Zeitschrift für Operations Research, Volume 27, page 39–54.

Fractional Programming[1])

By *S. Schaible*, Edmonton[2])

Received October 1982

Abstract: Following a comprehensive bibliography recently published in this journal, we review major results in fractional programming. The emphasis is on structural properties of fractional programs and their algorithmic implications. We limit the discussion to those types of ratio optimization problems for which we see a significant interest in the applications. It is attempted to provide a theoretical framework for future research in this area.

Zusammenfassung: Im Anschluß an die kürzliche Veröffentlichung einer umfassenden Bibliographie zur Quotientenprogrammierung in dieser Zeitschrift wird in der vorliegenden Arbeit ein Überblick über wichtige Ergebnisse auf diesem Gebiet der nichtlinearen Programmierung gegeben. Es stehen dabei strukturelle Eigenschaften von Quotientenprogrammen sowie deren Bedeutung für Lösungsverfahren im Mittelpunkt der Untersuchung. Die Diskussion beschränkt sich auf solche Quotientenprogramme, die für die Anwendungen von größerem Interesse sind. Es wird in der Arbeit versucht einen theoretischen Rahmen zu entwickeln, der für weitere Untersuchungen zur Quotientenprogrammierung hilfreich sein kann.

1. Introduction

Some decision problems in management science as well as other extremum problems give rise to the optimization of ratios. Constrained ratio optimization problems are commonly called fractional programs. They may involve more than one ratio in the objective function.

One of the first fractional programs (though not called so) was discussed as early as 1937 in von Neumann's classical paper on an expanding economy [see *von Neumann*, 1937, 1945]. In this economic equilibrium model the growth rate, i.e. the smallest of several output-input ratios is to be maximized. Von Neumann proposed a duality theory for this fractional program.

---

[1]) Invited survey.

[2]) Prof. Dr. *Siegfried Schaible*, Department of Finance and Management Science, Faculty of Business, University of Alberta, Edmonton, Canada T6G 2G1. This research was supported by J.D. Muir Fund, Faculty of Business, University of Alberta.

0340-9422/83/010039-54$2.50

Since that time, but mainly after the classical paper by *Charnes/Cooper* [1962] about 500 publications have appeared in fractional programming. A comprehensive bibliography as of March 1982 was recently published in this journal [see *Schaible*, 1982a].

Fractional programming has meanwhile been recognized by "International Abstracts in Operations Research," "Mathematical Reviews," and "Zentralblatt für Mathematik" as a separate entity within the area of nonlinear programming like quadratic programming or convex programming.

However, the discussion of fractional programs in textbooks is still quite limited and often restricted to the linear case. In 1978 a monograph solely devoted to fractional programming appeared [see *Schaible*, 1978]. In it a detailed survey of (potential) applications is given and theoretical and algorithmic results for concave fractional programs are presented.

In the present paper we review some major fractional programming results. Here we mainly focus on the theory and on solution strategies that result from the theory. For a review of applications and computational results see *Schaible/Ibaraki* [1983].

The material is organized as follows:
Section 2 provides notation and definitions; in Section 3 we briefly list some of the major application areas of fractional programming; Section 4 is devoted to the theory of singleratio problems whereas Section 5 deals with multiratio problems. In both sections we discuss the linear and the concave case. Nonconcave fractional programs are addressed in Section 6. Finally Section 7 deals with solution strategies resulting from the structural properties of fractional programs.

The bibliography at the end is not meant to be representative for the various efforts to build up a theory of fractional programming. In this paper we restrict our attention to those types of fractional programs for which we see a significant chance of being actually used. But even here the intention was not to include all the results, but to provide a framework for future work in fractional programming. A comprehensive bibliography is given in *Schaible* [1982a].

2. Notation and Definitions

Suppose f, g, h_j $(j = 1, \ldots, m)$ are realvalued functions which are defined on the set X of R^n. Let $h = (h_1, \ldots, h_m)^T$. We consider the ratio

$$q(x) = \frac{f(x)}{g(x)} \tag{1}$$

over the set

$$S = \{x \in X: h(x) \leqslant 0\}. \tag{2}$$

We assume that $g(x)$ is positive on X. If $g(x)$ is negative then
$q(x) = (-f(x))/(-g(x))$ may be used instead. The nonlinear program

(P) $\sup \{q(x): x \in S\}$ (3)

is called a *fractional program.*

In some applications more than one ratio appears in the objective function. Examples of such models are

$$\sup \{\sum_{i=1}^{p} q_i(x) : x \in S\} \tag{4}$$

and

$$\text{(GP)} \quad \sup \{\min_{1 \leqslant i \leqslant p} q_i(x): x \in S\}. \tag{5}$$

Here $q_i(x) = f_i(x)/g_i(x)$ $(i = 1, \ldots, p)$ where f_i, g_i are realvalued functions on X with $g_i(x) > 0$. Problem (GP) is sometimes referred to as a *generalized fractional program* [*Crouzeix/Ferland/Schaible; Jagannathan/Schaible*]. Both problems (4) and (5) are related to the multiobjective fractional program

$$\max_{x \in S} (q_1(x), \ldots, q_p(x)) \tag{6}$$

[*Choo; Warburton*].

3. Applications

In this section we briefly list major application areas of fractional programming. For more details and references see *Schaible* [1978, 1981] and *Schaible/Ibaraki* [1983]. Earlier applications are reviewed in *Grunspan* [1971].

