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Physical Uniformities on the State Space of Nonrelativistic Quantum Mechanics

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Uniformities describing the distinguishability of states and of observables are discussed in the context of general statistical theories and are shown to be related to distinguished subspaces of continuous observables and states, respectively. The usual formalism of quantum mechanics contains no such physical uniformity for states. Using recently developed tools of quantum harmonic analysis, a natural one-to-one correspondence between continuous subspaces of nonrelativistic quantum and classical mechanics is established, thus exhibiting a close interrelation between physical uniformities for quantum states and compactifications of phase space. General properties of the completions of the quantum state space with respect to these uniformities are discussed.

1. INTRODUCTION

It is a trivial observation that hardly any measurement in physics is absolutely precise. Yet the consequences of this fact for the philosophy of science, the formal methodology of physics, or the construction of particular physical theories are far from being fully understood. For example, what does it mean to say that one theory approximates another, or is a limiting case? Which parts of a theory can be tested by finitely many experiments of finite accuracy? What is the physical content of the frequently used idealization that measurements can in principle be refined indefinitely?

In a first step towards making such questions precise, Ludwig[(1)] has singled out uniform structures, or "physical uniformities", as the appropriate tool for describing imprecision in physics. In the present article this concept will be applied to statistical theories. It has been noted by Ludwig long ago

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that each physical uniformity on the state space of a statistical theory can be characterized by the space of observables, or affine functionals on the state space, which are continuous with respect to this uniformity. In this sense the usual quantum mechanical formalism contains no physical uniformity on the state space, which would correspond to a distinguished subspace $\mathscr{D} \subset \mathscr{B}(\mathscr{H})$ of observables. (This deficiency is shared by any purely W^*-algebraic, and dually by any C^*-algebraic description of physical systems.)

In the attempt to construct such uniformities, one may look for guidance in the more transparent analogous problem for classical mechanics on phase space. Surprisingly, this already solves the problem, since there is a natural one-to-one correspondence between the sets of physical uniformities on the state spaces of quantum and of classical mechanics (Theorem 5.6).

2. STATISTICAL THEORIES

Very often in science one encounters the phenomenon that "the same measurement" applied to "the same situation" produces varying results, which, however, appear at a reproducible rate as the experiment is repeated many times. The theoretical tool for describing such repeated experiments are *statistical theories*, which therefore need to contain three basic ingredients: First, there has to be a description of the "situations" to which the measurements are applied or, more precisely, of the procedures according to which "systems" are prepared or selected. Second, there must be a description of the procedures for setting up the measuring devices and for determining which result out of some preassigned outcome set has occurred in each individual experiment. (For our present purposes it suffices to consider measuring devices with only two different outcomes, say "+" and "−".) The third basic ingredient is the statistical function, assigning to each type of individual experiments (defined by some preparing procedure and some measuring procedure) the asymptotic rate or "probability" with which the outcome "+" occurs.

Different preparing procedures may induce the same rates on all available measuring devices. Such procedures are called statistically equivalent and the equivalence classes are called *states*. Similarly a class of statistically equivalent measuring procedures is called an *effect*. In the sequel the sets of states and effects are usually denoted by K and L. It is very convenient to embed these sets into linear spaces (in the sequel called $B \supset K$ and $D \supset L$) such that the statistical function becomes a bilinear map $\langle \cdot, \cdot \rangle$: $B \times D \to \mathbb{R}$. The typical structure resulting from this construction is described in the following definition. (For a more complete derivation, see Ludwig,[2] Neumann,[3] and Werner.[4])

Definition 2.1. A *statistical duality* $\langle B, D\rangle$ consists of a Banach space B, base-normed by a base $K \subset B$, an order-unit Banach space D with order unit 1 and unit order interval $L := [0, 1] \subset D$, and a bilinear form $\langle \cdot, \cdot \rangle$: $B \times D \to \mathbb{R}$, placing B and D in norm and order duality. Thus the following relations hold: $(x \in B, f \in D)$

$$K = \{\rho \in B \mid \rho \geqslant 0, \langle \rho, 1\rangle = 1\}; \quad L = \{f \in D \mid 0 \leqslant f \leqslant 1\}$$

$$\|x\| = \inf\{\lambda_1 + \lambda_2 \mid x = \lambda_1 \rho_1 - \lambda_2 \rho_2 ; \lambda_i \geqslant 0, \rho_i \in K\} = \sup_{f \in D} \frac{|\langle x, f\rangle|}{\|f\|}$$

$$\|f\| = \inf\{\lambda \mid -\lambda 1 \leqslant f \leqslant \lambda 1\} = \sup_{x \in B} \frac{|\langle x, f\rangle|}{\|x\|}$$

$$x \geqslant 0 \Leftrightarrow x \in \|x\| \cdot K \Leftrightarrow \forall_{f \geqslant 0} \langle x, f\rangle \geqslant 0$$

$$f \geqslant 0 \Leftrightarrow f \in \|f\| \cdot L \Leftrightarrow \forall_{x \geqslant 0} \langle x, f\rangle \geqslant 0$$

It is often convenient to work with the (canonical) complexifications of B and D, which satisfy analogous relations and will be denoted by the same letters. Familiar examples for D are $\mathscr{C}(X)$, $\mathscr{L}^\infty$, $\mathscr{B}(\mathscr{H})$, C^*-algebras $\mathscr{A}$ and W^*-algebras $\mathscr{M}$, and for B: $\mathscr{C}'(X)$, $\mathscr{L}^1$, $\mathscr{T}(\mathscr{H})$, $\mathscr{A}^*$ and $\mathscr{M}_*$ together with the obvious dualities.

