}$ is a *minimal form* for u iff n is minimal. We call n the *length* $\lambda(u)$ of u. Minimal forms are reduced.

LEMMA 2.1. *If $u=a_1^{\alpha_1}\ldots a_n^{\alpha_n}$ and $u=b_1^{\beta_1}\ldots b_m^{\beta_m}$ is a minimal form of u, then $m\leqq n$ and it is possible to get $b_1^{\beta_1}\ldots b_m^{\beta_m}$ from $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$ applying* (R2) *and* (R3) *only.*

Lemma 2.1 immediately implies

LEMMA 2.2. *Let $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$ and $b_1^{\beta_1}\ldots b_m^{\beta_m}$ be minimal forms of u. Then*

(i) *$n=m$ and it is possible to transform one minimal form into the other using* (R3) *only.*

(ii) *$\{a\in A\colon a=a_i$ for some $i\}=\{a\in A\colon a=b_i$ for some $i\}$.*

By Lemma 2.2 (ii) it is possible to speak about *the* letters of u. Let $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$ be a minimal form of u and a be a letter of u. By Lemma 2.2 (i) the following definitions are correct:

The *a-sequence* of u is the sequence $a^{\alpha_{i_1}}\ldots a^{\alpha_{i_m}}$ where $a^{\alpha_{i_j}}=a_{i_j}^{\alpha_{i_j}}$ is the j-th a-power in $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$.

If $[a,b]\neq 1$, we can similarly define the $\{a,b\}$-sequence $c_1^{\gamma_1}\ldots c_m^{\gamma_m}$ of u where $c_j^{\gamma_j}$ is the j-th occurrence of a power of a or b in $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$.

Furthermore, if $[a,b]\neq 1$ and $[b,c]\neq 1$ and u has a minimal form $\ldots a^{\alpha}\ldots b^{\beta}\ldots c^{\gamma}\ldots$, it is possible to say "b^{β} lies between a^{α} and c^{γ}", "b^{β} is on the right of a^{α}", and so on, since by Lemma 2.2 (i) this is true in every minimal form of u.

We make the following convention: $u=w_1w_2\ldots w_n$ is a minimal form means that the w_i's are minimal forms and the concatenation of the w_i's is a minimal form of u. We suppose that $w_i\neq e$ if there is no other assumption.

a is a *first (last) letter* of power α of u iff u has a minimal form $a^{\alpha}u'$ (resp. $u'a^{\alpha}$).

u is cyclic reduced iff u has no minimal form $a^{\alpha}u'a^{\beta}$ with $\alpha<0<\beta$ or $\beta<0<\alpha$.

We are now interested in the cancellation of some letter a in products uv.

LEMMA 2.3. (i) *Semifree groups are torsion free.*

(ii) *Let a be a first (last) letter of power α of u, and let ε be any natural >0. If there is some b with $[a,b]\neq 1$ in u, then a is a first (last) letter of power α of u^{ε}. Otherwise a is a first (last) letter of power $\alpha\cdot\varepsilon$ of u^{ε}. u and u^{ε} contain the same letters.*

If $a^{\alpha_1}\ldots a^{\alpha_n}$ is the a-sequence of u, $a^{\beta_1}\ldots a^{\beta_m}$ is the a-sequence of v, and $a^{\alpha_1}\ldots a^{\alpha_n}a^{\beta_1}\ldots a^{\beta_m}$ or $a^{\alpha_1}\ldots a^{\alpha_n+\beta_1}\ldots a^{\beta_m}$ is the a-sequence of uv, then we say that there is no full cancellation of an a-power between u and v.

LEMMA 2.4. *If $a^{\alpha_1}\ldots a^{\alpha_n}$ is the a-sequence of u and $a^{\beta_1}\ldots a^{\beta_m}$ is the a-sequence of v and $\alpha_n+\beta_1\neq 0$, then there is no full cancellation of an a-power between u and v.*

If B is any subset of a group G, then Gp (B) is used to denote the subgroup of G generated by B. The following lemma is easliy proved.

LEMMA 2.5. *If (A,R) is semifree and $B\subseteqq A$, then* Gp $(B)=(B,R_B)$, *where $R_B=R\cap\{[a,b]\colon a,b\in B\}$.*

Let W be any subset of a semifree group (A,R) and C be the set of letters that occur in some element of W. W is called to be *connected* iff for every $a,b\in C$ there are $c_1,\ldots,c_n\in C$ such that $[a,c_1]\neq 1$, $[c_i,c_{i+1}]\neq 1$ for $1\leqq i<n$, and $[c_n,b]\neq 1$. Then W is connected iff the graph $\langle C,\varrho\rangle$ is connected where $\varrho(a,b)$ iff $[a,b]\neq 1$ for $a,b\in C$.

W is *strongly connected* iff wWw^{-1} is connected for every w that contains letters of C only.