We find the following types of applications in the literature: economic applications, non-economic applications, indirect applications.

a) *Economic Applications*

Ratios to be optimized often describe some kind of an efficiency measure for a system. Among others the following examples are found [for more examples see *Kornbluth/Steuer*, 1981b]:

relative usage of material, productivity, profit/capital, profit/revenue, return/cost, return/risk, cost/time, expected cost/time, profit/time, liquidity, earnings per share, dividend per share.

These ratios arise in resource allocation, transportation, production, finance, maintenance or applications of stochastic processes or Markov renewal programs.

b) *Non-Economic Applications*

Outside management science fractional programs occur in information theory, applied linear algebra and approximation theory for example.

c) *Indirect Applications*

Finally, fractional programs are sometimes encountered as part of a solution procedure for another optimization problem. Examples are approximations to numerically untractable portfolio selection problems, dual location problems, subproblems in large-scale programming and deterministic substitutes of stochastic programs. Since some of these models arise in a variety of planning situations fractional programs occur indirectly in many different contexts.

After this brief review of applications we turn now to theoretical results.

4. Concave Singleratio Programs

The focus in fractional programming has been on the objective function and not on the constraint set S. As far as S is concerned it is assumed in most publications that S is a convex set of R^n. Accordingly, we will require in this paper that the domain X of all functions in (3), (4), (5) and (6) is a (nonempty) convex set and the constraint functions h_j are convex on X. This implies convexity of the feasible region S.

In this section we discuss the singleratio problem (P) in (3). In many applications of (3) the ratio $q(x)$ in the objective satisfies the following concavity/convexity assumption:

$$K: \begin{cases} f \text{ is concave and } g \text{ is convex on } X; \\ f \text{ is positive on } S \text{ if } g \text{ is not affine (linear plus constant).} \end{cases}$$

A ratio problem (3) is called a *concave fractional program* if condition K is satisfied. For such nonlinear programs Proposition 1–4 below can be proved [see *Mangasarian*, 1969a; *Avriel*[3])]:

Proposition 1: The objective function $q(x)$ in a concave fractional program is semistrictly quasiconcave on S. It is strictly quasiconcave if either f is strictly concave or g is strictly convex.

Proposition 2: If f, g are differentiable in a concave fractional program then the objective function $q(x)$ is pseudoconcave on S. It is strictly pseudoconcave there if either f is strictly concave or g is strictly convex.

From Proposition 1 and 2 we conclude:

Proposition 3: In a concave fractional program (P) any local maximum is a global maximum, and (P) has at most one maximum if f is strictly concave or g is strictly convex. In a differentiable concave fractional program a solution of the Karush-Kuhn-Tucker conditions is a maximum of (P).

A special case of a concave fractional program is the *linear fractional program* where f, g are affine functions and S is a convex polyhedron:

---

[3]) Instead of "strictly quasiconcave" and "strongly quasiconcave" as in *Avriel* [1976] we use the terms "semistrictly quasiconcave" and "strictly quasiconcave" respectively which were introduced in *Avriel* et al. [1981] to streamline the various definitions of concave and generalized concave functions.

$$\sup\left\{\frac{c^Tx+\alpha}{d^Tx+\beta}: Ax \leqslant b, x \geqslant 0\right\}. \tag{7}$$

Here $c, d \in R^n$, $b \in R^m$, $\alpha, \beta \in R$, A is a $m \times n$ matrix and T denotes the transpose.

More generally (P) is called a *quadratic fractional program* if f, g are quadratic functions and S is a convex polyhedron.

As we see concave, linear and quadratic fractional programs are extensions of concave, linear and quadratic programs where a convex, affine or quadratic denominator, respectively, is introduced.

For linear fractional programs we have in addition to Proposition 1–3:

Proposition 4: In a linear fractional program (7) the objective function is quasiconvex on the feasible region S, and therefore a maximum is attained at a vertex of S if S is nonempty and bounded.

We have seen that concave and linear fractional programs have several important properties in common with concave and linear programs, respectively.

In the following we want to discuss the possibilities of reducing concave fractional programs to a concave program. Later in this section we then introduce duality for these quasiconcave programs.

In order to relate an optimization problem to a concave program one might try to find either a suitable range transformation or a variable transformation [*Avriel*]. It was shown in *Schaible* [1971] that any quadratic program that is quasiconcave can be reduced to a concave program with help of a range transformation. Unfortunately, the same is not possible for fractional programs that are quasiconcave. In fact, one can prove that in case of the simple linear ratio $q(x) = x_1/x_2$ there does not exist any strictly increasing function H such that $H(q(x))$ is concave on an open convex set in the halfspace $S = \{x \in R^2 : x_2 > 0\}$.

However, a subclass of concave fractional programs can be reduced to an equivalent concave program by a range transformation [*Schaible*, 1974]:

Proposition 5: If in a concave fractional program $(g(x))^{\epsilon}$ is still convex on S for some $\epsilon \in (0, 1)$ then the equivalent problem

$$\sup\ \{-q(x)^{-t} : x \in S\} \tag{8}$$

is a concave program for $t \geqslant \epsilon/(1-\epsilon)$.

