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Upper Bound of the Speed of Convergence of Moment Density Estimators for Stationary Point Processes

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Summary: The speed of convergence of moment density estimators for stationary point processes is studied. Under relevant assumptions the order of magnitude for its upper bound is the same as in the i.i.d. case, when the process is Brillinger-mixing. The case of convariance density estimators is also considered.

1. Introduction

Let P be some stationary point process on the space $\mathbf{R}^d$. In accordance with *Krickeberg* [1982] the following notations are used for the various considered measures.

$\nu^{(k)}$ is the moment measure of order k,

$\dot{\nu}^{(k)}$ is the factorial moment measure of order k,

$\gamma^{(k)}$ is the cumulant measure of order k,

$\dot{\gamma}^{(k)}$ is the factorial cumulant measure of order k.

As P is stationary, all these measures, when they exist, can be desintegrated in the sense that, for $\nu^{(k)}$ for instance, there exists $\nu'^{(k)}$ such that formally

$$\nu^{(k)}(dx_1, \ldots, dx_k) = \nu'^{(k)}(du_1, \ldots, du_{k-1})\, dx_k \qquad (1.1)$$

where $u_i = x_i - x_k$, $i = 1, \ldots, k-1$.

$\nu'^{(k)}$ is the reduced moment measure of order k. In the sequel, a "prime" denotes a reduced measure. Expression (1.1) shows that, for the knowledge of the process, it is equivalent to estimate $\nu^{(k)}$ or $\nu'^{(k)}$.

The processes considered in that paper are mixing in the sense of Brillinger that is, for each $k = 2, 3, \ldots$, the reduced factorial cumulant exists and is a σ-finite measure on $\mathbf{R}^{d(k-1)}$.

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In order to obtain asymptotical results, the domain of observation is choosen as member of a regular family $\{G_r\}_{r \in \mathbf{R}_+}$ growing to $\mathbf{R}^d$ where r tends to infinity.

In this paper, an upper bound for the local estimation speed of density estimators for reduced moments of any order is computed. Then, the global estimation speed of a covariance density estimator is evaluated.

2. Density Estimation

Assume that the reduced factorial moment measure exists and admits a density $p^{(k)}$ with respects to $\lambda^{\otimes (k-1)}$, Lebesgue measure on $\mathbf{R}^{d(k-1)}$. Let g be a bounded continuous function on $\mathbf{R}^{d(k-1)}$, with integral 1. The family of functions $\{g_r\}_{r \in \mathbf{R}_+}$ is defined by

$$g_r(\mathrm{v}) = \beta_r^{-d(k-1)} g(\beta_r^{-1} \mathrm{v}). \tag{1.2}$$

where $r \to \beta_r$ is a nonnegative application on $\mathbf{R}_+$, β_r decreasing to 0 as r tends to infinity. Clearly, $\lambda^{\otimes (k-1)}(g_r) = 1$, and when r tends to infinity, g_r (v) tends to the Dirac measure at point v in the sense of distributions.

2.1 Recalls on Existing Estimators

Brillinger [1975], and then *Krickeberg* [1982] proposed estimators of $p^{(k)}(u)$, when u is a point of continuity for $p^{(k)}$.

Brillinger estimator

$$\tilde{p}_r^{(k)}(u, \mu) = \lambda(G_r)^{-1} \int_{\mathbf{R}^{dk} \setminus \Delta_k} \prod_{i=1}^{k} 1_{G_r}(x_i) g_r(x_1 - x_k - u_1, \ldots, x_{k-1} - x_k - u_{k-1}) \mu(dx_1), \ldots, \mu(dx_k) \tag{2.1}$$

where $\Delta_k = \{x \in \mathbf{R}^{dk};\ \exists\, i \text{ and } j \text{ such that } x_i = x_j\}$

Krickeberg estimator

$$\hat{p}_r^{(k)}(u, \mu) = \lambda(G_r)^{-1} \int_{\mathbf{R}^{dk} \setminus \Delta_k} 1_{G_r}(x_k) g_r(x_1 - x_k - u_1, \ldots, x_{k-1} - x_k - u_{k-1}) \mu(dx_1) \ldots \mu(dx_k). \tag{2.2}$$