LEMMA 2.6. *If (A, R) is semifree and $A=\bigcup_{1\le i\le k} A_i$ with $[a_i, a_j]=1$ for $a_i\in A_i$, $a_j\in A_j$, $i\ne j$, then $(A, R)=\bigoplus_{1\le i\le k} \mathrm{Gp}(A_i)$.*

3. Nielsen transformations

Let G be any group. An elementary Nielsen transformation works on vectors $(u_1, u_2, \ldots)$ where $u_i\in G$. It is one of the following transformations:

(T1) Replace some u_i by u_i^{-1}.
(T2) Replace some u_i by u_iu_j where $j\ne i$.
(T3) Delete some u_i where $u_i=1$.

A finite product of such transformations is a *Nielsen transformation;* it is *regular* if there is no factor of type (T3), and *singular* otherwise. (The definitions above are taken from [3].)

LEMMA 3.1. *Every regular Nielsen transformation has an inverse.*

LEMMA 3.2. *If $(u_1, \ldots, u_n)$ is carried in $(v_1, \ldots, v_m)$ by a Nielsen transformation, then $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ generate the same subgroup.*

LEMMA 3.3. *If $(u_1, \ldots, u_n)$ is a basis of a free subgroup F of G and it is carried in $(v_1, \ldots, v_m)$ by a regular Nielsen transformation, then $n=m$ and $(v_1, \ldots, v_m)$ is a basis of F.*

LEMMA 3.4. *u and v generate a cyclic subgroup of G iff it is possible to apply a singular Nielsen transformation on (u, v).*

To check freeness we use the following well-known

LEMMA 3.5. *Let $u_1, \ldots, u_n$ be elements of G. Then $(u_1, \ldots, u_n)$ is a basis for a free subgroup of G iff $w(u_1, \ldots, u_n)\ne 1$ for every reduced work $w(x_1, \ldots, x_n)$.*

4. The main proof

THEOREM 4.1. *Let u and v be elements of a semifree group $G=(A, R)$ such that $\{u, v\}$ is strongly connected. If u and v do not generate a cyclic subgroup, then (u, v) is a basis of a free subgroup.*

By Lemma 2.5 we can assume

(1) The letters occurring in u or v are axactly the elements of A.

By a transformation on (u, v) a finite product of elementary Nielsen transformations and inner automorphisms is meant. Such a transformation is regular iff there is no factor of type (T3), and singular otherwise. Since u and v do not generate a cyclic subgroup of G, Lemma 3.4 implies

(2) All possible transformations of (u, v) are regular.

If σ is a regular transformation on (u, v), then Gp (u, v) is free with basis (u, v) iff Gp $(\sigma(u), \sigma(v))$ is free with basis $(\sigma(u), \sigma(v))$.

This follows from Lemmas 3.1 and 3.3. Using Lemma 3.5 for a suitable transformation σ it will be shown that Gp $(\sigma(u), \sigma(v))$ is free with basis $(\sigma(u), \sigma(v))$. By (2) σ is regular and, therefore, as stated above, Gp (u, v) is free with basis (u, v).

Remark that regular Nielsen transformations and inner automorphisms "commute". Therefore, if $\{u, v\}$ is strongly connected, then by (1) and (2) $\{\sigma(u), \sigma(v)\}$ is strongly connected for every regular transformation σ. Using a suitable regular transformation, (1), (2), and Lemma 2.5 again, we can suppose

(3) For every transformation σ, A is the set of letters occurring in $\sigma(u)$ or $\sigma(v)$. A is connected.

Case 1. There exist some letter $a \in A$ and a transformation σ such that $\sigma(u) = a^\alpha u' a^\beta$ and $\sigma(v) = a^\gamma v' a^\delta$ are minimal forms, where u' and v' contain letters that do not commute with a, $\beta+\gamma \neq 0$, $\alpha+\delta \neq 0$, $\beta-\delta \neq 0$, $\gamma-\alpha \neq 0$, and the exponents $\alpha, \beta, \gamma, \delta$ need not differ from 0.

By Lemmas 3.5, 2.3, and 2.4 Gp $(\sigma(u), \sigma(v))$ is free with basis $(\sigma(u), \sigma(v))$.

Case 2. There exist some transformation σ and some letter a such that $\sigma(u) = a^\alpha u'$, a is not a letter of u' and $\sigma(v)$, and every letter of u' commutes with a (resp. u and v are exchanged). Assume that $\sigma(u) = a^\alpha u'$. By (3) there is some letter d of $\sigma(v)$ with $[a, d] \neq 1$. By assumption d is not a letter of $\sigma(u)$. By Lemma 2.3 (ii) every $\sigma(u)^\varepsilon$ contains a and not d and every $\sigma(v)^\varepsilon$ contains d and not a. If $w(x_1, x_2)$ is any nontrivial reduced word then the $\{a, d\}$-sequence of $w(\sigma(u), \sigma(v))$ is the concatenation of the a-sequences of the $\sigma(u)$-powers in w and the d-sequences of the $\sigma(v)$-powers in w. Lemma 3.5 implies the assertion.

