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Best Unbiased Estimators for the Parameters of a Two-Parameter Pareto Distribution[1])

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Summary: For a two-parameter Pareto distribution *Malik* [1970] has shown that the maximum likelihood estimators of the parameters are jointly sufficient. In this article the maximum likelihood estimators are shown to be jointly complete. Furthermore, unbiased estimators for the two parameters are obtained and are shown to be functions of the jointly complete sufficient statistics, thereby establishing them as the best unbiased estimators of the two parameters.

1. Introduction

The two-parameter form of the Pareto distribution has the distribution function given by

$$F_X(x) = 1 - (k/x)^a, \quad 0 < k \leqslant x, \quad a > 0, \tag{1.1}$$

where k is the location parameter and a is the shape parameter.

The Pareto distribution has been found to be suitable for approximating the right tails of distributions with positive skewness. It has also been found to adapt to several socio-economic, physical and biological phenomena. *Steindl* [1965] has described a wide variety of situations which conform to one or the other forms of the Pareto Law. *Johnson/Kotz* [1970] have given a brief description of most of the research work published up to 1970.

The mean and the variance for (1.1) are given by

$$E(X) = \frac{ak}{a-1}, \quad a > 1,$$

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and

$$\mathrm{Var}(X) = \frac{ak^2}{(a-1)^2\,(a-2)}, \quad a > 2.$$

For a random sample $x_1, x_2, \ldots, x_n$ of size n, from (1.1), *Quandt* [1966] has discussed estimation of the parameters a and k using the standard techniques. In particular he obtained the maximum likelihood estimators to be

$$\hat{k} = \min_{1 < i < n} x_i \tag{1.2}$$

$$\hat{a} = n\,[\sum_{i=1}^{n} \log (x_i/\hat{k})]^{-1}. \tag{1.3}$$

The expected values and variances are

$$E(\hat{a}) = \frac{an}{n-2}, \quad n > 2, \tag{1.4}$$

$$E(\hat{k}) = \frac{ank}{an-1}, \quad n > 1/a, \tag{1.5}$$

$$\mathrm{Var}(\hat{a}) = \frac{a^2 n^2}{(n-2)^2\,(n-3)}, \quad n > 3 \tag{1.6}$$

and

$$\mathrm{Var}(\hat{k}) = \frac{ank^2}{(an-2)\,(an-1)^2}, \quad n > 2/a. \tag{1.7}$$

Malik [1970] has derived the distributions of the maximum likelihood estimators $\hat{a}$ and $\hat{k}$, and they are given by

$$g(\hat{a}) = \frac{(an)^{n-1}}{\Gamma(n-1)\,(\hat{a})^n} \cdot e^{-an/\hat{a}}, \quad \hat{a} > 0, \tag{1.8}$$

and

$$h(\hat{k}) = ank^{an}\,(\hat{k})^{-an-1}, \hat{k} > 0. \tag{1.9}$$

Malik has also shown that $\hat{a}$ and $\hat{k}$ are stochastically independent and that $\hat{a}$ is a sufficient statistic for a if k is known, $\hat{k}$ is a sufficient statistic for k if a is known and $(\hat{a}, \hat{k})$ is a joint set of sufficient statistics for (a, k) when both a and k are unknown. *Malik* [1966] has also derived the exact moments of the order statistics from the Pareto distribution.

An unbiased estimator, and its variance, for a are given by

$$\tilde{a} = \frac{n-2}{n} \cdot \hat{a}, \quad n > 2 \tag{1.10}$$

and

$$\operatorname{Var}(\tilde{a}) = a^2/(n-3), \; n > 3. \tag{1.11}$$

All the estimators of k, found by *Quandt* [1966] are biased. In this article we develop an unbiased estimator, $\tilde{k}$, of k. We also show that $\hat{a}$ and $\hat{k}$ are both complete and $(\hat{a}, \hat{k})$ are jointly complete and furthermore $(\tilde{a}, \tilde{k})$ is the best unbiased estimator of (a, k).

2. Development of Unbiased Estimator for k

To find an unbiased estimator for k we need the following Lemma.