In the above discussion we have defined the equality of states as statistical equivalence $(\rho_1 = \rho_2 :\Leftrightarrow \forall_{f \in L} \langle \rho_1, f\rangle = \langle \rho_2, f\rangle)$. Therefore the experimental verification of "$\rho_1 = \rho_2$" requires a series of "monitoring experiments"[5] with different effects $f \in L$. However, it is only possible to perform finitely many such experiments (say, with effects $f_1 \cdots f_N$) of finite accuracy (say ε). Thus one can only verify operationally statements like

$$(\rho_1, \rho_2) \in U_{\varepsilon, f_1 \cdots f_N} := \{(x_1, x_2) \in K \mid \forall_{i=1}^N | \langle x_1 - x_2, f_i\rangle| \leqslant \varepsilon\}$$

The sets $U_{\varepsilon, f_1 \cdots f_N}$ $(\varepsilon > 0, f_i \in L)$ generate a uniform structure on K, which will be denoted by $\sigma(K, L)$. ($\sigma(K, L)$ is the restriction to K of the uniformity on B induced by the weak vector space topology $\sigma(B, D)$.) The same remarks apply to the possibility of distinguishing effects, which is described by the uniformity $\sigma(L, K)$ on L. We shall emphasize the close relationship of $\sigma(K, L)$ and $\sigma(L, K)$ to the operational identification of states and effects by calling them *physical uniformities* in contrast to other uniformities defined on K and L (e.g., by the norms) but lacking such interpretation.

Suppose now that the space D is given together with two families K_1 and K_2 of states on D (positive normalized linear functionals on D) and that K_1 and K_2 are $\sigma(D', L)$-dense in each other (i.e., $\forall_{\rho_1 \in K_1} \forall_{U \in \sigma(D', L)}$

$\exists_{\rho_2 \in K}(\rho_1, \rho_2) \in U$ and vice versa.) Then it is clear that no experiment is capable of distinguishing elements of K_1 from elements of K_2. Thus K_1 and K_2 can be called *physically equivalent*. This definition goes back to Haag and Kastler.[5] (In their set-up D is the C^*-algebra of quasilocal observables and $K_i = K_{\pi_i}$ are the sets of normal states associated with the two representations π_i of X.) This type of physical equivalence is a universal phenomenon in axiomatically minded mathematical physics. For example, two geometries in which all points have rational coordinates or can be constructed with ruler and compass are physically equivalent with respect to the uniformity induced by the euclidean metric. Usually each equivalence class of theories contains one canonical "maximal" representative, which in the cases considered so far is simply the Hausdorff completion[6] of any member of the class: Two uniform spaces are "dense in each other" iff their completions are identical. In the case of geometry one arrives in this way at the usual analytic geometry in $\mathbb{R}^2$. The purpose of the following proposition is to identify the completions of K and L for the physical uniformities $\sigma(K, L)$ and $\sigma(L, K)$.

Proposition 2.2. Let $\langle B, D \rangle$ be a statistical duality. Then:

(1) B' is an order unit space, whose unit order interval $\bar{L}$ can be identified with the $\sigma(L, K)$-completion of L. $\bar{L}$ is $\sigma(\bar{L}, K)$-compact.

(2) D' is a base-normed space, whose base $\bar{K}$ can be identified with the $\sigma(K, L)$-completion of K. $\bar{K}$ is $\sigma(\bar{K}, L)$-compact.

(3) B coincides with the set of $\sigma(L, K)$-uniformly continuous affine functions on L, vanishing at $0 \in L$.

(4) D coincides with the set of $\sigma(K, L)$-uniformly continuous affine functions on K.

(5) By $\sigma(D, B)$ and $\sigma(B, D)$-continuity, $\langle \cdot, \cdot \rangle$ extends from $B \times D$ to the canonical bilinear forms on $B \times B'$ and $D' \times D$, but not in general to $D' \times B'$.

Proof. (1) The dual of a base-normed space is an order unit space (Nagel[7], 3.1). $\bar{L}$ is a $\sigma(B', B)$-closed subset of the unit sphere of B' and hence compact and complete by the Banach–Alaoglu theorem. Since the $\sigma(B', B)$ and $\sigma(B', K)$ uniformities coincide on bounded sets, $\bar{L}$ must be the completion of L, if L is $\sigma(B', B)$-dense in $\bar{L}$. This follows readily from the bipolar theorem in $\langle B, B' \rangle$, applied to the unit sphere $2L - 1 \in D$:

$$(2L-1)^0 = \{x \in B \mid f \in D, \|f\| \leqslant 1 \Rightarrow \langle x, f \rangle \leqslant 1\}$$
$$= \{x \in B \mid \|x\| \leqslant 1\} = (2\bar{L} - 1)^0$$

(3) By definition of $\sigma(L, K)$ and since $B = \operatorname{lin} K$, every $x \in B$ is uniformly continuous. Suppose φ: $L \to \mathbb{R}$ is affine and uniformly continuous.

Then by continuity, φ extends to a $\sigma(\bar{L}, K)$-continuous function $\bar{\varphi}: \bar{L} \to \mathbb{R}$ and by linearity to a linear functional on B'. Let $N = \{f \in B' \mid \bar{\varphi}(f) = 0\}$. Since $\bar{\varphi}$ is $\sigma(\bar{L}, K)$-continuous and hence $\sigma(\bar{L}, B)$-continuous on $\bar{L}$, the intersection of N with the unit sphere of B' is $\sigma(B', B)$-closed. By the Banach–Dieudonné theorem, N is $\sigma(B', B)$-closed. Hence $\bar{\varphi}$ is $\sigma(B', B)$-continuous and can be represented by a unique element of B.

(2) and (4) are completely analogous (compare Nagel,[7] 3.1 and 1.3).

(5) The extensions to $B \times B'$ and $D' \times D$ are trivial. Examples for the last statement will be evident from the further discussion. ∎

Proposition 2.2.2 amounts to the statement, that the state spaces $K(B_1)$ and $K(B_2)$ of two statistical theories $\langle B_1, D\rangle$, $\langle B_2, D\rangle$ are always physically equivalent, since both are dense in $\bar{K}$ in the physical uniformity, and $\bar{K}$ depends only on D. From this point of view (favored by Haag and Kastler[5]) it is natural to work directly with the statistical duality $\langle D', D\rangle$, or in the algebraic framework, with the duality $\langle \mathscr{A}^*, \mathscr{A}\rangle$ for a C^*-algebra $\mathscr{A}$.