As an application consider the fractional program

$$\sup\ \left\{q(x) = \frac{f(x)}{x^TDx} : x \in S\right\}$$

where f is a positive concave function and $g(x) = x^TDx$ is a positive convex quadratic form on S. Then the equivalent problem

$$\sup\ \{-q(x)^{-1} : x \in S\}$$

has a concave objective function since $\sqrt{g(x)}$ is still convex, and thus we can use $t \geqslant 1$ in Proposition 5.

It is also possible to relate a very large class of concave fractional programs to a concave program by separating numerator and denominator with help of a parameter λ [*Jagannathan; Dinkelbach*; for similar approaches see *Geoffrion*, 1967; *Ritter; Cambini*]. Consider the parametric problem

$$(\mathrm{P}_\lambda) \quad \sup \{f(x) - \lambda g(x) : x \in S\}. \tag{9}$$

Suppose f, g are continuous and S is a (nonempty) compact set. Then

$$F(\lambda) = \max \{f(x) - \lambda g(x) : x \in S\}$$

is a strictly decreasing, continuous function on R where $F(\lambda) \to +\infty\ (-\infty)$ if $\lambda \to -\infty\ (+\infty)$. Let $\bar{\lambda}$ denote the unique zero of $F(\lambda)$. It is easily seen [*Jagannathan; Dinkelbach*]:

Proposition 6: The optimal solutions of $(\mathrm{P}_{\bar{\lambda}})$ and (P) are the same and $\bar{\lambda} = q(\bar{x})$ where $\bar{x}$ is an optimal solution of (P).

If (P) is a concave fractional program then (P_λ) is a parametric concave program for each λ in the range of $q(x)$. Hence, *any* concave fractional program considered above can be reduced to a parametric concave program.

In the transformations above, variables were not changed. We will now show that by a suitable transformation of variables *every* concave fractional program reduces to a *parameter-free* concave program. For the special case of a linear fractional program this transformation was suggested in *Charnes/Cooper* [1962]. It was extended to the concave case in *Schaible* [1973, 1976a].

We differentiate between problems with affine and nonaffine denominators.

Proposition 7: A concave fractional program (P) with an affine denominator can be reduced to the concave program

$$(\mathrm{P}'_=) \quad \sup \{tf(y/t) : th_j(y/t) \leqslant 0 \quad j = 1, \ldots, m, \ \ tg(y/t) = 1, \ y/t \in X, t > 0\} \tag{10}$$

by the transformation

$$y = \frac{1}{g(x)} x, \ \ t = \frac{1}{g(x)}. \tag{11}$$

In the proof one shows that (11) is a one-to-one mapping of S onto the feasible region of $(\mathrm{P}'_=)$, and that condition K and convexity of S imply that $(\mathrm{P}'_=)$ is a concave program.

In the special case of a linear fractional program (7), $(\mathrm{P}'_=)$ becomes the linear program

$$\sup \{c^T y + \alpha t : Ay - bt \leqslant 0, \ d^T y + \beta t = 1, y \geqslant 0, t > 0\} \tag{12}$$

[*Charnes/Cooper*, 1962]. Note that the strict inequality $t > 0$ can be replaced by $t \geqslant 0$ if (7) has an optimal solution.

In the case of a concave fractional program with a nonaffine denominator we relax the equality $tg(y/t) = 1$ in (10) to the inequality $tg(y/t) \leqslant 1$. Then the following is true:

Proposition 8: A concave fractional program (P) with a nonaffine denominator is equivalent to the concave program

$$(\mathrm{P}') \quad \sup\{tf(y/t)\colon th_j(y/t) \leqslant 0 \;\; j=1,\ldots,m, \;\; tg(y/t) \leqslant 1, y/t \in X, \, t>0\} \tag{13}$$

applying the transformation (11).

In the proof it is shown that at an optimal solution of (P′) the inequality $tg(y/t) \leqslant 1$ is satisfied as an equality.

If g is not affine the additional constraint $tg(y/t) \leqslant 1$ in (P′) is nonlinear. Such a nonlinear constraint can be avoided if the numerator f is affine by applying transformation (11) to the problem $\sup\{-g(x)/f(x)\colon x \in S\}$ which is equivalent to (P).

We illustrate this by the following example. In stochastic programming the maximum probability model [*Charnes/Cooper*, 1963] gives rise to the concave fractional program

$$\sup \left\{ q(x) = \frac{e^T x - k}{\sqrt{x^T V x}} : Ax \leqslant b, x \geqslant 0 \right\}, \tag{14}$$

where V is positive definite.

This is equivalent to

$$\sup \left\{ \frac{-\sqrt{x^T V x}}{e^T x - k} : Ax \leqslant b, \; x \geqslant 0 \right\}$$

assuming $e^T x - k > 0$ on the feasible region. Applying transformation (11) we then obtain the concave program

$$\sup \{-\sqrt{y^T V y} : Ay - bt \leqslant 0, e^T y - kt = 1, y \geqslant 0, t > 0\}$$

which can be reduced to the concave quadratic program

$$\sup \{-y^T V y\colon Ay - bt \leqslant 0, e^T y - kt = 1, y \geqslant 0, t > 0\}. \tag{15}$$

In concave programming the concept of duality plays a crucial role in both theory and applications. For nonconcave programs such as concave fractional programs standard concave programming duality concepts are not useful since basic duality relations are no longer true, even in linear fractional programming [*Schaible*, 1976c]. Therefore, duality has to be defined in a new way.