Both of them are biased estimators of $p^{(k)}(u)$:

$$E(\tilde{p}_r^{(k)}(u,\mu)) = \int_{G_r^{k-1}} g(v_1,\ldots,v_{k-1})\, p^{(k)}(u_1+\beta_r v_1,\ldots,u_{k-1}+ \beta_r v_{k-1})\, dv_1 \ldots dv_{k-1}.$$

$$E(\hat{p}_r^{(k)}(u,\mu)) = \int_{\mathbf{R}^{d(k-1)}} g(v_1,\ldots,v_{k-1})\, p^{(k)}(u_1+\beta_r v_1,\ldots,u_{k-1}+ \beta_r v_{k-1})\, dv_1 \ldots dv_{k-1}.$$

Brillinger [1975] states a theorem of convergence in law for $\tilde{p}^{(k)}(u,\mu)$ on $\mathbf{R}$. That theorem can be extended to $\mathbf{R}^d$ for both estimators. To prove it, it is necessary to study the behaviour of the so defined estimator cumulants. In the sequel, only the case of $\hat{p}_r^{(k)}$ is considered, that of $\tilde{p}_r^{(k)}$ being very similar.

2.2 Cumulants of $\hat{p}_r^{(k)}$

Assuming the existence of the moments of the process up to the order hk, it is possible to write the cumulant of order of $\hat{p}_r^{(k)}(u)$ [*Jolivet*].

The moment of order h is written as

$$\lambda(G_r)^{-h} \sum_{l=1}^{kh} \sum_{P_l} \sum_{s=1}^{l} \sum_{Q_s} \int_{\mathbf{R}^{dl}} \prod_{m=1}^{l} \prod_{j\in p_m} \delta(x_m - z_j) \cdot \prod_{i=1}^{h} 1_{G_r}(z_{ik}) \cdot \tag{2.3}$$

$$\cdot\, 1_{\Delta_k^c}(\underset{\sim}{z}) \cdot g_r(z_{(i-1)k+1} - z_{ik} - u_1, \ldots, z_{ik-1} - z_{ik} - u_{k-1}) \cdot$$

$$\cdot \prod_{r=1}^{s} \dot{\gamma}^{(\# q_r)}(dx_{\beta_1},\ldots,dx_{\beta_{\# q_r}})$$

where $\sum_{P_l}$ is the sum on all the partitions of $\{1,\ldots,kh\}$ in l subsets $p_1, p_2, \ldots, p_l$ and $\sum_{Q_s}$ is the sum on all the partitions of $\{1,2,\ldots,l\}$ in s subsets $q_1, q_2, \ldots, q_s$.

The cumulant of order h is the sum of indecomposable integrals – that is: they can not be written as a product of integrals on natural decompositions of $(\mathbf{R}^d)^l$ – included in the expression of the moments. Among these terms, only those with argument of any function g_r formed by k different points of $\mathbf{R}^d$ are nonnul. On the other hand, as g_r tends to the Dirac measure, $g_r(u)\, g_r(v)$ tends to a nonnul distriburion only when $u = v$. The only terms to take into account in the asymptotic estimate of the cumulant are then of the type:

$$\lambda (G_r)^{-h} \int_{\mathbf{R}^{dnk}} \prod_{i=1}^{I} 1_{G_r} (x_k^i) g_r^{n_i} (x_1^i - x_k^i - u_1, \ldots, x_{k-1}^i - x_k^i - u_{k-1}) \cdot$$

$$\cdot \check{\gamma}^{(\# q_1)} (dy_1^1, \ldots, dy_{\# q_1}^1) \ldots \check{\gamma}^{(\# q_s)} (dy_1^s, \ldots, dy_{\# q_s}^s).$$