By Proposition 2.2.1, the dual arguments apply to theories $\langle B, D_1\rangle$, and $\langle B, D_2\rangle$ over a given state space, and the physical equivalence of $L(D_1)$ and $L(D_2)$. We are thus led to consider theories $\langle B, B'\rangle$, which under the assumption that B' is a C^*-algebra must be of the form $\langle \mathscr{M}_*, \mathscr{M}\rangle$ for a W^*-algebra $\mathscr{M}$ and its unique predual $\mathscr{M}_*$.

There are obvious technical advantages in working with the asymmetric dualities $\langle D', D\rangle$ and $\langle B, B'\rangle$: Since $\bar{K}$ and $\bar{L}$ are compact, they contain an abundance of extreme points, so that one may use suitable variants of Choquet theory to decompose arbitrary elements into "simpler" constituents. In the algebraic framework the various decompositions into pure states in $\mathscr{A}^*$ and the spectral theorem in $\mathscr{M}$ are of this type. However, the normal state space of $\mathscr{M}$ need not contain any pure states and $\mathscr{A}$ need not contain any projections, so that analogous decompositions in $\mathscr{M}_*$ and $\mathscr{A}$ are in general not possible. Yet some structural information is always lost in the transition from $\langle B, D\rangle$ to an asymmetric duality $\langle B, B'\rangle$ or $\langle D', D\rangle$: Although the elements of K and $\bar{K}$ cannot be separated from each other by a finite application of the measurements from L, the new idealized states in $\bar{K}\backslash K$ formally admit much finer discrimination among the elements of L, i.e., $\sigma(L, \bar{K})$ is generally finer than $\sigma(L, K)$. Proposition 2.2.3 shows that it is indeed precisely the description of physical distinguishability of effects that is forgotten by passing from $\langle B, D\rangle$ to $\langle D', D\rangle$: if only $\langle D', D\rangle$ and the uniformity $\sigma(L, K)$ are known, B can be reconstructed. Proposition 2.2.4 is the dual result for $\langle B, B'\rangle$ and Proposition 2.2.5 further emphasizes this point: If $\rho_\alpha \to \rho$ and $f_\beta \to f$ converge in $\sigma(K, L)$ and $\sigma(L, K)$, respectively, the limits $\langle \rho, f_\beta\rangle = \lim_\alpha \langle \rho_\alpha, f_\beta\rangle$ and $\langle \rho_\alpha, f\rangle = \lim_\beta \langle \rho_\alpha, f_\beta\rangle$ are well-defined, but in general $\langle\rho,\cdot\rangle$ is no longer $\sigma(L,K)$-continuous, so that $\lim_\beta\langle\rho,f_\beta\rangle$ need not exist. Thus, one may either consider idealized states in $\bar{K}$ or idealized effects in $\bar{L}$, but these two idealizations are in general not compatible.

In the above discussion we were led to consider asymmetric dualities by a notion of physical equivalence referring only to one side of the symmetrical structure $\langle B,D\rangle$. It is possible to restore this symmetry and to define a notion of physical equivalence of arbitrary statistical theories, each given in terms of measuring and preparing procedures. The "canonical representative" of each equivalence class can be characterized and under mild assumptions these representatives are precisely the statistical dualities defined above. The sets of preparing and measuring procedures then correspond to norm-dense subsets of K and L. (For details see Werner[4].) In particular, when one assumes that for some members of the equivalence class the sets of procedures are countable (the text describing a procedure has finite length), B and D must both be norm-separable. Since a W^*-algebra is norm separable iff it is finite-dimensional, this property usually does not hold in asymmetric dualities.

Very often in mathematical physics only one side of the duality $\langle B,D\rangle$ is given. For example, a space D of observables may have been constructed or else assumed to be given, but not the physical uniformity for the observables or, equivalently, the state space $K\subset B$. Restoring the symmetry between states and effects, i.e., choosing a dense subspace $B\subset D'$ is not merely a mathematical problem: It requires the solution of the physical problem, which subclass of states in $\bar{K}\subset D'$ can actually be repared. This last statement is of course to be taken with grain of salt: so many idealizing assumptions usually enter the construction of a physical theory, that the identification of a linear functional in D' with certain "real" physical processes is rather naive. With these reservations the above statement should be taken as "the choice of B requires a characterization of those states. which can reasonably be expected to be preparable in a relatively simple way." By choosing B, we take back the extreme idealization that "essentially" all states in $\bar{K}$ can be prepared. At the same time we gain a physical uniformity on D, which relativizes the idealizations made in the construction of D by providing a measure of the "imprecision"up to which mathematical points in D are to be taken seriously as the theoretical images of "real" measuring devices. To concern oneself with the choice of B is more than a topological fancy, and in a different guise this problem is well-known in mathematical physics: If $D=\mathscr{A}$ is a C^*-algebra, it can be formulated as the choice of "physically relevant" representations π of $\mathscr{A}$. (B is then the set of normal states associated with π, and the dual B' is the von Neumann algebra $\pi(\mathscr{A})''$.) For example, in statistical mechanics the representation of the quasilocal algebra determines the temperature at infinity, and in quantum field theory, the choice of π for a certain algebra (C^*-inductive limit of local factors) distinguishes interacting from free theories.

This paper is concerned with the case of ordinary quantum mechanics, where the usual duality $\langle \mathscr{T}(\mathscr{H}), \mathscr{B}(\mathscr{H}) \rangle$ is of the asymmetric type $\langle B, B' \rangle$. But before studying the possible choices of $D \subset B'$ here, it is useful to develop a better understanding of the classical case.

3. CLASSICAL THEORIES

For classical theories, the interplay between the space B and D is especially transparent. As the distinguishing feature of classical state spaces I shall take the property that any two convex decompositions of the same state have a common refinement. Equivalently, we may say that B has the Riesz decomposition property: $\{U_i\}$, $\{V_j\} \subset B$; $U_i, V_j \geqslant 0$ and $\sum_{i=1}^n U_i = \sum_{j=1}^m V_j$ imply the existence of positive $W_{ij} \in B$ such that $U_i = \sum_{j=1}^m W_{ij}$ and $V_j = \sum_{i=1}^n W_{ij}$. This property is known[8] to imply that B and its dual B' are vector lattices. Together with a corresponding property for D this is formalized in the following definition and proposition:

Definition 3.1. A statistical duality $\langle B, D \rangle$ is called *classical*, if B has the Riesz decomposition property and D is a sublattice of B'.

