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On the divergence of spherical sums of double Fourier–Haar series

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Let $f \in L(I^2)$, $I = [0,1]$ and let $a_{m,n}(f)$ be the Fourier coefficient of the function f with respect to the double Haar system $\{\chi_m(x)\chi_n(y)\}_{m,n=1}^{\infty}$. Furthermore, for $R > 0$ let

$$\mathrm{S}_R(f,x,y) = \sum_{m^2+n^2 \le R^2} a_{m,n}(f)\chi_m(x)\chi_n(y)$$

be the spherical partial sums of the double Fourier–Haar series of the function f. We denote by $|A|$ the 2-dimensional Lebesgue measure of the set A. As usually, $L(\ln^+ L)^\gamma(I^2)$, $\gamma \ge 0$, is the class of all measurable functions defined on I^2 and satisfying the condition

$$\iint_{I^2} |f(x,y)| \left(\ln^+ |f(x,y)|\right)^\gamma \, dxdy < \infty$$

where $\ln^+ |u|$ equals to $\ln |u|$ for $|u| \ge 1$ and 0 in the other cases.

In this work we prove

Theorem 1. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a function satisfying the conditions $\Phi(u) \uparrow \infty$ as $u \to \infty$ and*

$$\Phi(u) = o(u \ln u). \tag{1}$$

Then there exists a function f on I^2 such that $\Phi \circ |f| \in L(I^2)$ and

$$\overline{\lim_{R\to\infty}}\, \mathrm{S}_R(f,x,y) = +\infty$$

almost everywhere on I^2.

By replacing o with O this statement is no longer valid, since it is well-known (see [3]) that $f \in L \ln^+ L(I^2)$ implies the convergence of $\mathrm{S}_R(f,x,y)$ almost everywhere on I^2 as $R \to \infty$.

A consequence of Theorem 1 is

Theorem 2. *Let $0 < \eta \le 1$. Then there exists a function $f \in L(\ln^+ L)^{1-\eta}(I^2)$ whose spherical Fourier–Haar partial sums diverge almost everywhere on I^2.*

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Notice that, in the case $\eta = 1$, in [4] one established for sufficiently small $\varepsilon > 0$ the existence of a function $f \in L(I^2)$ and a set $E \subset I^2$ such that $|E| > 1 - \varepsilon$ and $\mathbf{S}_R(f, x, y)$ is divergent on E as $R \to \infty$.

For definitions and results concerning the above mentioned topics, see the surveys [2], [6], [7], [8].

Proof of Theorem 1. In the sequel we shall use constructions similar to those applied in [4], [5].

Let

$$k(j) = 2^{2j}, \quad j = 1, 2, \ldots, \tag{2}$$

$$b_{r,1}^{(1)} = [0, 2^{r-k(1)}] \times [0, 2^{-r}], \quad \text{for } r = 1, \ldots, k(1) - 1, \tag{3}$$

$$B_1^{(1)} = \bigcup_{r=1}^{k(1)-1} b_{r,1}^{(1)}, \tag{4}$$

$$d_1^{(1)} = \bigcap_{r=1}^{k(1)-1} b_{r,1}^{(1)}, \tag{5}$$

$$F_1^{(1)} = I^2 \setminus B_1^{(1)}, \tag{6}$$

$$\lambda(k) = (2^{k+1} - k)2^{-k-1}, \tag{7}$$

$$\gamma_1 = 1. \tag{8}$$

Then

$$|b_{r,1}^{(1)}| = 2^{-k(1)}, \quad r = 1, \ldots, k(1) - 1, \tag{9}$$

$$|B_1^{(1)}| = 2^{-(k(1)+1)} k(1), \tag{10}$$

$$|d_1^{(1)}| = 2^{2(1-k(1))}, \tag{11}$$

$$|F_1^{(1)}| = \lambda(k(1)). \tag{12}$$

Now we can represent $F_1^{(1)}$ in the form

$$F_1^{(1)} = \bigcup_{\ell=\gamma_1+1}^{\gamma_2} Q_\ell^{(1)}, \tag{13}$$

where the $Q_\ell^{(1)}$, $\ell = \gamma_1 + 1, \ldots, \gamma_2$, are equal squares congruent to $d_1^{(1)}$ and have pairwise disjoint inners.