Case 3. Not Case 1, not Case 2, and there is some letter $a \in A$ and a transformation σ such that a is not a letter of both $\sigma(u)$ and $\sigma(v)$.

We show by induction on $\lambda(\sigma(u)) + \lambda(\sigma(v))$:

(4) If a and σ fulfil the condition of Case 3, then there is some w such that a is not a letter of both $w\sigma(u)w^{-1}$ and $w\sigma(v)w^{-1}$, and $w\sigma(u)w^{-1}$ and $w\sigma(v)w^{-1}$ are cyclic reduced.

Assume that $\sigma(u)$ and $\sigma(v)$ are not both cyclic reduced and a is an element of $\sigma(u)$. We consider the more difficult case when $\sigma(u)$ has a minimal form $c^\alpha u' c^{-\beta}$ with $\alpha, \beta > 0$ or $\alpha, \beta < 0$. Since the condition of Case 1 is not fulfilled $\sigma(v)$ has w. l. o. g. a minimal form $c^\alpha v'$ or $c^\gamma v''$, where c commutes with every letter of v'' ($\gamma = 0$ is possible). First we suppose $\sigma(v) = c^\alpha v'$. Then $c \neq a$, because c is an element of $\sigma(v)$. This implies that a is not a letter of $c^{-\alpha}\sigma(v)c^\alpha$. Furthermore

$$\lambda(c^{-\alpha}\sigma(u)\, c^\alpha) + \lambda(c^{-\alpha}\sigma(v)\, c^\alpha) = \lambda(u' c^{\alpha-\beta}) + \lambda(v' c^\alpha) <$$

$$< \lambda(c^\alpha u' c^{-\beta}) + \lambda(c^\alpha v') = \lambda(\sigma(u)) + \lambda(\sigma(v)).$$

The assertion follows from the induction hypothesis. If $\sigma(v)=c^{\gamma}v''$, where c commutes with every letter of v'', then $\gamma\neq 0$ implies $c\neq a$. Therefore $c^{-\alpha}\sigma(v)c^{\alpha}$ does not contain a for every γ. We have

$$\lambda(c^{-\alpha}\sigma(u)\,c^{\alpha})+\lambda(c^{-\alpha}\sigma(v)\,c^{\alpha}) = \lambda(u'c^{\alpha-\beta})+\lambda(c^{\gamma}v'')<$$
$$<\lambda(c^{\alpha}u'c^{-\beta})+\lambda(c^{\gamma}v'') = \lambda(\sigma(u))+\lambda(\sigma(v)).$$

The assertion follows as above from the induction hypothesis.

The situation of Case 3 implies furthermore:

(5) There are letters a and b in A such that $[a, b]\neq 1$ and either a is not in $\sigma(v)$ and b is in $\sigma(v)$ or a is not in $\sigma(u)$ and b is in $\sigma(u)$.

To prove (5) assume that Case 3 is given and a is an element of $\sigma(u)$. We get a and b with (5) since the following procedure must break off. Suppose that the letters $a_0=a, a_1,\dots, a_k$ are chosen such that

(5′) $\{a_0, \dots, a_k\}$ is connected, every a_i is an element of $\sigma(u)$ but not an element of $\sigma(v)$, and every a_i commutes with all elements of $\sigma(v)$.

Since $\{\sigma(u), \sigma(v)\}$ is connected, there exist some c in $\sigma(u)$ or in $\sigma(v)$ and some j with $0\leqq j\leqq k$ and $[a_j, c]\neq 1$. By (5′) c is not an element of $\sigma(v)$. Then either c and some letter of $\sigma(v)$ fulfil (5) or $a_0, \dots, a_k, c$ satisfy (5′). Since $\{\sigma(u), \sigma(v)\}$ is connected, the letters of $\sigma(u)$ cannot satisfy (5′), therefore the procedure breaks off, and we get some a, b with (5).

Applying at first (4) and then (5), and using u instead of $w\sigma(u)w^{-1}$ and v instead of $w\sigma(v)w^{-1}$ for convenience, we can suppose w. l. o. g.

(6) There is some letter a in u that is not a letter of v and some letter b in v such that $[a, b]\neq 1$. u and v are cyclic reduced.

To verify the condition of Lemma 3.5 we show the following:

(7) For every nontrivial reduced word $w(x_1, x_2)$ the a-sequence of $w(u, v)$ is the concatenation of the a-sequences of u in $w(u, v)$.