The various assumptions:

g bounded, continuous, integrable

P mixing in the sense of Brillinger

integral on $\mathbf{R}^{dnk}$ indecomposable

enable the proof that such a term is asymptotically a $O\,[\lambda (G_r)^{1-h} \beta_r^{d(k-1)(I-h)}]$. Accordingly, the leading terms of the sum constituing the cumulant are those with $I = 1$ and then their sum is

$$[\lambda (G_r) \beta_r^{d(k-1)}]^{1-h} \int_{\mathbf{R}^{d(k-1)}} g^h (v_1, \ldots, v_{k-1}) p^{(k)} (u_1 + \beta_r v_1, \ldots, u_{k-1} + \beta_r v_{k-1}) dv_1 \ldots dv_{k-1}.$$

Then, asymptotically, the cumulant of order h of $\hat{p}_r^{(k)} (u)$ behaves as

$$[\lambda (G_r) \beta_r^{d(k-1)}]^{1-h} p^{(k)} (u) \int_{\mathbf{R}^{d(k-1)}} g^h (v_1, \ldots, v_{k-1}) dv_1 \ldots dv_{k-1}$$

if $p^{(k)}$ is continuous in u.

3. Overestimation of the Convergence Speed

3.1 Criterion Selection

As commonly used in density estimation works, the selected criterion of the quality of the estimator is the expectation of the absolute difference between $p^{(k)}$ and its estimator, to the power h, for some h greater than 1, that is

$$E\,[\,|\,\hat{p}_r^{(k)} (u_1, \ldots, u_{k-1}) - p^{(k)} (u_1, \ldots, u_{k-1})\,|^h].$$

That quantity is overestimated by the sum of two terms, one for the bias and the other for the random variations.

$$R_h (\hat{p}^{(k)} (u), p^{(k)} (u)) = E\,[\,|\,\hat{p}_r^{(k)} (u) - p^{(k)} (u)\,|^h]$$

$$\leqslant 2^{h-1} [E\,|\,\hat{p}_r^{(k)} (u) - \bar{p}_r^{(k)} (u)\,|^h + |\,\bar{p}_r^{(k)} (u) - p^{(k)} (u)\,|^h]$$

where

$$\bar{p}_r^{(k)}(u) = E_P(\hat{p}_r^{(k)}(u)).$$

3.2 Overestimation of the Bias

The bias overestimation no introduces any new problem with respect to point processes. The framework adopted will be reasonable for applications, that is without too many restrictive assumptions, as in the following lemma [*Doukhan/Ghindes*].

Let Df be the differential of f and $\| \mathrm{Df}(x) \|$ the norm of the linear application $y \to \langle y, \mathrm{Df}(x) \rangle$, that is:

$$\| \mathrm{Df}(x) \| = \sup_{\|y\| \leqslant 1} | \langle y, \mathrm{Df}(x) \rangle |.$$

Let g_β be the application $x \to \frac{1}{\beta^d} g\left(\frac{x}{\beta}\right)$.

Lemma 3.1: Let g be a measurable function of $\mathbf{R}^d$ into $\mathbf{R}$, non negative and with sum 1; let β be a positive real number and f a measurable function of $\mathbf{R}^d$ into $\mathbf{R}^m$ with one of the following properties:

(i) f is differentiable in the distribution sense
(ii) f is differentiable almost everywhere with locally bounded directional derivatives
(iii) f is locally lipschitz.

It is assumed, in addition, that the differential application Df defined almost everywhere under these assumptions is such that:

$$\| \mathrm{Df} \|_q^q = \int \| \mathrm{Df}(x) \|^q \, dx < \infty$$

and

$$\| \mathrm{Df} \|_\infty = \sup_{x \in \mathbf{R}^d} \{ \| \mathrm{Df}(x) \| \} < \infty \quad \text{if } q = +\infty.$$

Then:

$$\int_{\mathbf{R}^d} | f(x) - f * g_\beta(x) |^q \, dx \leqslant \beta^q \cdot \int_{\mathbf{R}^d} | \| x \|^q g(x) | \, dx \cdot \| \mathrm{Df} \|_q^q \tag{3.1}$$

$$\sup_{x \in \mathbf{R}^d} | f(x) - f * g_\beta(x) | \leqslant \beta \cdot \int_{\mathbf{R}^d} | \| x \| g(x) | \, dx \cdot \| \mathrm{Df} \|_\infty \quad \text{if } q = +\infty. \tag{3.2}$$

With more assumptions on the function f and on the kernel g, the preceding lemma can be extended to functions with derivations up to the order s, generalizing results obtained on $\mathbf{R}$ [*Bretagnolle/Huber*].