Lemma 2.1: For a two-parameter Pareto distribution (1.1) the following result holds.

$$E(1/\hat{a}^j) = \frac{\Gamma(n+j-1)}{(an)^j\,\Gamma(n-1)}, \; j \text{ being any positive integer less than } n.$$

Proof: Using (1.8) we have

$$E(1/\hat{a}^j) = \int_0^\infty \frac{1}{\hat{a}^j} \cdot \frac{(an)^{n-1}}{\Gamma(n-1)\,\hat{a}^n} \cdot e^{-an/\hat{a}} \cdot d\hat{a}.$$

The Lemma is established by making the substitution

$$x = an/\hat{a}.$$

Consider the expression z, given below, as an estimator for k,

$$z = [1 - C/\hat{a}] \cdot \hat{k} \quad \text{where } C \text{ is some suitable constant.} \tag{2.1}$$

Using (1.4) and (1.5) and the fact that $\hat{a}$ and $\hat{k}$ are independent, we have by using Lemma 2.1,

$$E(z) = E(\hat{k}) - CE(\hat{k}/\hat{a})$$

$$= \frac{ank}{an-1} - CE(\hat{k}) \cdot E(1/\hat{a})$$

$$= \frac{ank}{an-1} - C\frac{ank}{an-1} \cdot \frac{n-1}{an}$$

$$E(z) = \frac{k}{an-1}[an - C(n-1)]. \tag{2.2}$$

From (2.2) it is clear that if we take C to be equal to $1/(n-1)$, then $E(z)$ becomes equal to k, thereby giving an unbiased estimator for k, namely

$$\tilde{k} = \left[1 - \frac{1}{(n-1)\hat{a}} \right] \cdot \hat{k}. \tag{2.3}$$

To find the variance of $\tilde{k}$ we make use of Lemma 2.1, the independence of $\hat{a}$ and $\hat{k}$ and $E(\hat{k}^2)$ given by *Malik* [1966],

$$E(\hat{k}^2) = \frac{an}{an-2} \cdot k^2, \quad n > 2/a.$$

Thus we have

$$E(\tilde{k}^2) = E\left\{ \left[1 - \frac{1}{(n-1)\hat{a}} \right]^2 \cdot \hat{k}^2 \right\}$$

$$= E(\hat{k}^2) \cdot E\left\{ 1 - \frac{2}{(n-1)\hat{a}} + \frac{1}{(n-1)^2 \hat{a}^2} \right\}$$

$$E(\tilde{k}^2) = \frac{[a(n-1)(an-2)+1]}{a(n-1)(an-2)} \cdot k^2, \quad n > 2/a.$$

Subtracting k^2 we get the variance of $\tilde{k}$ given by

$$\operatorname{Var}(\tilde{k}) = \frac{k^2}{a(n-1)(an-2)}, \quad n > 2/a. \tag{2.4}$$

We thus have developed a set of unbiased estimators $(\tilde{a}, \tilde{k})$ for (a, k) defined by (1.10) and (2.3). Also it is noted that these unbiased estimators are functions of the joint sufficient statistic $(\hat{a}, \hat{k})$. Next we prove the completeness of $\hat{a}$ and $\hat{k}$ and the joint completeness of $(\hat{a}, \hat{k})$.

Theorem 2.1: If a is known, the statistic $\hat{k}$ given by (1.2) is complete.

Proof: Let $\phi(\hat{k})$ be some statistic such that

$$E\{\phi(\hat{k})\} = 0 \text{ for all } k \in (0, \infty).$$

Using (1.9) we have

$$E\{\phi(\hat{k})\} = \int_k^\infty \phi(\hat{k}) \cdot ank^{an} (\hat{k})^{-an-1} \cdot d\hat{k} = 0 \text{ for all } k \in (0, \infty)$$

$$\Rightarrow \int_k^\infty \phi(\hat{k}) (\hat{k})^{-an-1} \, d\hat{k} = 0 \text{ for all } k \in (0, \infty). \tag{2.5}$$

Let $\phi(\hat{k}) = \phi^+(\hat{k}) - \phi^-(\hat{k})$ where ϕ^+ and $\phi-$ denote the positive and negative parts of $\phi(\hat{k})$ respectively. Define

$$\nu^{\pm}(A) = \int_A \phi^{\pm}(\hat{k})(\hat{k})^{-an-1} d\hat{k} \quad A = [k, \infty].$$

Then (2.5) implies $\nu^+(A) = \nu^-(A)$ for all A, hence $\phi^+ = \phi^-\ [\lambda]$ i.e. $\phi(\hat{k}) = 0\ [\lambda]$ implying completeness of $\hat{k}$. [λ = Lebesgue measure on $(0, \infty)$].

Theorem 2.2: The statistic $\hat{a}$ given by (1.3) is complete.