Firstly, we consider the case that $w(x_1, x_2)$ contains positive (negative, resp.) powers of x_1 only. For $w(x_1, x_2)=x_1^n$ (7) is true since u is cyclic reduced and by the conditions "not Case 1" and "not Case 2", a cannot be a first and a last letter of u at the same time. If (7) were false, there would be some subword $u^{\gamma}v^{\delta}u^{\varepsilon}$ so that the last a-power of u^{γ} can "touch" the first a-power of u^{ε} after some applications of (R2) and (R3) (Lemma 2.1). Since (7) is fulfilled for x_1^n we can suppose $\gamma=\varepsilon=1$ (-1, resp.). If there is one a-power in u only, let $u=u_1a^{\alpha}u_3$ be a minimal form of u. Then for every letter c of $u_3v^{\delta}u_1$, $[a, c]=1$, by assumption. We can replace (u, v) by $(u_1^{-1}uv^{\delta}u_1, u_1^{-1}vu_1)=$ $=(a^{\alpha}(u_3v^{\delta}u_1), u_1^{-1}vu_1)$ using a suitable transformation. Bur $\sigma(u)=a^{\alpha}(u_3v^{\delta}u_1)$ and $\sigma(v)=u_1^{-1}vu_1$ fulfil the condition of Case 2, a contradiction.

Otherwise, there is some minimal form $u_1a^{\alpha}u_2a^{\beta}u_3$ of u, where a does not occur in u_1 and u_3. As above $[a, c]=1$ for every letter c of $u_3v^{\delta}u_1$. There is some transformation τ such that $\tau(u)=u_1^{-1}uv^{\delta}u_1=a^{\alpha}u_2(u_3v^{\delta}u_1)a^{\beta}$ and $\tau(v)=u_1^{-1}vu_1$. a does not occur in $u_1^{-1}vu_1\cdot u_1^{-1}vu_1$ contains b of (6) by Lemma 4.2 below, and in $u_2(u_3v^{\delta}u_1)$

there is some d with $[a, d]\neq 1$, since such a d is in u_2, and $[a, c]=1$ for every c of $u_3 v^\delta u_1$. Therefore we have Case 1, a contradiction.

Now let $w(x_1, x_2)$ be any nontrivial reduced word. We find a minimal form $w_1(x_1, x_2)x_2^{\beta_1}w_2(x_1, x_2)x_2^{\beta_2}...x_2^{\beta_{l-1}}w_l(x_1, x_2)$ of $w(x_1, x_2)$ where:

(i) x_1 is a last letter of $w_i(x_1, x_2)$ for $1\leq i<l$.

(ii) x_1 is a first letter of $w_i(x_1, x_2)$ for $1<i\leq l$.

(iii) There are either only positive or only negative powers of x_1 in $w_i(x_1, x_2)$ for $1\leq i\leq l$.

(iv) If the powers of x_1 are positive in w_i, then the powers of x_1 in w_{i+1} are negative $(1\leq i<l)$. If the powers of x_1 are negative in w_i, then the powers of x_1 are positive in w_{i+1} $(1\leq i<l)$.

Let us consider $w_i(u, v)v^\beta w_{i+1}(u, v)$. Then $w_i(x_1, x_2)=w_i'(x_1, x_2)x_1^\gamma$, $w_{i+1}(x_1, x_2)=x_1^\delta w_{i+1}'(x_1, x_2)$ with $\lambda(w_j(x_1, x_2))>\lambda(w_j'(x_1, x_2))$ for $j=i, i+1$. Assume w. l. o. g. $\gamma>0>\delta$. ($\delta>0>\gamma$ is similar.) Let $u=u'a^\alpha r$ be a minimal form, where a is not a letter of r, and every power in r occurs in every minimal form of u on the right of a^α. Since (7) holds for $w_i(u, v)$ and $w_{i+1}(u, v)$, we get $w_i(u, v)=\overline{w}_i(u, v)a^\alpha r$, $w_{i+1}(u, v)= =r^{-1}a^{-\alpha}\overline{w}_{i+1}(u, v)$. Now it is suffcient to show that $rv^\beta r^{-1}$ contains the letter b of assumption (6). This follows from

LEMMA 4.2. *Assume that the letter b occurs in a word y, and y is cyclic reduced. Then b is a letter of ryr^{-1} for every r.*

PROOF. The lemma will be proved by induction on $\lambda(r)$. If there is any cancellation of a full power of b in ryr^{-1}, assume w. l. o. g. that $y=sb^\beta y'$ and $r=r'b^{-\beta}s^{-1}$ are minimal forms, where s does not contain b and $s=e$ is possible. Then $ryr^{-1}= =r'(y'sb^\beta)r'^{-1}$. Since y is cyclic reduced, there is no cancellation in $(y'sb^\beta)$, b is a letter of $(y'sb^\beta)$, and $(y'sb^\beta)$ is cyclic reduced. Since $\lambda(r')<\lambda(r)$, by induction hypothesis $ryr^{-1}=r'(y'sb^\beta)r'^{-2}$ contains b. Q. E. D.

Case 4. Not Case 1 and for every transformation σ, each of $\sigma(u)$ and $\sigma(v)$ contains all letters of A.