Let $D^s f(u)$ be the differential of order s at point u and $\| D^s f(u) \|$ its norm as s-linear form on $\mathbf{R}^d$ · g, function on $\mathbf{R}^d$, is said strong symmetric if $g(x_1, \ldots, x_d) = g(\alpha_1 x_1, \ldots, \alpha_d x_d)$ for every set $(\alpha_1, \ldots, \alpha_d)$ of $\{-1, 1\}^d$.

Lemma 3.2: Let g be a continuous bounded strong symmetric kernel such that

$$\int_{\mathbf{R}^d} g(x)\,dx = 1$$

$\int_{\mathbf{R}^d} x_1^{r_1} \ldots x_d^{r_d}\, g(x)\,dx = 0$ for each set of d positive integers $r_1, \ldots, r_d$, $0 < \sum_{i=1}^{d} r_i < s$

$$\int_{\mathbf{R}^d} \| x \|^s \,|\, g(x) \,|\, dx < \infty.$$

Let f be an s times differentiable function such that

$$\| D^s f \|_q^q = \int_{\mathbf{R}^d} \| D^s f(u) \|^q\, du < \infty$$

$$\| D^s f \|_\infty = \sup_{u \in \mathbf{R}^d} \| D^s f(u) \| < \infty.$$

Then

$$\| f - f * g_\beta \|_q^q \leqslant \beta^{sq} \cdot \| D^s f \|_q^q \cdot [\int_{\mathbf{R}^d} |^s g(u) \,|\, du]^q \tag{3.3}$$

with

$$^s g(u) = \int_0^1 \| \frac{u}{\zeta} \|^s \frac{(1-\zeta)^{s-1}}{\zeta (s-1)!}\, g\left(\frac{u}{\zeta}\right) d\zeta.$$

and

$$\| f - f * g_\beta \|_\infty \leqslant \beta^s \cdot \| D^s f \|_\infty \int_{\mathbf{R}^d} |^s g(u) \,|\, du. \tag{3.4}$$

As on $\mathbf{R}$, the proof of that lemma rests on the overestimation of the L^q-norm of the convolution product of the rest of order s of the Taylor expansion of f by the kernel $^s g$.

Kernels satisfying the above assumptions exist: one can choose products of Parzen s-kernel on $\mathbf{R}$.

3.3 Asymptotic Overestimation of the Random Part and Optimal choice of β_r

As mentioned earlier, the cumulant of order h of $\hat{p}_r^{(k)}(u)$ have the same behaviour as

$$[\lambda (G_r)\, \beta_r^{d(k-1)}]^{1-h}\, p^{(k)}(u) \int_{\mathbf{R}^{d(k-1)}} g^h (v_1, \ldots, v_{k-1})\, dv_1 \ldots dv_{k-1}.$$

If $C_1, C_2, \ldots, C_k$ are the cumulants up to the order h of $\hat{p}_r^{(k)}(u)$, the moment of order h, $\bar{M}_h$, of $\hat{p}_r^{(k)}(u)$ can be written as

$$\sum_{i=1}^{h} \sum_{\substack{n_1, \ldots, n_i \in \{2, \ldots, h\} \\ n_1 + \ldots + n_i = h}} N_h(n_1, \ldots, n_i)\, C_{n_1} \ldots C_{n_i}$$

where $N_h(n_1, \ldots, n_i)$ is the number of partitions of $\{1, 2, \ldots, h\}$ in i subsets with respective cardinal numbers $n_1, \ldots, n_i$.