We then have the following representation theorem:

Proposition 3.2. Let $\langle B, D \rangle$ be a classical statistical duality such that B and D are separable. Then there is a unique compact metrizable space Ω and a Borel measure μ on Ω with support Ω, unique up to quasi-equivalence, such that

$$B \simeq \mathscr{L}^1(\Omega, \mu); \; D \simeq \mathscr{C}(\Omega) \quad \text{and} \quad \langle \rho, f \rangle = \int \mu(d\omega)\, \rho(\omega)\, f(\omega)$$

Outline of proof. (For details see Werner.[4]) $D \simeq \mathscr{C}(\Omega)$ is Kakutani's representation theorem for AM-spaces[9]; metrizability of Ω is equivalent to the separability of D. Thus B is a space of measures on Ω. Since D determines order and norm on B, B must be a sublattice of D'. Since B is separable, one may pick a norm-dense sequence $\{\mu_n\}$ in the base K of B. Then all measures in B must be absolutely continuous with respect to $\mu = \Sigma 2^{-n} \mu_n$. Thus B can be identified with a sublattice of $\mathscr{L}^1(\Omega, \mu)$, which is necessarily of the form $\mathscr{L}^1(\Omega, \Sigma_0, \mu)$ for a suitable sub-σ-algebra Σ_0 of the Borel algebra Σ. Then the condition that D is a sublattice of B' and not merely a lattice in its own ordering implies $\Sigma = \Sigma_0$. Since B determines the order on D, the support of μ has to be Ω. ∎

It is easy now to construct examples for Proposition 2.2.5: let Ω be the unit interval with its usual topology and let μ be the Lebesgue measure. Then for any point $\omega \in \Omega$ one may find a sequence $\{\rho_n\} \subset K \subset \mathscr{L}^1(\Omega, \mu)$ and an element $f \in \bar{L} \subset \mathscr{L}^\infty(\Omega, \mu)$ such that ρ_n converges in $\sigma(\bar{K}, L)$ to the point measure ε_ω on ω and the sequence $\langle \rho_n, f \rangle$ is not convergent. Thus $\varepsilon_\omega \in \bar{K}$ has no extension by continuity from L to $\bar{L}$, although there are of course many Hahn–Banach extensions of this functional.

Suppose now that only a *measure space* (Ω, Σ, μ) and thus an asymmetric duality $\langle B, B' \rangle = \langle \mathscr{L}^1, \mathscr{L}^\infty \rangle$ is given. Then from the point of view of $\langle B, B' \rangle$ it makes no sense to talk about individual points $\omega \in \Omega$. In fact one may add to Ω or delete from it any number of points without changing the structure of B as long as these modifications are of μ-measure zero: the structure of B depends only $\Sigma/\text{mod}\,\mu$, i.e., the extreme points of $\bar{L} \subset B'$. Moreover all separable spaces $\mathscr{L}^1(\Omega, \Sigma, \mu)$ over atomless measure spaces are isomorphic. Thus from the measure theoretic point of view there is no difference between $(\mathbb{R}, dx)$ and $(\mathbb{R}^n, d^n x)$. On the other hand, the difference between two theories in which systems are described by one and n real parameters respectively is intuitively quite clear and lies in the different topologies of $\mathbb{R}$ and $\mathbb{R}^n$. Under an isomorphism of $\mathscr{L}^1(\mathbb{R}^n, d^n x)$ with $\mathscr{L}^1(\mathbb{R}, dx)$ "points" will end up "close" to each other, which may nevertheless be distinguished clearly from each other by the measurement of one of the n real parameters. The above Proposition shows that the selection of a distinguished class of measurements $D \subset \mathscr{L}^\infty(\Omega, \Sigma, \mu)$ is indeed tantamount to the introduction of a topology for Ω and by Proposition 2.2, to the introduction of a topology for the set $K \subset B$ of statistical states.

Analogous remarks apply to the dual situation where only a *topological space* Ω and thus an asymmetric duality $\langle D', D \rangle = \langle \mathscr{C}(\Omega)', \mathscr{C}(\Omega) \rangle$ is given. The points $\omega \in \Omega$ are intrinsic to this structure: they may be identified with the pure statistical states, i.e., the extreme points of the base $\bar{K} \subset D'$. On the other hand, there is no notion of "almost sure equality" of Borel subsets which could be used to define a (reasonably small) σ-algebra of "events", constituting a "logic" of the system. These structures depend on the choice of a distinguished measure $\mu \in \mathscr{C}(\Omega)'$ or equivalently of a physical uniformity on D, induced by $\mathscr{L}^1(\Omega, \mu) \simeq B \subset D'$ in the manner described in Section 2.

It was demonstrated in Section 2 that the main problem of this paper, the construction of physical uniformities for the state space of quantum mechanics, amounts to the choice of a subspace $D \subset \mathscr{B}(\mathscr{H}) = B'$ of "continuous" observables. By the above remarks the analogous problem for classical theories can be understood as the introduction of topologies for a given measure space. It will turn out that in the nonrelativistic case the solutions to the quantum mechanical and the classical problem are in one-to-one correspondence. Therefore the following observations will also be of direct importance to the quantum mechanical problem.

For any given measure space (Ω, Σ, μ), $\mathscr{L}^\infty(\Omega, \Sigma, \mu)$ is a commutative

W^*-algebra and hence isomorphic to $\mathscr{C}(X)$ for some compact, totally disconnected space X. The topology of X hardly describes any notion of "physical distinguishability", and it will indeed be irrelevant for the following considerations except for the fact that the choice of a subspace $D \subset \mathscr{L}^\infty(\Omega, \Sigma, \mu)$ amounts to the choice of a subset of $\mathscr{B}(X)$, the lattice of all bounded real valued functions on X.