Let a be any element of A. Define $\lambda_a(w)$ to be the length of the a-sequence of w. By the assumption above we have $\lambda_a(\sigma(u))>0$ and $\lambda_a(\sigma(v))>0$ for every transformation σ. Now we take σ in such a way that $\lambda_a(\sigma(u))+\lambda_a(\sigma(v))$ is minimal. For convenience we use u instead of $\sigma(u)$ and v instead of $\sigma(v)$. Therefore

(8) $\lambda_a(u)+\lambda_a(v)$ is minimal with respect to transformations.

We need

(9) $\lambda_a(xx)=2\lambda_a(x)$ for every $x\in\{u, v, u^{-1}, v^{-1}\}$.

To prove (9) assume w. l. o. g. $x=u$. Firstly consider $\lambda_a(x)=1$ and $\lambda_a(xx)<2\lambda_a(x)$. There is some minimal form $u_1a^\alpha zu_3$ of x, where every letter of u_3 is on the right of a^α in every minimal form of u, every letter of u_1 is on the left of a^α in every minimal form of u, and every letter of z commutes with a. Then $\lambda_a(uu)=1$ implies $u_1^{-1}=u_3$. $u_1^{-1}uu_1=a^\alpha z$ does not contain every letter of A, a contradiction to the assumption of Case 4.

Now suppose $\lambda_a(u)\geq 2$ and $\lambda_a(uu)<2\lambda_a(u)$. Similarly as above you get a minimal form $u_1a^\alpha u_2a^\beta u_1^{-1}$ of u. Then $u_1^{-1}uu_1=a^\alpha u_2a^\beta$. Since we are not in Case 1 and every letter of A occurs in $u_1^{-1}vu_1$, w.l.o.g. $a^\alpha v'$ is a minimal form of $u_1^{-1}vu_1$. It follows

$$\lambda_a(a^{-\alpha}u_1^{-1}uu_1a^\alpha)+\lambda_a(a^{-\alpha}u_1^{-1}vu_1a^\alpha) = \lambda_a(u_2a^{\alpha+\beta})+\lambda_a(v'a^\alpha)<\lambda_a(u)+\lambda_a(v),$$

a contradiction to (8).

Let us assume that there is at most consolidation but not full cancellation of a-powers. That means:

(10) For every $x, y\in\{u, u^{-1}, v, v^{-1}\}$ with $x\neq y^{-1}$ either
(i) $\lambda_a(xy)=\lambda_a(x)+\lambda_a(y)$ or
(ii) $x=w_1a^\alpha w_2$, $y=w_1a^\beta w_3$ are minimal forms, where a is not a letter of w_1 and $\alpha-\beta\neq 0$.

(10) and $\lambda_a(u)$, $\lambda_a(v)>1$ imply that $w(u, v)\neq 1$ for any nontrivial reduced $w(x_1, x_2)$.

It remains to settle the case when (w.l.o.g.) a^α is the only a-power in u. If for every $x, y\in\{u, v, u^{-1}\}$ (10) (i) is true, then there is nothing to do. Otherwise, by (9) w.l.o.g. u and v satisfy (10) (ii).

Applying $w_1^{-1}...w_1$, w.l.o.g. $u=a^\alpha w_2$, $v=a^\beta w_3$ are minimal forms of u and v. Then a is not a last letter of u, since otherwise $[a, c]=1$ for every letter c of u and by the assumption of Case 4 for every $c\in A$. This would be a contradiction because A is connected. Similarly, a is not a last letter of v if $\lambda_a(v)=1$. If $\lambda_a(v)>1$, then a is not a last letter of v, too. Otherwise, we would have minimal forms $u=a^\alpha w_2$, $v=a^\beta w_3' a^\gamma$, where by (10) $\alpha, \beta, \gamma\neq 0$, $\alpha+\gamma\neq 0$, $\alpha-\beta\neq 0$. Then the condition of Case 1 is fulfilled, a contradiction.

Therefore, $\lambda_a(uv)=\lambda_a(vu)=\lambda_a(u)+\lambda_a(v)$. If $\lambda_a(uv^{-1})=\lambda_a(u)+\lambda_a(v)$, the assertion follows. Otherwise, $\lambda_a(uv^{-1})=\lambda_a(v)$ and we get a contradiction in the following way:

If $\lambda_a(v)=1$, then $uv^{-1}=a^{\alpha-\beta}z$, where every letter of z commutes with a. If σ is the transformation with $\sigma(u)=u$, $\sigma(v)=uv^{-1}$, the assumption of Case 4 is violated, a contradiction.