Taking into account the results of 2.2, if h is even, $\bar{M}_h$ is equivalent to

$$N_h(2, \ldots, 2)\, C_2^{h/2}$$

that is to say

$$\frac{h!}{2^{h/2}(h/2)!}\,[p^{(k)}(u) \int_{\mathbf{R}^{d(k-1)}} g^2(\mathrm{v})\, d\mathrm{v}]^{h/2}\, (\lambda(G_r)\,\beta_r^{d(k-1)})^{-h/2}.$$

If h is odd,

$$E[|\hat{p}_r^{(k)}(u) - \bar{p}_r^{(k)}(u)|^h] \leqslant [E(\hat{p}_r^{(k)}(u) - \bar{p}_r^{(k)}(u))^{h+1}]^{h/(h+1)}.$$

The right-hand side of the inequality is equivalent to

$$\left[\frac{(h+1)!}{2^{(h+1)/2}((h+1)/2)!}\right]^{h/(h+1)} [p^{(k)}(u) \int_{\mathbf{R}^{d(k-1)}} g^2(\mathrm{v})\, d\mathrm{v}]^{h/2}\, (\lambda(G_r)\,\beta_r^{d(k-1)})^{-h/2}.$$

Putting together this result and that of lemma 3.2, choosing $\lambda(G_r)$ and β_r bound by the relation

$$(\lambda(G_r)\,\beta_r^{d(k-1)})^{h/2}\,\beta_r^{sh} = Q$$

with Q some fixed real number, we have the

Theorem 3.3: If the point process P on $\mathbf{R}^d$ is Brillinger-mixing and if the density $p^{(k)}$ of the reduced factorial moment of order k exists and is continuous in u; if

$p^{(k)}$ is s time differentiable and if the norm $\| D^s p^{(k)} (v) \|$ is bounded; if, an the other hand, the kernel g verifies the hypothesis of lemma 3.2 then

$$\overline{\lim_{r\uparrow\infty}}\ \lambda (G_r)^{s\,h/(2s+d(k-1))}\, E\, |\, \hat{p}_r^{(k)} (u) - p^{(k)} (u)\, |^h \tag{3.5}$$
$$\leqslant C \left[(p^{(k)}(u))^{h/2}\, Q^{-(d(k-1)/(2s+d(k-1))} + \| D^s p^{(k)} \|_h^h\, Q^{(2s/(2s+d(k-1))} \right]$$

where C depends only on s, h and on the kernel g.

Remarks:

(1) Under the assumptions of the lemma 3.1, a similar result is obtained with $s = 1$.

(2) because the reduced moment measures are not σ-finite in the classical examples, it will not be natural to introduce such an assumption in view to obtain a global overestimation. A global overestimation result will be proved later for the co-variance measure which is, by hypothesis, σ-finite.

4. Case of the Covariance Measure

The cumulant of order 2, the covariance, being of a great interest in the applications and also being rather simple to manage, we shall consider the problem of the estimation of its density.

4.1 Estimation of the Density

It is assumed that the cumulants of the process admit a density up to the order 4.

z being the density of the itensity of the process, the density $q^{(2)}$ of the covariance measure is estimated by

$$\hat{q}_r^{(2)} (u) = \lambda (G_r)^{-1} \int_{\mathbf{R}^{2d} \setminus \Delta_2} 1_{G_r} (x_2)\, g_r (x_1 - x_2 - u)\, (\mu (dx_1) -$$
$$- z\lambda (dx_1))\, (\mu (dx_2) - z\lambda (dx_2)). \tag{4.1}$$

The variance of $\hat{q}_r^{(2)} (u)$ is given by

$$\lambda (G_r)^2 \text{ var } (\hat{q}_r^{(2)} (u)) =$$

$$\int_{\mathbf{R}^{4d}} 1_{G_r}(x_2)\, 1_{G_r}(x_4)\, g_r(x_1 - x_2 - u)\, g_r(x_3 - x_4 - u)\, q^{(4)}(x_1 - x_4, x_2 - x_4, x_3 - x_4)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$+ \int_{\mathbf{R}^{4d}} 1_{G_r}(x_2)\, 1_{G_r}(x_4)\, g_r(x_1 - x_2 - u)\, g_r(x_3 - x_4 - u)\, q^{(2)}(x_1 - x_3)\, q^{(2)}(x_2 - x_4)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$+ \int_{\mathbf{R}^{4d}} 1_{G_r}(x_2)\, 1_{G_r}(x_4)\, g_r(x_1 - x_2 - u)\, g_r(x_3 - x_4 - u)\, q^{(2)}(x_1 - x_4)\, q^{(2)}(x_2 - x_3)\, dx_1\, dx_2\, dx_3\, dx_4$$