Similar considerations applied to each of the squares $Q_\ell^{(1)}$, $\ell = \gamma_1 + 1, \dots, \gamma_2$ taking into account (2)–(13) give

$$|B_\ell^{(1)}| = 2^{-k(1)-1}k(1)|Q_\ell^{(1)}|, \quad \ell = \gamma_1 + 1, \dots, \gamma_2, \tag{14}$$

$$\Big|\bigcup_{\ell=\gamma_1+1}^{\gamma_2} B_\ell^{(1)}\Big| = 2^{-k(1)-1}k(1)|F_1^{(1)}|, \tag{15}$$

$$|d_\ell^{(1)}| = 2^{2(1-k(1))}|Q_\ell^{(1)}|, \quad \ell = \gamma_1 + 1, \dots, \gamma_2, \tag{16}$$

$$\Big|\bigcup_{\ell=\gamma_1+1}^{\gamma_2} d_\ell^{(1)}\Big| = 2^{2(1-k(1))}|F_1^{(1)}| = 2^{2(1-k(1))}\lambda(k(1))|Q_1^{(1)}|. \tag{17}$$

Set

$$F_2^{(1)} = F_1^{(1)} \setminus \bigcup_{\ell=\gamma_1+1}^{\gamma_2} B_\ell^{(1)}. \tag{18}$$

From (14), (15) and (18) it follows

$$|F_2^{(1)}| = \lambda(k(1))|F_1^{(1)}| = \lambda^2(k(1))|Q_1^{(1)}|. \tag{19}$$

Continuing this procedure we obtain the sequence $\{F_i^{(1)}\}_{i=1}^{\infty}$. Let $F_1 = \bigcap_{i=1}^{\infty} F_i^{(1)}$. Then (see (19))

$$|F_1| = 0. \tag{20}$$

Let us denote by $\mathcal{D}_1$ the collection of all inner squares, i.e.

$$\mathcal{D}_1 = \{d_\ell^{(1)} : \ell = 1, 2, \dots\}, \tag{21}$$

and by $\mathcal{B}_1$ the family of all rectangles $b_{r,\ell}^{(1)}$ i.e.

$$\mathcal{B}_1 = \{b_{r,\ell}^{(1)} : r = 1, \dots, k(1) - 1;\ \ell = 1, 2, \dots\}, \tag{22}$$

and let their unions be

$$D_1 = \bigcup_{\ell=1}^{\infty} d_\ell^{(1)}, \tag{23}$$

$$B_1 = \bigcup_{\ell=1}^{\infty} B_\ell^{(1)}. \tag{24}$$

Then we get (see (16), (17), (23))

$$|D_1| = 2^{2(1-k(1))}(1 + \lambda(k(1)) + \lambda^2(k(1)) + \cdots)|Q_1^{(1)}| = \tag{25}$$
$$= \frac{2^{2(1-k(1))}}{1 - \lambda(k(1))}|Q_1^{(1)}|.$$

Furthermore, for every rectangle $b \in \mathcal{B}_1$ we have (see (21), (22), (23))

$$D_1 \cap b \in \mathcal{D}_1 \tag{26}$$

and

$$|D_1 \cap b| = 2^{2-k(1)}|b|. \tag{27}$$

If $\bar{b}$ is the symmetric image of the rectangle $b \in \mathcal{B}_1$ with respect to its right or top side then

$$|D_1 \cap \bar{b}| \leq 2^{3-k(1)}(k(n))^{-1}|b|. \tag{28}$$

This completes the first step of the construction.

In the second step we divide each set $B_\ell^{(1)}$, $\ell = 1, 2, \ldots$, into squares being equal to its corresponding inner squares $d_\ell^{(1)}$ (while the partitioning squares have no common inner points). For each of these squares we can repeat all the considerations of the first step only by replacing the terms $k(1)$ by $k(2)$. In particular, in this manner we obtain the families $\mathcal{D}_2$ and $\mathcal{B}_2$ of inner squares and rectangles, respectively, and the corresponding unions D_2 and B_2 of these squares and rectangles (cf. (21)–(24)).

In the n-th step (with corresponding $k(n)$), with the notations $\mathcal{D}_n$, $\mathcal{B}_n$ and D_n, B_n for the families of inner squares and rectangles and their respective unions and by setting $F_n = I^2 \setminus B_n$ we get

$$|D_n| \leq 2^{3-k(n)}(k(n))^{-1}|Q_1^{(1)}| = 2^{3-k(n)}(k(n))^{-1}. \tag{29}$$

For an arbitrary $b \in \mathcal{B}_n$ (see (20), (25)–(28)) we obtain

$$D_n \cap b \in \mathcal{D}_n, \tag{30}$$

$$|D_n \cap b| = 2^{2-k(n)}|b|, \tag{31}$$

$$|D_n \cap \bar{b}| \leq 2^{3-k(n)}(k(n))^{-1}|b|, \tag{32}$$

$$|F_n| = 0, \tag{33}$$

whence it follows

$$|B_n| = 1. \tag{34}$$

Let a natural number $q \geq 1$ be given. By condition (1) we have

$$\Phi(2^{k(n)}q) = o(2^{k(n)}q \ln(2^{k(n)}q)) \quad \text{as} \quad n \to \infty.$$

Therefore there exists a natural number $n(q) > q$ such that

$$\Phi(2^{k(n(q))}q) < 2^{k(n(q))}q^{-2} \ln(2^{k(n(q))}q),$$

and consequently,

$$\Phi(2^{k(n(q))}q)2^{-k(n(q))}(n(q))^{-1} < \tag{35}$$

$$< 2^{1-k(n(q))}(\ln(2^{k(n(q))}q))^{-1}\Phi(2^{k(n(q))}q) < 2q^{-2}.$$

In view of the conditions of the theorem we can choose the sequence $n(1)$, $n(2), \ldots$ to be strictly increasing.