If $\lambda_a(v)>1$, then we have a minimal form $uv^{-1}=a^\gamma w' a^{-\beta}$. By $\lambda_a(uv^{-1})=\lambda_a(v)$, $\gamma\neq 0$. If $\alpha\neq\gamma$, the conditions of Case 1 are fulfilled for u and uv^{-1} (remember that $\alpha\neq\beta$). If $\alpha=\gamma$, then

$$\lambda_a(a^{-\alpha}(uv^{-1})\,a^\alpha)+\lambda_a(a^{-\alpha}ua^\alpha) = \lambda_a(w'a^{\alpha-\beta})+\lambda_a(w_2a^\alpha)<$$
$$<\lambda_a(uv^{-1})+\lambda_a(u) = \lambda_a(v)+\lambda_a(u).$$

This contradicts (8).

Contrary to (10) it remains to suppose that there are $x, y\in\{u, v, u^{-1}, v^{-1}\}$, $x\neq y^{-1}$ with minimal forms $x=w_1a^\alpha w_2$, $y=w_1a^\alpha w_3$, where w_1 does not contain the letter a and $w_1=e$ is possible. By (9), $x\neq y$. Applying $w_1^{-1}...w_1$ we can assume w.l.o.g.

(11) $u=a^\alpha u'$ and $v=a^\alpha v'$, where a is not a first letter of u', v'.

Then $\lambda_a(u)$, $\lambda_a(v)>1$, because otherwise $\lambda_a(u^{-1}v)+\lambda_a(u)<\lambda_a(u)+\lambda_a(v)$ or $\lambda_a(u^{-1}v)+\lambda_a(v)<\lambda_a(u)+\lambda_a(v)$, a contradiction to (8).

We have $\lambda_a(uv)=\lambda_a(vu)=\lambda_a(u)+\lambda_a(v)$. By $\lambda_a(u)\geqq 2$, $\lambda_a(v)\geqq 2$ there are minimal forms $u=a^\alpha u_1a^\beta u_2$ and $v=a^\alpha v_1a^\gamma c_2$, where a is not a letter of u_2 and v_2, every power in u_2 occurs in every minimal form of u on the right of a^β, every power in v_2 occurs in every minimal form of v on the right of a^γ, and $u_2, v_2\neq e$.

Assume w.l.o.g. $\lambda_a(u) \geqq \lambda_a(v)$. By (8), $\lambda_a(u^{-1}v)$, $\lambda_a(v^{-1}u) \geqq \lambda_a(u) \geqq 2$.

If $u_2 \neq v_2$ or $\beta \neq \gamma$, then full cancellation of a-powers is only possible in products $u^{-1}v$. (Remark that $u_2^{-1}v_2 \neq 1$ implies the existence of some letter d in $u_2^{-1}v_2$ with $[a, d] \neq 1$ by the construction of u_2 and v_2.)

$u^{-1}v$ has a minimal form $u_2^{-1}a^{-\beta}za^{\gamma}v_2^{-1}$ since $\lambda_a(u^{-1}v) \geqq \lambda_a(u) \geqq 2$. We can write every nontrivial word $w(u, v)$ as a product $w_1w_2...w_n$ with $w_i = u^{-1}v$, or $w_i = v^{-1}u$, or $w_i \in \{u, v, u^{-1}, v^{-1}\}$ and $w_i = u$ implies $w_{i-1} \neq v^{-1}$, $w_i = v$ implies $w_{i-1} \neq u^{-1}$, $w_i = u^{-1}$ implies $w_{i+1} \neq v$, and $w_i = v^{-1}$ implies $w_{i+1} \neq u$. Then there is no full cancellation of an a-power between w_i and w_{i+1}. It follows $w(u, v) \neq e$. If $u_2 = v_2$ and $\beta = \gamma$, we prove

(12) $\lambda_a(uv^{-1}) > \lambda_a(u)$ and $\lambda_a(u^{-1}v) > \lambda_a(u)$.

By (8) uv^{-1} has a normal form $a^{\alpha}za^{-\alpha}$. Then $\lambda_a(uv^{-1}) \leqq \lambda_a(u)$ would imply

$$\lambda_a\big(a^{-\alpha}(uv^{-1})\,a^{\alpha}\big) + \lambda_a(a^{-\alpha}va^{\alpha}) < \lambda_a(uv^{-1}) + \lambda_a(v) \leqq \lambda_a(u) + \lambda_a(v),$$

contrary to (8).

$u^{-1}v$ has a minimal form $u^{-1}v = u_2^{-1}a^{-\beta}za^{\beta}u_2$ by (8), $u_2 = v_2$ and $\beta = \gamma$. Then $\lambda_a(u^{-1}v) \leqq \lambda_a(u)$ would imply

$$\lambda_a(a^{\beta}u_2u^{-1}vu_2^{-1}a^{-\beta}) + \lambda_a(a^{\beta}u_2vu_2^{-1}a^{-\beta}) < \lambda_a(u^{-1}v) + \lambda_a(v) \leqq \lambda_a(u) + \lambda_a(v),$$

a contradiction to (8).