Let X be any set. Then any subset $D \subset \mathscr{B}(X)$ defines an initial uniform structure $\sigma(X, D)$ on X, generated by the entourages

$$U_{\epsilon, f_1 \cdots f_n} := \{(x_1, x_2) \mid \forall_{i=1}^n \, |f_i(x_1) - f_i(x_2)| \leqslant \varepsilon\}$$

for $\varepsilon > 0$ and $f_1 \cdots f_n \in D$. Again, if the elements of X are interpreted as physical states or objects and the functions $f \in D$ describe the response of certain measurement devices, $\sigma(X, D)$ may be called a physical uniformity. The Hausdorff completion[6] of $(X, \sigma(X, D))$ will be called the D-completion of X and denoted by $\hat{X}^D$ or, more precisely by i_D: $X \to \hat{X}^D$. Since every function in D is bounded, $\hat{X}^D$ is a compact space. The canonical map i_D identifies indistinguishable objects and the unique uniformity on $\hat{X}^D$ compatible with its topology describes the distinguishability of the equivalence classes of such objects. In particular i_D is injective iff $\sigma(X, D)$ is Hausdorff and iff D separates points of X. The set of all $\sigma(X, D)$-uniformly continuous real valued functions on X is naturally isomorphic to $\mathscr{C}(\hat{X}^D)$. By an application of the Stone–Weierstrass theorem to $\mathscr{C}(\hat{X}^D)$ we conclude that two subsets $D_1, D_2 \subset \mathscr{B}(X)$ induce the same uniformity $\sigma(X, D_1) = \sigma(X, D_2)$ iff $D_1 \cup \{1\}$ and $D_2 \cup \{1\}$ generate the same norm-closed subalgebra (or linear sublattice) of $\mathscr{B}(X)$. We arrive thus at a one-to-one correspondence between norm-closed subalgebras of $\mathscr{B}(X)$, precompact uniform structures on X and mappings i: $X \to Y$ with dense range in a compact space Y. Under this correspondence $Y = \hat{X}^D$ is metrizable iff $\sigma(X, D)$ has a countable base and iff D is norm-separable.

In the case considered above the space X was already equipped with a topology. It is easy to see that in this case the mapping i_D is continuous iff every $f \in D$ is continuous. If D is the algebra of all bounded continuous functions, $\hat{X}^D$ is known as the Stoně–Čech compactification. More generally, $i_D: X \to \hat{X}^D$ is called a compactification if i_D is a homeomorphism onto $i_D(X)$. Clearly it is necessary for this that X is a completely regular space and D consists of continuous functions. Then $(i_D, \hat{X}^D)$ is a compactification iff D separates points from closed sets.[10] Moreover, if X is locally compact we may replace this condition either by the requirement that $i_D(X)$ is open in $\hat{X}^D$ or that the algebra generated by D contains all functions going to a constant outside of compact sets.[11] (These results will later be applied to phase space.)

In general there are many different compactifications of a given topological space. The following example shows how the choice between different compactifications can be related to different classes of measuring procedures: Let $X = \mathbb{R}^3$ with its usual topology be interpreted as the set of spatial points. Consider first the method of distinguishing points by means of measuring rods of finite accuracy, applied during finite expeditions into space. A typical entourage of the associated uniformity $\mathcal{N}_1$ consists of the set of pairs (x_1, x_2) which remain undistinguishable for some given expedition. Note that in each case most of the universe remains uncharted. These uncharted regions form a Cauchy filter for the uniformity $\mathcal{N}_1$, whose unique limit point in the completion of $(X, \mathcal{N}_1)$ may be denoted by "∞". It is easy to see that this completion is equal to $X \cup \{\infty\}$, the Alexandroff *one-point compactification* of X. As an alternative method consider optical measurements of finite angular resolution, carried out from finitely many observatories. Apart from the filters converging to an ordinary point in X, obtained by observing this point from two different angles with increasing accuracy, the typical Cauchy filters of this uniformity are finite intersections of cones with common axis and nonzero opening angle, based at arbitrary points in space. The corresponding compactification is homeomorphic to a closed ball in $\mathbb{R}^3$, whose boundary is just the celestial sphere. In Section 6 we shall be led to consider the analogous *ball compactification* of $\mathbb{R}^n$. A convenient metrization of the associated uniformity is given by $d(x, y) := |x/(1+|x|) - y/(1+|y|)|$, where $|x|$ denotes the euclidean norm in $\mathbb{R}^n$. Translations and nonsingular linear transformations are Lipshitz-continuous with respect to this metric and hence extend by continuity to the compactification. Important properties of this compactification are that the affine group of $\mathbb{R}^n$ acts on it continuously and that the sphere at infinity is pointwise invariant under translations.

4. CRITERIA FOR THE CHOICE OF PHYSICAL UNIFORMITIES ON STATES

The examples at the end of the preceding section show that each "physical" uniformity is closely related to a class of measuring devices by means of which the objects in question are to be tested and thereby distinguished. For constructing the uniformities in these examples, we did not make use of an explicit physical description of the measuring instruments. Only qualitative features, particularly those related to the resolution of very distant points, entered the construction.

Since for quantum mechanical measuring processes detailed theoretical descriptions are usually not available, we need to rely on similar qualitative ideas concerning measurement in our endeavor to construct physical uniformities for quantum states. The following list of criteria for the choice of a subspace $D \subset B'$, and hence of a physical uniformity on states, is due to Ludwig. Since the parallels between classical and quantum theories will play a crucial role in the sequel, these criteria will be formulated for arbitrary statistical theories, and are hence applicable to both cases. Thus we shall assume throughout that B is a base-normed Banach space and that $D \subset B'$ is a norm-closed subspace containing $1 \in B'$.

Condition 1. $D \cap [0, 1]$ is $\sigma(B', B)$-dense in $[0, 1] \subset B$.

Condition 2. D is $\sigma(B', B)$-dense in B.