Proof: Let $\phi(\hat{a})$ be some statistic such that

$$E\{\phi(\hat{a})\} = 0 \text{ for all } a \in (0, \infty).$$

Using (1.8) we have

$$E\{\phi(\hat{a})\} = \int_0^\infty \phi(\hat{a}) \frac{(an)^{n-1}}{\Gamma(n-1)(\hat{a})^n} \cdot e^{-an/\hat{a}} \cdot d\hat{a} = 0 \text{ for all } a \in (0, \infty)$$

$$\Rightarrow \int_0^\infty \phi(\hat{a}) \cdot (\hat{a})^{-n} \cdot e^{-an/\hat{a}}\, d\hat{a} = 0 \text{ for all } a \in (0, \infty).$$

Letting $\frac{n}{\hat{a}} = x$ we have

$$\int_0^\infty g(x) \cdot x^{n-2} \cdot e^{-ax}\, dx = 0 \text{ for all } a \in (0, \infty). \tag{2.6}$$

We observe that for some $C(a) > 0, f_a(x) = C(a)\, x^{n-2}\, e^{-ax}, x > 0$ defines a one parameter exponential family with the natural parameter space $\theta = (0, \infty)$ and hence is complete, establishing completeness of $\hat{a}$.

Theorem 2.3: The family of joint distributions

$$\{F_{a,k}^{\hat{a},\hat{k}}; (a, k) \in \theta \times \theta\}, \text{ where } \theta = (0, \infty),$$

is complete.

Proof: Since $\hat{a}$ and $\hat{k}$ are independent, the joint density of $(\hat{a}, \hat{k})$ is given by

$$f_{a,k}(\hat{a}, \hat{k}) = \frac{(an)^n\, k^{an}\, \hat{k}^{-an-1} \cdot e^{-an/\hat{a}}}{\Gamma(n-1)\, \hat{a}^n}, \quad \hat{k} \geqslant k > 0, \hat{a} > 0.$$

Let $\phi(\hat{a}, \hat{k})$ be some statistic such that

$$E\{\phi(\hat{a}, \hat{k})\} = 0 \text{ for all } (a, k) \in \theta \times \theta.$$

This gives

$$\int_0^\infty \int_k^\infty \frac{(an)^n\, k^{an}\, \hat{k}^{-an-1} \cdot e^{-an/\hat{a}}}{\Gamma(n-1)\,\hat{a}^n} \cdot \phi(\hat{a}, \hat{k})\, d\hat{k}\, d\hat{a} = 0 \text{ for all } (a, k) \in \theta \times \theta.$$

Using Fubini's theorem we can write the double integral as

$$\int_k^\infty ank^{an}\, \hat{k}^{-an-1} \left\{ \int_0^\infty \phi(\hat{a}, \hat{k}) \frac{(an)^{n-1} \cdot e^{-an/\hat{a}}}{\Gamma(n-1)\,\hat{a}^n}\, d\hat{a} \right\} d\hat{k} = 0 \text{ for all } (a, k) \in \theta \times \theta.$$

The integral in braces is some function of $\hat{k}$ and a, say $h(\hat{k}, a)$, thus we have

$$\int_k^\infty ank^{an}\, \hat{k}^{-an-1} \cdot h(\hat{k}, a)\, d\hat{k} = 0 \text{ for all } (a, k) \in \theta \times \theta. \tag{2.7}$$

By Theorem 2.1 $\lambda\,\{\hat{k} : h(\hat{k}, a) \neq 0\} = 0 \quad \forall\, a > 0$ which implies (Fubini)

$$\lambda \otimes \lambda\, \{(a, \hat{k}) : h(\hat{k}, a) \neq 0\} = 0$$

hence (Fubini)

$$\lambda\, \{a : h(\hat{k}, a) \neq 0\} = 0 \text{ for } \lambda\text{-almost all } \hat{k}.$$

Continuity of $h(\hat{k}, \cdot)$ implies $h(\hat{k}, \cdot) \equiv 0$ for λ-almost all $\hat{k}$, and by Theorem 2.2

$$\lambda\, \{\hat{a} : \phi(\hat{a}, \hat{k}) \neq 0\} = 0 \text{ for } \lambda\text{-almost all } \hat{k},$$

hence (Fubini)

$$\lambda \otimes \lambda\, \{(\hat{a}, \hat{k}) : \phi(\hat{a}, \hat{k}) \neq 0\} = 0$$

which proves the theorem.

Theorem 2.4: The statistic $(\tilde{a}, \tilde{k})$ is the best unbiased estimator of (a, k).

Proof: Theorem 2.3 establishes that $(\hat{a}, \hat{k})$ is a set of jointly complete sufficient statistic. The estimator $(\tilde{a}, \tilde{k})$ is a function of $(\hat{a}, \hat{k})$ and is unbiased for (a, k); therefore $(\tilde{a}, \tilde{k})$ is the unique set of unbiased estimators that is a function of the jointly complete sufficient statistics [*Zacks*; Theorem 3.1.2, p. 104] and its ellipse of concentration is contained in the ellipse of concentration of every other unbiased estimator of (a, k) [*Cramer*, p. 493]. Thus $(\tilde{a}, \tilde{k})$ is the unbiased estimator of (a, k).

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