(12) implies

(13) $\lambda_a(xy) > \max\big(\lambda_a(x), \lambda_a(y)\big)$ for every $x, y \in \{u, v, u^{-1}, v^{-1}\}$ with $x \neq y^{-1}$.

If z has a minimal form $r_0a^{\alpha_1}r_1a^{\alpha_2}r_2...r_{l-1}a^{\alpha_l}r_l$, where $(a^{\alpha_1}, a^{\alpha_2}, ..., a^{\alpha_l})$ is the a-sequence of z and $r_0 = e$, $r_l = e$ is possible, define $m(z)$ to be the subword $a^{\alpha_{k+1}}$ if $l = 2k+1$ and $m(z) = a^{\alpha_k}r_ka^{\alpha_{k+1}}$ if $l = 2k$.

Let $w = z_1^{\varepsilon_1}...z_n^{\varepsilon_n}$ with $z_i \in \{u, v\}$, $\varepsilon_i \in \{1, -1\}$ be any nontrivial reduced word. Then by (13) the subwords $m(z_i)$ will not be cancelled in w. That means if $m(z_i) = a^{\alpha_{k+1}}$, $m(z_i)$ remains in w, if $m(z_i) = a^{\alpha_k}r_ka^{\alpha_{k+1}}$, only consolidation of a^{α_k} and $a^{\alpha_{k+1}}$ is possible. Hence $w \neq 1$. Q.E.D.

5. Consequences

THEOREM 1.2. *Let u, v be elements of a semifree group (A, R). Then $[u, v] \neq 1$ implies that $\{u, v\}$ is a basis of a free subgroup of rank 2.*

PROOF. Applying a suitable inner automorphism we can suppose:

$(*)$ For every w that contains letters of u and v only the same letters as in u and v occur in wuw^{-1} and wvw^{-1}.

By Lemma 2.5 it is possible to assume that A is the set of letters in u and v. If A is connected, then by $(*)$ $\{u, v\}$ is strongly connected and the assertion follows from Theorem 4.1.

Otherwise, $A = \bigcup_{1 \leqq i \leqq k} A_i$ with $k > 1$, every A_i is connected, and $[a_i, a_j] = 1$ for $a_i \in A_i$ and $a_j \in A_j$ with $i \neq j$. Then $(A, R) = \bigoplus_{1 \leqq i \leqq k} \mathrm{Gp}\,(A_i)$ by Lemma 2.6. Let $u =$

$=u_1u_2\dots u_k$ and $v=v_1v_2\dots v_k$ be minimal forms with $u_i, v_i \in \mathrm{Gp}(A_i)$. Since $[u, v]\neq 1$, there must be some j with $[u_j, v_j]\neq 1$. Since A is the set of all letters in u and v, by Lemma 2.2 (ii) A_j is the set of all letters of u_j and v_j. Therefore, $\{u_j, v_j\}$ is strongly connected by $(*)$. Applying Theorem 4.1 the assertion follows. Q.E.D.

Further consequences of Theorem 4.1 are the results of [1].

THEOREM 5.1. *Let u, v be elements of a semifree group (A, R). Then $[u, v]=1$ iff there are elements w, w_i, and integers α_i, β_i $(1\leqq i\leqq n)$ such that:*

(i) *If $i\neq j$, then every letter of w_i commutes with every letter of w_j.*

(ii) *Every w_i is connected.*

(iii) $u=w_1\prod_{1\leqq i\leqq n} w_i^{\alpha_i}w^{-1}$ *and* $v=w\prod_{1\leqq i\leqq n} w_i^{\beta_i}w^{-1}$.

THEOREM 1.1. *Every Abelian subgroup of a semifree group is free-Abelian.*

6. Counterexamples

Unfortunately, it is not possible to sharpen Theorem 1.2. We need the following result of Baumslag:

THEOREM 6.1 (BAUMSLAG [2]). *For all elements u, v, w of a free group it holds: $[u, v]=w^n\neq 1$ with $n\geqq 1$ implies $n=1$.*

THEOREM 6.2 *Let G be the semifree group*

$$(\{a, b, c\},\quad \{[a, c]\}) \oplus (\{x, y, z\},\quad \{[y, z]\})$$

and G_1 be $\mathrm{Gp}(\{ax, by, cz\})$. *Then G_1 is not semifree.*

PROOF. Since $[ax, cz]=[x, z]\neq 1$ and $[by, cz]=[b, c]\neq 1$, G_1 is not Abelian and therefore not cyclic. Furthermore, $[x, z]$, $[b, c]\in G_1$, $[[x, z], [b, c]]=1$, but $[x, z]$, $[b, c]$ do not generate a cyclic subgroup of G. Then $\mathrm{Gp}(\{[x, z], [b, c]\})$ is not a cyclic subgroup of G_1 and therefore G_1 is not free. By Theorem 1.2, it is not possible to generate G_1 by fewer than three elements.