In case of a concave fractional program the equivalence to a concave program can be used to introduce duality. A classical dual of the equivalent concave program (P′) ($(\mathrm{P}'_{=})$ for affine g) can be used to define a dual of (P) in a meaningful way. We want to illustrate this.

Consider the concave fractional program

$$(\mathrm{P}) \quad \sup \left\{ q(x) = \frac{f(x)}{g(x)} : x \in X, h(x) \leqslant 0 \right\}. \tag{16}$$

The Lagrangean dual of the equivalent program (P′) (after retransforming variables into x) is

$$\text{(D)} \quad \inf \quad \sup_{x \in X} \left\{ \frac{f(x) - v^T h(x)}{g(x)} : v \geqslant 0 \right\}. \tag{17}$$

This reduces to the classical Lagrangean dual of a concave program if $g(x) \equiv 1$ [see *Geoffrion*, 1971 for duality in concave programming].

Because of the equivalence of (P) and (P′) shown in Proposition 8 classical duality relations can be extended to the pair (P) and (D). In particular we find for the optimal value in (P) and (D):

$$\sup(\text{P}) \leqslant \inf(\text{D}); \tag{18}$$

if (P) is feasible, sup (P) is finite and a constraint qualification [*Mangasarian*, 1969a] holds then

$$\sup(\text{P}) = \min(\text{D}). \tag{19}$$

Also, other duality relations such as converse duality theorems can be proved [for more details see *Schaible*, 1976c, 1978].

If all functions f, g, h_j in (16) are differentiable on X and X is an open set, then Wolfe's dual [*Mangasarian*, 1969a] may be applied to (P′) (or ($\text{P}'_=$) if g is affine). This yields

$$\begin{aligned} & \mu \to \inf \\ & -\nabla f(x) + (\nabla h(x))^T v + \mu \nabla g(x) = 0 \\ (\text{D}_\text{W}) \; & -f(x) + (h(x))^T v + \mu g(x) \geqslant 0 \\ & x \in X, \;\; v \in R^m, v \geqslant 0, \mu \geqslant 0 \end{aligned} \tag{20}$$

(μ not signrestricted if g is affine).
As in the nondifferentiable case weak, strong and converse duality relations can be established [*Schaible*, 1976a, 1978].

In the special case of a linear fractional program (7) we obtain the following dual program

$$\begin{aligned} & \mu \to \inf \\ & A^T v + \mu d \geqslant c \\ & -b^T v + \mu\beta \geqslant \alpha \\ & v \geqslant 0, \;\; \mu \in R. \end{aligned} \tag{21}$$

The dual of a concave quadratic fractional program with an affine denominator is a linear program with one additional concave quadratic constraint. Several duality theorems were derived for linear and quadratic fractional programs that extend those in linear and quadratic programming [*Schaible*, 1976c, 1978].

Like in concave programming the dual fractional program can be used to determine the sensitivity of an optimal solution with regard to right-hand-side changes. Consider the differentiable fractional program

$$\max \left\{ q(x) = \frac{f(x)}{g(x)} : x \in X,\ h_j(x) \leq b_j \quad j = 1, \dots, m \right\}. \tag{22}$$

Let $\bar{q}(b)$ denote the optimal value as a function of $b = (b_1, \dots, b_m)^T$. The b_j's may represent capacities and $\bar{q}(b)$ the maximal return on investment for example.

Now let $\bar{x}$ be an optimal solution of (22) for $b = b^0$ and $\begin{pmatrix} \bar{x} \\ \bar{v} \\ \bar{\mu} \end{pmatrix}$ an optimal solution of the dual (D_W). Then under mild additional assumptions

$$\left. \frac{\partial \bar{q}(b)}{\partial b_j} \right|_{b=b_0} = \frac{1}{g(\bar{x})} \bar{v}_j \qquad j = 1, \dots, m, \tag{23}$$

[see *Schaible*, 1978]. Hence the marginal increases of $\bar{q}(b)$ with respect to b_j are proportional to the dual variables $\bar{v}_j$. The value of $\partial \bar{q}/\partial b_j$ can be calculated from (23) once a dual optimal solution is known. Applications of (23) have been discussed in *Schaible* [1978].

We mention that sensitivity analysis for the special case of linear fractional programs has been extensively studied by *Bitran/Magnanti* [1976]. For further results see the references there.

There have been suggested several approaches to introduce duality in linear or concave fractional programming. It can be shown that many of the resulting duals are classical duals of the equivalent concave program (P') or ($P'_=$) [for details see *Schaible*, 1976a, 1976c, 1978]. Very recently new concepts of duality in fractional programming have been proposed by *Flachs/Pollatschek* [1982], *Deumlich/Elster* [1980], *Craven* [1981], *Mond/Weir* [1981], *Passy* [1981] which challenge further discussions.

5. Concave Multiratio Programs

We now turn to the multiratio problems (4), (5) and (6). Much less is known about such optimization problems than in the singleratio case. Since we obtained strong results for those fractional programs (P) which satisfy condition K it seems to be natural to make the same assumption in the multiratio case. In many applications of (4), (5) or (6) condition K is indeed satisfied by the ratios $q_i(x)$ $(i = 1, \dots, p)$.

However, so far not much is known about the properties of (4) even if K holds. Certainly the objective function in (4) is no longer quasiconcave in general, and therefore nonglobal local maxima may exist [*Schaible*, 1977]. On the other hand, some helpful results could be obtained for the two-ratio problem. These properties may be of interest in simultaneous optimization of absolute and relative terms [*Schaible*, 1982b].