Condition 1 is equivalent to the requirement that $\langle B, D \rangle$ is a statistical duality in the sense of Definition 2.1 and obviously implies Condition 2. However, it is often easier to check Condition 2, which only means that the functionals in D separate points in B. If B' is a W^*-algebra and D is a subalgebra, both conditions are equivalent by virtue of Kaplansky's density theorem. The following condition has also been discussed in Section 2 and is equivalent to the metrizability of the physical uniformity:

Condition 3. D is norm separable.

These conditions together still admit an immense variety of different spaces D. Further restrictions can be formulated with respect to further structures given on B. As an additional structure we shall consider only a symmetry group G, which will be taken later as the Galilei group (or some group related to it). If G is a topological group and β: $G \to \mathrm{Aut}(B)$ is a representation of G by positive normalized linear maps, we may formulate:

Condition 4. For each $g \in G$, the adjoint β'_g: $B' \to B'$ of β_g satisfies: $B'_g D \subset D$.

Condition 5. For each $\rho \in B$ and $f \in D$ the function $g \to \langle \beta_g \rho, f \rangle$ is continuous.

Condition 6. For $f \in D$ the function $g \to \beta'_g f$ is norm continuous.

Condition 4 postulates that the class of measurements corresponding to D is closed under the symmetry operations described by G. Condition 5 is automatically satisfied if β_g is strongly continuous on B (i.e., satisfies the analogue of Condition 6) and is implied by Condition 6. Surprisingly, the strong continuity condition Condition 6 already follows from Conditions 3, 4 and 5. This was proven by Ludwig[2], Theorem VI.1.2.1, using a category argument for G and we include the relevant result here for reference:

Lemma 4.1. Let D be a Banach space and $\beta': G \to \mathscr{L}(D)$ a representation of a locally compact group by linear operators on D. Let M be a $\sigma(D', D)$-dense subset of the unit sphere of D' and $U \subset G$ a neighborhood of the identity, on which $\|\beta'_g\|$ is uniformly bounded. Then for any $f \in D$ the function $g \to \beta'_g f$ is norm continuous on G if and only if for all $\rho \in M$ the function $g \to \langle \rho, \beta'_g f \rangle$ is continuous on U, and $\{\beta'_g f \mid g \in U\}$ is norm separable.

Condition 5 may also be replaced by the stronger condition, that all functions $g \to \langle \beta_g \rho, f \rangle$ are uniformly continuous with respect to some "physical uniformity" on G, which is coarser than the group-uniformity. In this way we can postulate that $\beta'_g f$ becomes less and less sensitive when $g \in G$ becomes "large." (Compare the examples of Section 3.) Thus we shall consider conditions of the following type:

Condition 7. For each $\rho \in B$ and $f \in D$ the function $g \to \langle \beta_g \rho, f \rangle$ belongs to some suitably specified subspace of $\mathscr{C}(G)$.

It will turn out later that for nonrelativistic classical or quantum theories a condition of this type characterizes D completely: If Conditions 4 and 6 are satisfied, D may be reconstructed from the norm-closed span of $\{\langle B.\rho, f \rangle \mid \rho \in B, f \in D\}$.

5. CORRESPONDING SPACES OF OBSERVABLES

The natural symmetry group for a system of n free distinguishable nonrelativistic particles is the nth direct power of the Galilei group. The subgroup of this group generated by translations and boosts will be denoted by X and is isomorphic to $\mathbb{R}^{2N}$ $(N = 3n)$. X can also be viewed as the *phase space* of the system and carries a symplectic from $\{\cdot, \cdot\}: X \times X \to \mathbb{R}$ given in the usual way by $\{x, x'\} = \{(p, q), (p', q')\} = \sum_{i=1}^{N} (p_i q'_i - p'_i q_i)$. The Haar measure on X will always be normalized as $dx = (2\pi)^{-N} dp^N dq^N$. Our main results will only refer to this group X of kinematical transformations, which will remain a part of the symmetry group also when the free Hamiltonian is replaced by an interacting one. Without much additional effort we shall obtain results also for (arbitrary subgroups of) the group of *linear symplectic transformations* on X, which will be denoted by G in the sequel.

For *classical systems* the statistical states are represented by elements of $\mathscr{B}_c = \mathscr{L}^1(X, dx) \equiv \mathscr{L}^1$ and the symmetry groups X and G are represented by $(\alpha_x f)(y) = f(x - y)$ and $(\beta_g f)(y) = f(g^{-1} y)$ for $f \in \mathscr{B}_c$, $x, y \in X$ and $g \in G$. Physical uniformities on classical states correspond to subspaces $\mathscr{D}_c \subset \mathscr{B}'_c = \mathscr{L}^\infty(X, dx)$.

The states of *quantum systems* are represented by elements of $\mathscr{B}_q =$

$\mathscr{T}(\mathscr{H}) \equiv \mathscr{T}$, the Banach space of trace class operators on a Hilbert space $\mathscr{H}$. The trace norm on $\mathscr{T}$ and the norm of $\mathscr{L}^1$ will both be denoted by $\|\cdot\|_1$. The group X is represented by $\alpha_x(A) = E(x) A E(-x)$ for $x \in X$, $A \in \mathscr{T}$, where $x \to E(x)$ is an irreducible representation of the Weyl commutation relations $E(x) E(y) = e^{i\{x,y\}/2} E(x+y)$. By von Neumann's uniqueness theorem there is a family $\{U_g\}_{g \in G}$ of unitaries of $\mathscr{H}$ satisfying $U_g E(x) U_g^* = E(gx)$. Since U_g represents G up to a factor, we may define a representation β of G on $\mathscr{T}$ by $\beta_g(A) = U_g A U_g^*$. Physical uniformities on quantum states correspond to subspaces $\mathscr{D}_q \subset \mathscr{B}_q' = \mathscr{B}(\mathscr{H})$.