$$+ \int_{\mathbf{R}^{3d}} 1_{G_r}(x_2)\, 1_{G_r}(x_3)\, g_r(x_1 - x_2 - u)\, g_r(x_1 - x_3 - u)\, q^{(3)}(x_1 - x_3, x_2 - x_3)\, dx_1\, dx_2\, dx_3$$

$$+ \int_{\mathbf{R}^{3d}} 1_{G_r}(x_2)\, 1_{G_r}(x_1)\, g_r(x_1 - x_2 - u)\, g_r(x_3 - x_1 - u)\, q^{(3)}(x_1 - x_3, x_2 - x_3)\, dx_1\, dx_2\, dx_3$$

$$+ \int_{\mathbf{R}^{3d}} 1_{G_r}(x_2)\, 1_{G_r}(x_3)\, g_r(x_1 - x_2 - u)\, g_r(x_2 - x_3 - u)\, q^{(3)}(x_1 - x_3, x_2 - x_3)\, dx_1\, dx_2\, dx_3$$

$$+ \int_{\mathbf{R}^{3d}} 1_{G_r}(x_2)\, g_r(x_1 - x_2 - u)\, g_r(x_3 - x_2 - u)\, q^{(3)}(x_1 - x_3, x_2 - x_3)\, dx_1\, dx_2\, dx_3$$

$$+ \int_{\mathbf{R}^{2d}} 1_{G_r}(x_1)\, 1_{G_r}(x_2)\, g_r(x_1 - x_3 - u)\, g_r(x_2 - x_1 - u)\, q^{(2)}(x_1 - x_2)\, dx_1\, dx_2$$

$$+ \int_{\mathbf{R}^{2d}} 1_{G_r}(x_2)\, g_r^2(x_1 - x_2 - u)\, q^{(2)}(x_1 - x_2)\, dx_1\, dx_2 .$$

The same type of considerations as developed in 2.2 enables the demonstration of the fact that, if the point process is Brillinger-mixing, and if $q^{(2)}$ is continuous in u, then

$$\operatorname{var}(\hat{q}_r^{(2)}(u)) = [\lambda(G_r)\beta_r^d]^{-1} q^{(2)}(u) \| g \|_2^2 + o\,[\lambda(G_r)\beta_r^d)^{-1}]. \tag{4.3}$$

4.2 Overestimation of the Risk

Let us now consider the evaluation of the global risk for u in a compact K of $\mathbf{R}^d$.

$$R_K(q_r^{(2)}, \hat{q}^{(2)}) = E_P \int_K (\hat{q}_r^{(2)}(u) - q^{(2)}(u))^2\, du$$

$$= \int_K \operatorname{var}(\hat{q}_r^{(2)}(u))\, du + \int_K (\bar{q}_r^{(2)}(u) - q^{(2)}(u))^2\, du$$

where $\bar{q}_r^{(2)}$ is the expectation of $\hat{q}_r^{(2)}$.

Then, it can be proved without any other assumption than that of Brillinger mixing that

$$\int_K \operatorname{var}(\hat{q}_r^{(2)}(u))\, du \leqslant$$

$$\frac{1}{\lambda(G_r)\beta_r^d}[\| g * g \|_\infty (\| q^{(4)} \|_1 + 4 \| q^{(3)} \|_1 + \| q^{(2)} \|_1) + \| g \|_2^2 \| q^{(2)} \|_1$$

$$+ \| g \|_\infty \| g \|_1 \| q^{(2)} \|_1^2 \lambda(K)] + \frac{2}{\lambda(G_r)} \| g * g \|_1 q^{(2)} \|_1^2 \tag{4.4}$$