Problem (5) is a much more tractable one than problem (4). Extending an earlier definition we call (5) a *concave generalized fractional program* if all ratios in (5) satisfy condition K. Since the minimum of semistrictly quasiconcave functions is semistrictly quasiconcave we have:

Proposition 9: In a concave generalized fractional program (GP) the objective function is semistrictly quasiconcave and hence any local maximum is a global maximum.

We now introduce duality for concave generalized fractional programs. In the previous section we saw that duality relations for singleratio programs can be obtained in a straightforward way with help of the transformation (11). Unfortunately, there has not been found any range or variable transformation that reduces an (arbitrary) concave generalized fractional program (GP) to a concave program. Therefore, duality cannot be introduced in the same simple fashion as in the singleratio case. Nevertheless, it is possible to obtain a dual and duality relations through convex analysis. We follow here the recent approach in *Jagannathan/Schaible* [1982]. Other approaches often lead to the same dual [see *Crouzeix/Ferland/Schaible*].

Consider the concave generalized fractional program

$$\text{(GP)} \quad \sup \left\{ \min_{1 \leqslant i \leqslant p} \frac{f_i(x)}{g_i(x)} : x \in X,\ h(x) \leqslant 0 \right\}. \tag{24}$$

Without loss of generality we can assume the following signrestriction that is somewhat stronger than in K: if at least one g_i is not affine then *all* f_i are positive on S. Furthermore we require that $-f_i$, g_i and h_j are lower semicontinuous on X and X is compact. Let $F = (f_1, \dots, f_p)^T$ and $G = (g_1, \dots, g_p)^T$. Now assume $S \neq \emptyset$. Then using an alternative theorem for convex inequalities one can derive problem

$$\text{(GD)} \quad \inf \left\{ \sup_{x \in X} \frac{u^T F(x) - v^T h(x)}{u^T G(x)} : u \geqslant 0,\ u \neq 0,\ v \geqslant 0 \right\} \tag{25}$$

which has the property that

$$\sup \text{(GP)} = \inf \text{(GD)} \tag{26}$$

holds. Problem (GD) can be considered as a dual of (GP).

The dual (GD) is again a generalized fractional program. It involves possibly infinitely many ratios. The objective function in (GD) is semistrictly quasiconvex, and hence any local minimum is a global minimum.

In the singleratio case ($p = 1$) the dual (GD) reduces to (D), see (17). Hence the different duality approaches in Section 4 and 5 lead to the same dual in the case $p = 1$.

Now consider the linear generalized fractional program

$$\sup \left\{ \min_{1 \leqslant i \leqslant p} \frac{c_{i.}^T x + \alpha_i}{d_{i.}^T x + \beta_i} : Ax \leqslant b,\ x \geqslant 0 \right\}. \tag{27}$$

Here $c_{i.}^T$ (respectively $d_{i.}^T$) denotes row i of a $p \times n$ matrix C (respectively D). Column j

of C (respectively D, A) is denoted by $c_{.j}$ (respectively $d_{.j}$, $a_{.j}$). Let $\alpha = (\alpha_1, \ldots, \alpha_p)^T$ and $\beta = (\beta_1, \ldots, \beta_p)^T$. Under the assumption $D \geqslant 0$ and $\beta > 0$ the dual of (27) can be written as follows:

$$\inf \left\{ \max \left[\frac{\alpha^T u + b^T v}{\beta^T u}, \max_{1 \leqslant j \leqslant n} \frac{c_{.j}^T u - a_{.j}^T v}{d_{.j}^T u} \right] : u \geqslant 0, u \neq 0, v \geqslant 0 \right\} \tag{28}$$

with the convention

$$\frac{\rho}{0} = \begin{cases} +\infty & \text{if } \rho > 0 \\ -\infty & \text{if } \rho \leqslant 0. \end{cases}$$

Note that compactness of X or S is not assumed in the linear case.

We see that the dual of a linear generalized fractional program is again a linear generalized fractional program. It involves finitely many ratios.

Duality relations for the pair (27), (28) are discussed in detail in *Crouzeix/Ferland/Schaible* [1981]. For $p = 1$ the dual (28) reduces to the dual linear fractional program in (21).

Recently also multiobjective fractional programs (6) have been studied [*Choo; Warburton; Kornbluth/Steuer*, 1981a; *Weber; Schaible*, 1983]. There again it is mostly assumed that all ratios $q_i(x)$ satisfy condition K with special attention given to linear ratios. Among others the geometric properties of the set of weakly (strongly) efficient solutions such as closedness and connectedness have been investigated. However more work is to be done in multiobjective fractional programming. This may also provide additional insight into the structure of problem (4) and (5). For applications of multiobjective fractional programming see *Kornbluth/Steuer* [1981b], *Schaible/Ibaraki* [1983].

6. Nonconcave Fractional Programs

It is true that in many applications of single-or multiratio programming the concavity/convexity assumption K is satisfied. However, there are other ratio programs of interest for which this is not so.

In the singleratio case sometimes the quotient of two concave or two convex functions or the quotient of a convex and a concave function is to be maximized. It was shown in *Schaible* [1976d] that all three problems can be solved by maximizing a related quasiconvex function after possibly transforming variables. In such a problem an optimal solution is an extreme point of the feasible region. There is more work to be done on ratio problems of this type, however.