Before proceeding further we have to describe an extended notion of convolutions needed to establish some close interrelations between the classical and the quantum mechanical representations of X. This notion has been introduced in Werner[12] and the reader is referred to that paper for further details and for proofs. Note that the convolution in $\mathscr{L}^1$ can be rewritten as $(f * g)(x) = [\int dy\, f(y)\, \alpha_y g](x) = \int dy\, f(y)(\alpha_x \beta_- g)(y)$, where β_- represents the phase space inversion $x \to -x$.Replacing integrals by traces we arrive at the following definitions:

Definition 5.1. Let $A, B \in \mathscr{T}$ and $f, g \in \mathscr{L}^1$. Then their *convolutions* are defined by:

$$(f * g)(x) = \int dy\, f(y)\, g(x-y)$$

$$(A * B)(x) = \operatorname{tr}(A(\alpha_x \beta_- B))$$

$$A * f = f * A = \int dy\, f(y)\, \alpha_y(A)$$

and their Fourier transforms are defined by:

$$(\mathscr{F} f)(x) = \int dy\, f(y)\, e^{i\{x,y\}}$$

$$(\mathscr{F} A)(x) = \operatorname{tr}(A E(x))$$

Convolution is defined by the same formulae if either $f \in \mathscr{L}^\infty$ or $A \in \mathscr{B}(\mathscr{H})$ (but not both). Note that $f * g$, $A * B$, $\mathscr{F} f$, and $\mathscr{F} A$ are functions on X, whereas $f * A$ is an operator. Basic properties are summarized in the following propositions:

Proposition 5.2. (1) Convolution is associative and commutative. The convolution of positive elements is positive.

For $T \in \mathscr{L}^1 \cup \mathscr{T}$ and $A \in \mathscr{L}^1 \cup \mathscr{T}$ or $A \in \mathscr{L}^\infty \cup \mathscr{B}(\mathscr{H})$:

(2) $\|T * A\|_1 \leqslant \|T\|_1 \|A\|_1$ and $\|T * A\| \leqslant \|T\|_1 \|A\|$;

(3) $\alpha_x(T * A) = (\alpha_x T) * A$ and $\beta_g(T * A) = (\beta_g T) * (\beta_g A)$;

(4) For $A, B \in \mathscr{L}^1 \cup \mathscr{T}$: $\mathscr{F}(A * B) = (\mathscr{F}A) \cdot (\mathscr{F}B)$.

Of the many analogues of classical results that may be proven for these extended convolutions and Fourier transforms, the following will be the most important for our purposes:

Proposition 5.3. (Wiener's approximation theorem) For $T \in \mathscr{T}$ the following conditions are equivalent:

(1) $\forall_{x \in X} (\mathscr{F}T)(x) \neq 0$;

(2) $T * \mathscr{T}$ is norm-dense in $\mathscr{L}^1$;

(3) $T * \mathscr{L}^1$ is norm-dense in $\mathscr{T}$;

(4) $A \in \mathscr{B}(\mathscr{H})$, $T * A = 0 \Rightarrow A = 0$;

(5) $f \in \mathscr{L}^\infty$, $T * f = 0 \Rightarrow f = 0$. If these conditions are satisfied, T is said to be *regular.*

As a first step towards the characterization of physical uniformities, we single out those spaces of quantum and classical observables, which satisfy Conditions 4 and 6 with respect to the group X:

Definition 5.4. A subspace $\mathscr{D}_c \subset \mathscr{L}^\infty$ (respectively, $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$) is called *continuous*, if it is norm-closed, invariant under α_x, and $\lim_{x \to 0} \|\alpha_x A - A\| = 0$ for all $A \in \mathscr{D}_c$ (respectively, $A \in \mathscr{D}_q$). The largest continuous subspaces will be denoted by $\mathscr{C}_c \subset \mathscr{L}^\infty$ and $\mathscr{C}_q \subset \mathscr{B}(\mathscr{H})$.

Note that $\mathscr{C}_c$ is simply the space of bounded uniformly continuous functions. Since $f \in \mathscr{C}_c$ may be uniformly approximated by functions of the form $\rho * f$ with $\rho \in \mathscr{L}^1$, it is easy to see that a continuous subspace $\mathscr{D}_c$ coincides with the norm-closed linear hull of the set of functions $x \to \langle \alpha_x \rho, f \rangle$ for $\rho \in \mathscr{L}^1, f \in \mathscr{D}_c$ and is hence completely determined by a condition of the form of Condition 7. However, the x-dependence of expectation values and hence Condition 7 may be formulated equally well in the quantum mechanical setting and hence we may use each continuous subspace $\mathscr{D}_c$ to formulate a constraint on subspaces $\mathscr{D}_q$ by postulating that all functions $x \to \mathrm{tr}((\alpha_x W)A)$ lie in $\mathscr{D}_c$ for $W \in \mathscr{T}$, $A \in \mathscr{D}_q$. This is equivalent to the condition $\mathscr{T} * \mathscr{D}_q \subset \mathscr{D}_c$ and the above definition of convolutions allows us to formulate the "reverse" condition. Thus we arrive at:

Definition 5.5. Two continuous subspaces $\mathscr{D}_c \subset \mathscr{L}^\infty$ and $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$ are said to be *corresponding*, if $\mathscr{T} * \mathscr{D}_c \subset \mathscr{D}_q$ and $\mathscr{T} * \mathscr{D}_q \subset \mathscr{D}_c$.

Theorem 5.6: (1) For any continuous subspace $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$ there is a unique corresponding continuous subspace $\mathscr{D}_c \subset \mathscr{L}^\infty$ and conversely.

(2) If $T \in \mathscr{E}$ is regular, $T * \mathscr{D}_c$ is norm-dense in $\mathscr{D}_q$ and $T * \mathscr{D}_q$ is norm-dense in $\mathscr{D}_c$.

(3) (Tauberian theorem) If $T \in \mathscr{E}$ is regular:

$$A \in \mathscr{C}_q, \qquad T * A \in \mathscr{D}_c \Rightarrow A \in \mathscr{D}_q$$

$$f \in \mathscr{C}_c, \qquad T * f \in \mathscr{D}_q \Rightarrow f \in \mathscr{D}_c$$

(4) $\mathscr{D}_c$ is an order unit subspace or is separable or separates points of $\mathscr{L}^1$ if and only if $\mathscr{D}_q$ has the analogous property.