Assume that G_1 is semifree. The facts above imply $G_1=(\{u_1, u_2, u_3\}, R)$, where R contains one or two $[u_i, u_j]$. Then w.l.o.g. either

(i) $G_1 = \mathrm{Gp}(\{u_1\})\oplus(\mathrm{Gp}(\{u_2\})*\mathrm{Gp}(\{u_3\}))$

or

(ii) $G_1 = \mathrm{Gp}(\{u_1\})*(\mathrm{Gp}(\{u_2\})\oplus\mathrm{Gp}(\{u_3\}))$.

Case (i). Let $ax=r_1s_1$, $by=r_2s_2$, $cz=r_3s_3$ with $r_i\in\mathrm{Gp}(\{u_1\})$ and $s_i\in\mathrm{Gp}(\{u_2\})*$ $*\mathrm{Gp}(\{u_3\})$ for $1\leqq i\leqq 3$. Since ax, by, cz do not pairwise commute, $s_i\neq 1$ for $1\leqq i\leqq 3$. As a subgroup of the free group $\mathrm{Gp}(\{u_2\})*\mathrm{Gp}(\{u_3\})$, $G_2=\mathrm{Gp}(\{s_1, s_2, s_3\})$ is a free subgroup of G. Then $[[s_1, s_3], [s_2, s_3]]=1$ implies the existence of some $v\in G_2$ and some integers n, m with $[s_1, s_3]=v^m$ and $[s_2, s_3]=v^n$. By the Theorem 6.1 of Baumslag $|m|=1$ and $|n|=1$. If $[s_1, s_3]=[s_2, s_3]$, then $[s_3, s_1s_2^{-1}]=1$ and therefore $[cz, ax(by)^{-1}]=$ $=1$, a contradiction by the definition of G. If $[s_1, s_3]=[s_2, s_3]^{-1}$, then $[x, y]=[ax, cz]=$ $=[s_1, s_3]=[s_2, s_3]^{-1}=[by, cz]^{-1}=[b, c]^{-1}$, a contradiction. Therefore, the only possibility of G_1 to be semifree is

Case (ii). Consider $[[x, z], [b, c]]=1$ and $[x, z], [b, c]\in G_1$. By Theorem 5.1 there are v, w and intergers n, m, n_2, m_2, n_3, m_3 such that $[x, z]=wv^n w^{-1}$ and $[b, c]=wv^m w^{-1}$, where u_1 is a letter of v or $[x, z]=wu_2^{n_2}u_3^{n_3}w^{-1}$ and $[b, c]=wu_2^{m_2}u_3^{m_3}w^{-1}$. W.l.o.g. we can suppose $w=e$. Otherwise, apply the inner automorphism $w^{-1}...w$ of G_1.

The first case above is impossible because otherwise v would be a nontrivial common element of the subgroups Gp $(\{x, y, z\})$ and Gp $(\{a, b, c\})$ of G.

If $[x, z]=u_2^{n_2}u_3^{n_3}$ and $[b, c]=u_2^{m_2}u_3^{m_3}$, assume that $u_2=r_2s_2$ and $u_3=r_3s_3$ with $r_i\in$ Gp $(\{x, y, z\})$ and $s_i\in$ Gp $(\{a, b, c\})$. Since $[x, z]\in$ Gp $(\{x, y, z\})$, it follows $s_2^{n_2}s_3^{n_3}=e$ and $[x, z]=r_2^{n_2}r_3^{n_3}$. By $[u_2, u_3]=1$ we have $[r_2, r_3]=1$. Therefore and by Theorem 1.1 Gp $(\{r_2, r_3\})$ is free-Abelian of rank at most two. If there is some r with $r_2=r^i$ and $r_3=r^j$, then $[x, z]=r^{in_2+jn_3}$. By Lemma 2.3 r is an element of the free group Gp $(\{x,z\})$. By the Theorem of Baumslag (6.1) $[x, z]=r$ or $[x, z]=r^{-1}$. Therefore, Gp $(\{[x, z]\})=$ Gp $(\{r_2, r_3\})$. Otherwise, (r_2, r_3) is a basis of free-Abelian group. Since $[b, c]=s_2^{m_2}s_3^{m_3}$ and $r_2^{m_2}r_3^{m_3}=1$, it follows $m_2=m_3=0$. But this contradicts $[b, c]=s_2^{m_2}s_3^{m_3}\neq 1$.

Analogously, we can show that Gp $(\{[b, c]\})=$ Gp $(\{s_2, s_3\})$. Therefore, Gp $(\{u_2\})\oplus$ Gp $(\{u_3\})\subseteqq [G_1, G_1]$. It follows that $G_1/[G_1, G_1]$ is a cyclic group. This contradicts the fact that rank $(G_1/[G_1, G_1])=3$. Q.E.D.

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ZENTRALINSTITUT FÜR MATHEMATIK
UND MECHANIK
DDR 108 BERLIN
MOHRENSTRASSE 39