Now we proceed to the construction of the required function f. For every $q = 1, 2, \ldots$ consider the functions

$$f_q(x,y) = \begin{cases} 2^{k(n(q))-1}q & \text{for } (x,y) \in D_{n(q)}, \\ 0 & \text{for } (x,y) \notin D_{n(q)}. \end{cases} \tag{36}$$

For an arbitrary rectangle $b \in \mathcal{B}_{n(q)}$ we have (see (29)–(33))

$$\frac{1}{|b|}\iint_b f_q = \frac{|D_{n(q)} \cap b|}{|b|} 2^{k(n(q))-1} q = 2q \tag{37}$$

and furthermore,

$$\frac{1}{|b|}\iint_{\bar{b}} f_q = \frac{|D_{n(q)} \cap \bar{b}|}{|b|} 2^{k(n(q))-1} q < \frac{4q}{k(n(q))}. \tag{38}$$

By (37) and (38), for every rectangle $b \in \mathcal{B}_n(q)$ the following estimate is fulfilled:

$$\frac{1}{|b|}\Big(\iint_b f_q - \iint_{\bar{b}} f_q\Big) > 2q - 8. \tag{39}$$

Let

$$f(x,y) = \sum_{q=1}^{\infty} f_q(x,y). \tag{40}$$

Then $\Phi \circ |f| \in L(I^2)$. Indeed, by setting

$$T = \bigcup_{q=1}^{\infty} D_{n(q)}, \quad U_q = D_{n(q)} \setminus \bigcup_{j>q} D_{n(j)}, \quad U = \bigcup_{q=1}^{\infty} U_q,$$

we obtain $|T| = |U|$ in view of the estimate (see (29))

$$\Big|\bigcup_{j=q}^{\infty} D_{n(j)}\Big| \le \sum_{j=q}^{\infty} |D_{n(j)}| \le 8 \sum_{j=q}^{\infty} 2^{-k(n(j))}[k(n(j))]^{-1} \to 0, \quad q \to \infty,$$

since $T = U \cup \big(\bigcap_{q=1}^{\infty} \bigcup_{j=q}^{\infty} D_{n(j)}\big)$.

Taking this into account we have (see (36), (40), (35))

$$\iint_{I^2} \Phi \circ |f| = \iint_U \Phi \circ |f| \le \sum_{q=1}^{\infty} \iint_{U_q} \Phi \circ \Big|\sum_{j=1}^{q} f_j\Big| \le$$

$$\le \sum_{q=1}^{\infty} \Phi(2^{k(n(q))}q)|D_{n(q)}| = 8 \sum_{q=1}^{\infty} 2^{-k(n(q))}[k(n(q))]^{-1}\Phi(2^{k(n(q))}q) < \infty.$$

To the proof of the divergence of the spherical partial sums of the Fourier–Haar series of the function f we shall use some considerations from [4, pp. 46-47] according to which (with our above notations) for almost all $(x, y) \in \bigcap_{q=1}^{\infty} B_{n(q))}$ there exist sequences $b_{n(q)}$, $R(q)$, L_q (in general depending on (x, y)) such that

$$(x, y) \in b_{n(q)} \in \mathcal{B}_{n(q)}, \tag{41}$$

$$\lim_{q \to \infty} R(q) = +\infty, \tag{42}$$

$$\lim_{q \to \infty} L_q = f(x, y), \tag{43}$$

$$\mathbf{S}_{R(q)}(f, x, y) = L_q + \frac{1}{2|b_{n(q)}|}\Big(\iint_{b_{n(q)}} f_q - \iint_{\bar{b}_{n(q)}} f_q\Big). \tag{44}$$

Now, since $|\bigcup_{q=1}^{\infty} B_{n(q))}| = 1$ (see (34)), taking into account (39), (41)–(44) we get the statement of Theorem 1.

Remark. For a comparison of the obtained results with the case of the multiple trigonometric system, we refer to [1] where, in particular, it is proved that for every $p \in [1, 2)$ there exists a function in the class $L^p(\mathbf{T}^2)$ such that the spherical sums of its Fourier series diverge in measure on the square $\mathbf{T}^2 = [0, 2\pi]^2$.

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О расходимости сферических сумм двойных рядов Фурье–Хаара

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Доказана следующая теорема.

Пусть $\Phi : [0, +\infty) \to [0, +\infty)$ — функция, удовлетворяющая при $u \to +\infty$ условиям $\Phi(u) \uparrow \infty$ и $\Phi(u) = o(u \ln u)$.

Тогда существует интегрируемая на $[0,1]^2$ функция f такая, что $\Phi \circ |f| \in L([0,1]^2)$, а сферические суммы ее двойного ряда Фурье–Хаара расходятся почти всюду на $[0,1]^2$.

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