Another type of a nonconcave fractional program arises in portfolio theory. *Ohlson/Ziemba* [1976] introduce the following approximation to a numerically untractable model:

$$\max \left\{ q(x) = \frac{c^T x}{(x^T W x)^\gamma} : x \in S \right\}. \tag{29}$$

Here $c \in R^n$, $c > 0$, W is a positive definite $n \times n$ matrix and $\gamma \in (0, 1/2)$. Both c and W are expressed in terms of the expected returns as well as the variances and covariances. The exponent γ is related to the risk aversion parameter of the utility function.

Since the denominator in (29) is neither convex nor concave this model belongs to none of the classes of problems discussed before. The analysis in *Schaible/Ziemba* [1982] shows among others: there is a large class of variance-covariance matrices often met in applications for which the ratio $q(x)$ in (29) is concave on the positive orthant of R^n provided $\gamma \in [0, \gamma_0]$; however $q(x)$ is not even quasiconcave there if $\gamma \in (\gamma_0, 1/2)$. The critical value γ_0 decreases when more risky data are involved or the number of securities is increased. The smaller γ_0 becomes the less likely it is for a given utility function that the approximating problem is a quasiconcave program [for additional results see *Schaible/Ziemba*, 1982].

A multiratio problem that does not satisfy condition K was recently encountered in *Hodgson/Lowe* [1982]. The authors discuss a material control problem for a warehouse in which the sum of set up cost, inventory carrying cost and material handling cost is to be minimized. In this way the optimal lot sizes and the optimal ordering of the various products in the warehouse are simultaneously determined.

In the suggested iterative procedure a fractional program (4) has to be solved at each step. Several of the ratios $q_i(x)$ do not satisfy condition K. What is worse the total cost function is not even quasiconvex. However it could be shown in *Schaible/Lowe* [1982] that by a suitable transformation of variables this problem can be reduced to a (strictly) convex minimization problem. We learn from this example that a fractional program which does not satisfy condition K may still be tractable.

7. Algorithms

There have been suggested several solution procedures in fractional programming. Most of them solve linear or, more generally, concave fractional programs (3). Such methods can be classified as follows:

I. direct solution of the quasiconcave program (P),
II. solution of the concave program (P′) or ($P'_=$),
III. solution of the dual program (D),
IV. solution of the parametric concave program (P_λ).

In the following we outline these methods.

Strategy I: Direct solution of the quasiconcave program (P)

As seen in Proposition 3 concave fractional programs have several important properties in common with concave programs due to the generalized concavity properties of (P). Pseudoconcave programs can be solved by some of the standard concave programming techniques as shown in *Zangwill* [1969], *Martos* [1975] and *Craven* [1978]. For example, the method by *Frank/Wolfe* [1956] can be applied where at each iteration the objective function is linearized. In case of fractional programs either the ratio as a whole is linearized [*Mangasarian*, 1969b] or the numerator and denominator separately

[*Meister/Oettli*]. Then a sequence of linear programs or linear fractional programs is to be solved if S is a convex polyhedron. In either linearization the solutions to the subproblem converge to a global maximum of (P).

If in a concave fractional program the objective function is quasiconvex as in linear fractional programming (Proposition 4) and if the feasible region is a compact convex polyhedron then an optimal solution is attained at a vertex and a simplex-like procedure can be applied to calculate a global maximum. By it a sequence of adjacent vertices of S with increasing values of $q(x)$ is determined. The method is finite under mild additional assumptions [for details see *Martos*, 1975].

Strategy II: Solution of the concave program (P′) or ($P'_{=}$)

Some of the concave programming algorithms are not suitable in pseudoconcave programming [*Martos*, 1975]. Concave fractional programs, however, can be reduced to a concave program by a transformation of variables (see Proposition 7 and 8). This enables us to get access to *any* concave programming algorithm [*Avriel; Gill/Murray*].

In case there is a special algebraic structure in the numerator and/or denominator, one may prefer to solve (P′) (($P'_{=}$)) rather than (P). As an example, we mention the maximum probability model (14). We saw that it can be reduced to a concave quadratic program (15) by applying the variable transformation (11). In this way the fractional program can be solved by a standard quadratic programming algorithm.

Furthermore, the transformation of a linear fractional program (7) yields a linear program (12). For problems (7) with a bounded feasible region *Wagner/Yuan* [1968] have shown that several other methods in linear fractional programming are algorithmically equivalent to solving the linear program (12) with the simplex method, in the sense that they generate the same sequence of feasible solutions. *Bitran* [1979] has done a numerical comparison of some of these solution procedures on randomly generated linear fractional programs. It seems that the algorithm by *Martos* [1964] is computationally superior to several other methods.

We have seen in Proposition 5 a restricted class of concave fractional programs can be reduced to a concave program by a range transformation. In contrast to the variable transformation (11) such a range transformation does not change the feasible region. It may therefore be particularly useful in fractional programs where variables are restricted to be integers. For algorithms in integer fractional programming, see for example *Bitran/Magnanti* [1976], *Granot/Granot* [1977], *Chandra/Chandramohan* [1980] and *Schaible* [1981].

Strategy III: Solution of the dual program (D)

Sometimes it may be advantageous to solve a dual program of (P′) rather than (P′) itself. For instance, for concave quadratic fractional programs with an affine denominator, the dual (20) becomes a linear program with one additional concave quadratic constraint [*Schaible*, 1976c].