If it is also assumed that

$$\| \int_{\mathbf{R}^{2d}} q^{(4)}(\cdot, x, y)\, dxdy \|_\infty + \| \int_{\mathbf{R}^d} q^{(3)}(\cdot, x)\, dx \|_\infty + \| q^{(2)} \|_\infty < \infty$$

then, the following relation is true

$$\int_K \operatorname{var}(\hat{q}_r^{(2)}(u))\, du$$

$$\leqslant \frac{1}{\lambda(G_r)\beta_r^d} \cdot \| g \|_2^2 \cdot \| q^{(2)} \|_1 +$$

$$+ \frac{1}{\lambda(G_r)}\Bigg[\| g * g \|_1 \cdot \Bigg(\| \int_{\mathbf{R}^{2d}} q^{(4)}(\cdot + x + y, x, y)\, dxdy \|_\infty + 2 \| q^{(2)} \|_1^2 +$$

$$+ 3 \int_{\mathbf{R}^d} q^{(3)}(x, \cdot)\, dx \|_\infty + 2 \| \int_{\mathbf{R}^d} q^{(3)}\left(\frac{x + \cdot}{2}, x\right) dx \|_\infty \Bigg) +$$

$$+ \| q^{(2)} \|_\infty \cdot \| q^{(2)} \|_1 \cdot \| g \|_1^2 \lambda(K) \Bigg]. \tag{4.5}$$

Asymptotically, the right hand sides of formula (4.4) and (4.5) are of the same order, and the choice of β_r will be equivalent to get the optimal speed. That choice is precised in an assumption framework leading to the second overestimation. We restrict ourself to the case, generally sufficient for the applications, where $q^{(2)}$, $q^{(3)}$, $q^{(4)}$ are respectively bound continuous on $\mathbf{R}^d$, $\mathbf{R}^{2d}$, $\mathbf{R}^{3d}$, which ensure that the L_∞ norms in formula (4.5) are finite.

Theorem 4.1:

If the point process P on $\mathbf{R}^d$ is Brillinger mixing
if the densities $q^{(2)}$, $q^{(3)}$, $q^{(4)}$ are bounded continuous
if $q^{(2)}$ is s times differentiable and if $\| D^s q^{(2)} \|_2^2$ exists
if, on the other hand, the kernel g verifies the hypothesis of lemma 3.2.
Then

$$\overline{\lim_{r\uparrow\infty}} \; \lambda (G_r)^{2s/(2s+d)} E_P \int_K (\hat{q}_r^{(2)}(u) - q^{(2)}(u))^2 \, du$$

$$\leqslant C \, [\| q^{(2)} \|_1 \, Q^{-d/(2s+d)} + \| D^s q^{(2)} \|_2^2 \, Q^{2s/(2s+d)}] \tag{4.6}$$

where C depends only on g and s.

The overestimation is optimal for

$$Q = \| g \|_2^2 \, \| {}^s g \|_1^{-2} \, \| q^{(2)} \|_1 \, \| D_s q^{(2)} \|_2^{-2} \; .$$

Then

$$\overline{\lim_{r\uparrow\infty}} \; \lambda (G_r)^{2s/(2s+d)} E_P \int_K (\hat{q}_r^{(2)}(u) - q^{(2)}(u))^2 \, du \tag{4.7}$$

$$\leqslant 2 \, \| g \|_2^{4s/(2s+d)} \, \| {}^s g \|_1^{2d/(2s+d)} \, \| q^{(2)} \|_1^{2s/(2s+d)} \, \| D^s q^{(2)} \|_2^{2d/(2s+d)}.$$

Remarks

(i) *Under the same assumptions on the $q^{(i)}$ and g, the same overestimation of the global risk is obtained on a family K_r of compact subsets of $\mathbf{R}^d$ such that $\lambda(K_r) = o(\beta_r^{-d})$.*

(ii) *Under the same assumptions, an asymptotic speed of same order is obtained taking $\lambda(K_r) = 0(\beta_r^{-d})$ but with a larger constant.*

(iii) *If the assumptions for g are preserved, but if $q^{(2)}$, $q^{(3)}$ and $q^{(4)}$ are only assumed integrable, a speed of the same order is obtained, with a different constant, K being fixed.*

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