An additional advantage of the dual approach is that the dual optimal solution provides insight into the sensitivity of a primal optimal solution (see (23)).

As in the singleratio case, the dual (DP) of a generalized fractional program (GP) may have computational advantages over the primal. For a linear problem (GP) the dual is again of this type (see (28)). In contrast to the primal, it has only nonnegativity constraints (assuming B is strictly positive).

Strategy IV: Solution of the parametric concave program (P_λ)

When the parametric problem (P_λ) in (9) is used, the zero $\bar{\lambda}$ of the strictly decreasing function

$$F(\lambda) = \max \{f(x) - \lambda g(x): x \in S\}$$

has to be calculated; see Proposition 6. The disadvantage of solving a parametric problem rather than the parameter-free program (P′) may be outweighted by other benefits. For instance, for a quadratic fractional program, the structure of the model is not well exploited when (P′) is used whereas (P_λ) is a concave quadratic program for each λ. This can be treated by standard techniques.

The zero of F may be calculated by parametric programming techniques or by methods that solve (P_λ) for discrete values $\lambda = \lambda_i$ converging to $\bar{\lambda}$. Such an iterative procedure was suggested by *Dinkelbach* [1967]. For details on convergence properties and modifications of Dinkelbach's algorithm, see *Ibaraki* et al. [1976], *Schaible* [1976b, 1978], *Ibaraki* [1981] and *Schaible/Ibaraki* [1983]. *Ibaraki* [1982] compared numerically several iterative procedures that find $\bar{\lambda}$.

An extension of Dinkelbach's algorithm to linear generalized fractional programs (5) was suggested by *Charnes/Cooper* [1977] [for an additional analysis see *Crouzeix/Ferland/Schaible*].

A more detailed presentation of different parametric procedures using (P_λ) is given in *Schaible/Ibaraki* [1983]. There also the relative efficiency of these methods is discussed.

In spite of all efforts in fractional programming it is probably fair to say that the structural properties of fractional programs have not yet been well enough exploited in algorithms. Furthermore, more computational comparisons of the various solution methods have to be performed in order to determine which technique is most efficient for a particular type of fractional program.

References

Avriel, M.: Nonlinear Programming: Analysis and Methods. Englewood Cliffs 1976.

Avriel, M., et al.: Introduction to Concave and Generalized Concave Functions. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 21–50.

Bitran, G.R.: Experiments with Linear Fractional Problems. Naval Research Logistics Quarterly **26**, 1979, 689–693.

Bitran, G.R., and *T.L. Magnanti*: Duality and Sensitivity Analysis for Fractional Programs. Operations Research **24**, 1976, 675–699.

Cambini, A.: An Algorithm for a Special Class of Fractional Programs. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 491–508.

Chandra, S., and *M. Chandramohan*: A Branch and Bound Method for Integer Nonlinear Fractional Programs. Zeitschrift für Angewandte Mathematik und Mechanik **60**, 1980, 735–737.

Charnes, A., and *W.W. Cooper*: Programming with Linear Fractional Functionals. Naval Research Logistics Quarterly 9, 1962, 181–186.

– : Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints. Operations Research 11, 1963, 18–39.

– : Goal Programming and Multi-objective Optimization, Part I. European Journal of Operational Research 1, 1977, 39–54.

Choo, E.U.: Multicriteria Linear Fractional Programming. Ph.D. Thesis, University of British Columbia, Vancouver 1980.

Craven, B.D.: Mathematical Programming and Control Theory. London 1978.

– : Duality for Generalized Convex Fractional Programs. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 473–489.

Crouzeix, J.P., *J.A. Ferland*, and *S. Schaible*: Duality in Generalized Linear Fractional Programming. Technical Report No. 399, Département d'informatique et de recherche opérationelle, Université de Montréal, 1981.

Deumlich, R., and *K.H. Elster*: Duality Theorems and Optimality Conditions for Nonconvex Optimization Problems. Mathematische Operationenforschung und Statistik, Ser. Optimization 11, 1980, 181–219.

Dinkelbach, W.: On Nonlinear Fractional Programming. Management Science 13, 1967, 492–498.

Flachs, J., and *M. Pollatschek*: Equivalence Between a Generalized Fenchel Duality Theorem and a Saddle-Point Theorem for Fractional Programs. Journal of Optimization Theory and Applications 37, 1982, 23–32.

Frank, M., and *P. Wolfe*: An Algorithm for Quadratic Programming. Naval Research Logistics Quarterly 3, 1956, 95–110.

Geoffrion, A.M.: Solving Bi-criterion Mathematical Programs. Operations Research 15, 1967, 39–54.

– : Duality in Nonlinear Programming: A Simplified Applications-Oriented Development. SIAM Review 13, 1971, 1–37.

Gill, P.E., and *W. Murray*: Numerical Methods for Constrained Optimization. London 1974.

Granot, D., and *F. Granot*: On Integer and Mixed Integer Fractional Programming Problems. Annals of Discrete Mathematics 1, 1977, 221–231.

Grunspan, M.: Fractional Programming: A Survey. Technical Report No. 50, Department of Industrial and Systems Engineering, University of Florida, Gainesville 1971.

Hodgson, T.J., and *T.J. Lowe*: Production Lot Sizing with Material Handling Cost Considerations. AIIE Transactions, 1982 (forthcoming).

Ibaraki, T.: Solving Mathematical Programming Problems with Fractional Objective Functions. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 441–472.

– : Parametric Approaches to Fractional Programs. Department of Applied Mathematics and Physics, Kyoto University, Kyoto 1982.

Ibaraki, T., et al.: Algorithms for Quadratic Fractional Programming Problems. Journal of the Operations Research Society of Japan 19, 1976, 174–191.

Jagannathan, R.: On Some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming. Management Science 12, 1966, 609–615.

Jagannathan, R., and *S. Schaible*: Duality in Generalized Fractional Programming via Farkas' Lemma. Journal of Optimization Theory and Applications, to appear.

Kornbluth, J.S.H., and *R.E. Steuer*: Multiple Objective Linear Fractional Programming. Management Science 27, 1981a, 1024–1039.

– : Goal Programming with Linear Fractional Criteria. European Journal of Operational Research 8, 1981b, 58–65.

Mangasarian, O.L.: Nonlinear Programming. New York 1969a.

– : Nonlinear Fractional Programming. Journal of the Operations Research Society of Japan 12, 1969b, 1–10.

Martos, B.: Hyperbolic Programming. Naval Research Logistics Quarterly **11**, 1964, 135–155. Originally published in Math. Institute of Hungarian Academy of Science **5**, 1960, 383–406 (Hungarian).

– : Nonlinear Programming: Theory and Methods. Amsterdam 1975.

Meister, B., and *W. Oettli*: On the Capacity of a Discrete, Constant Channel. Information and Control **11**, 1967, 341–351.

Mond, B., and *T. Weir*: Generalized Concavity and Duality. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 263–279.

Neumann, J. von: Über ein ökonomisches Gleichgewichtssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes. Ergebnisse eines Mathematischen Kolloquiums. Hrsg. v. K. Menger. Leipzig–Wien 1937, Heft 8, 73–83.

– : A Model of General Economic Equilibrium. Review of Economic Studies **13**, 1945, 1–9.

Ohlson, J.A., and *W.T. Ziemba*: Portfolio Selection in a Log-normal Market when the Investor has a Power Utility Function. Journal of Financial and Quantitative Analysis **11**, 1976, 57–71.

Passy, U.: Pseudo-duality and Non-convex Programming. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 239–261.

Ritter, K.: A Parametric Method for Solving Certain Nonconcave Maximization Problems. Journal of Computer and System Sciences **1**, 1967, 44–54.

Schaible, S.: Beiträge zur quasikonvexen Programmierung. Doctoral Dissertation, Köln 1971.

– : Fractional Programming: Transformations, Duality and Algorithmic Aspects. Technical Report 73–9, Department of Operations Research, Stanford University, Stanford 1973.

– : Maximization of Quasiconcave Quotients and Products of Finitely Many Functionals. Cahiers du Centre d'Etudes de Recherche Opérationelle **16**, 1974, 45–53.

– : Fractional Programming. I, Duality. Management Science **22**, 1976a, 858–867.

– : Fractional Programming. II, On Dinkelbach's Algorithm. Management Science **22**, 1976b, 868–873.

– : Duality in Fractional Programming: A Unified Approach. Operations Research **24**, 1976c, 452–461.

– : Minimization of Ratios. J. of Optimization Theory and Applications **19**, 1976d, 347–352.

– : On the Sum of a Linear and Linear-fractional Function. Naval Research Logistics Quarterly **24**, 1977, 691–693.

– : Analyse und Anwendungen von Quotientenprogrammen. Meisenheim 1978.

– : A Survey of Fractional Programming. Generalized Concavity in Optimization and Economics. Ed. by S. Schaible and W.T. Ziemba. New York 1981, 417–440.

– : Bibliography in Fractional Programming. Zeitschrift für Operations Research **26** (7), 1982a.

– : Simultaneous Optimization of Absolute and Relative Terms. Faculty of Business Administration and Commerce, University of Alberta, Edmonton 1982b.

– : Bicriteria Quasiconcave Programs. Cahiers du Centre d'Etudes de Recherche Opérationelle **25**, 1983 (forthcoming).

Schaible, S., and *T. Ibaraki*: Invited Review: Fractional Programming. European J. of Operational Research, 1983 (forthcoming).

Schaible, S., and *T.J. Lowe*: A Note on a Material Control Problem. Krannert Graduate School of Management, Purdue University, West Lafayette 1982.

Schaible, S., and *W.T. Ziemba*: On the Concavity of the Sum of Lognormals is Lognormal Approximation in Portfolio Theory. Graduate School of Management, University of California, Los Angeles, 1982.

Schaible, S., and *W.T. Ziemba* (eds.): Generalized Concavity in Optimization and Economics. New York 1981.

Wagner, H.M., and *J.S.C. Yuan*: Algorithmic Equivalence in Linear Fractional Programming. Management Science **14**, 1968, 301–306.

Warburton, A.R.: Topics of Multiple Criteria Optimization. Ph.D. Thesis, University of British Columbia, Vancouver 1981.

Weber, R.: Pseudomonotonic Multiobjective Programming. Discussion Paper, Department of Economics, University of Saarland, Saarbrücken 1982.

Zangwill, W.I.: Nonlinear Programming: A Unified Approach. Englewood Cliffs 1969.