By virtue of this theorem the problem of constructing physical uniformities on the state space of nonrelativistic quantum mechanics is completely reduced to the analogous problem for classical mechanics. In particular, we may associate with each continuous subspace $\mathscr{D}_q$ a completion $\hat{X}^{\mathscr{D}_c}$ of phase space as described in Section 3. Often the properties of the compact space $\hat{X}^{\mathscr{D}_c}$ will help to make intuitive the type of "closeness" described by the physical uniformity $\sigma(\mathscr{E}, \mathscr{D}_q)$ in much the same way as the properties of the compactifications of $\mathbb{R}^3$ discussed in Section 3 reflected the sensitivity of certain classes of measuring procedures. This connection will be studied in more detail in the next section.

The following result shows that correspondence also preserves the Conditions 4 and 6 when formulated with respect to subgroups of G such as rotations, spatial dilations, particle permutations, and the time evolutions of free or harmonically bound systems.

Proposition 5.7. Let $\mathscr{D}_c$ and $\mathscr{D}_q$ be corresponding continuous subspaces and $i: X \to \hat{X}$ the $\mathscr{D}_c$-completion of X. Let $H \subset G$ be a subgroup equipped with its left invariant uniformity. Then:

(1) $\beta_H \mathscr{D}_q \subset \mathscr{D}_q$ iff $\beta_H \mathscr{D}_c \subset \mathscr{D}_c$ iff for each $g \in H$ the map $\tau_g: x \to gx$ extends by continuity to a homeomorphism $\hat{\tau}_g: \hat{X} \to \hat{X}$.

(2) $\{\beta_g\}_{g \in H}$ is, in addition, strongly continuous on $\mathscr{D}_q$ iff this is valid on $\mathscr{D}_c$ iff the map $\hat{\tau}: H \times \hat{X} \to \hat{X}$ is uniformly continuous.

A basic example of a pair of corresponding continuous spaces is given by the space $\mathscr{K}_q \subset \mathscr{B}(\mathscr{H})$ of *compact operators* and the space $\mathscr{K}_c \subset \mathscr{L}^\infty$ of *continuous functions on X vanishing at infinity*. By adjoining units we obtain the corresponding order unit space $\mathscr{K}_q \oplus \mathbb{C}1$ and $\mathscr{K}_c \oplus \mathbb{C}1$. Both spaces satisfy Conditions 1, 3, 4, and 6 for the whole symplectic affine group on X. Accordingly, this group acts continuously on the associated completion of phase space, which is the one-point compactification. Not only the symplectic group is strongly continuous on $\mathscr{K}_q \oplus \mathbb{C}1$: This space can be characterized as the largest subspace $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$ such that, for any strongly continuous representation β: $G \to \mathrm{Aut}(\mathscr{T})$ of any topological group G, all functions $g \to \beta_g' A$ $(A \in \mathscr{D}_q)$ are norm continuous. Thus the uniformity $\sigma(\mathscr{T}, \mathscr{K}_q \oplus \mathbb{C}1)$ is a candidate for a physical uniformity on quantum states independently of the symmetry group under consideration. We have seen in Section 2 how the elements of the $\sigma(\mathscr{T}, \mathscr{K}_q \oplus \mathbb{C}1)$-completion $\bar{K}$ of the ordinary state space can be considered as idealized states. In the present case Propostion 2.2 makes it easy to calculate this space: since the dual of $\mathscr{K}_q$ is the trace class $\mathscr{T}$ itself, we have $(\mathscr{K}_q \oplus \mathbb{C}1)' = \mathscr{T} \oplus \mathbb{C}\phi_\infty$, where ϕ_∞ is the functional $\langle \phi_\infty, C + \lambda 1 \rangle = \lambda$. Consequently the space $\bar{K} = \{W \oplus (1 - \mathrm{tr}\, W)\phi_\infty \mid W \in \mathscr{T}, W \geqslant 0, \ \mathrm{tr}\, W \leqslant 1\}$ is obtained from the ordinary state space by adjoining a single new extreme point ϕ_∞. The state ϕ_∞ has the property that for any $A \in \mathscr{K}_q \oplus \mathbb{C}1$, $W \in \mathscr{T}$ and any one-parameter group U_t whose generator has absolutely continuous spectrum $\lim_{t \to \infty} \mathrm{tr}(U_t W U_t^* A) = \mathrm{tr}\, W \cdot \langle \phi_\infty, A \rangle$. (The existence of all these limits again characterizes $\mathscr{K}_q \oplus \mathbb{C}1$.) For other subspaces $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$ the $\sigma(\mathscr{T}, \mathscr{D}_q)$-completion of state space will contain more than one "state at infinity". The following section is devoted to a study of this structure.

6. COMPLETIONS OF THE QUANTUM MECHANICAL STATE SPACE

Throughout this section $\mathscr{D}_c$ and $\mathscr{D}_q$ denote corresponding continuous order unit subspaces and i: $X \to \hat{X}$ denotes the $\mathscr{D}_c$-completion of phase space. $K = \{W \in \mathscr{T} \mid W \geqslant 0, \ \mathrm{tr}\, W = 1\}$ will be the usual state space of quantum mechanics, whose $\sigma(\mathscr{T}, \mathscr{D}_q)$-completion $\bar{K}$ will be identified with the base in $\mathscr{D}_q'$.

The classical counterpart of $\bar{K}$ is easily characterized: If $\mathscr{D}_c$ is a C^*-algebra, we have $\mathscr{D}_c \simeq \mathscr{C}(\hat{X})$ so that the base of $\mathscr{D}_c'$ is the set of Radon probability measures on $\hat{X}$. In the quantum case many new problems arise. For example: will the pure states in K, given by one-dimensional projections on $\mathscr{H}$, remain pure in the larger set $\bar{K}$ or will some admit a further convex decomposition? The following two results show that under fairly general circumstances such a decomposition is impossible.

Proposition 6.1. (Lebesgue decomposition) Let B' be a W^*-algebra and $D \subset B'$ a $\sigma(B', B)$-dense C^*-subalgebra. Then there is a projection P: $D' \to D'$ such that P and $1 - P$ are both positive and PD is the canonical image of B in D'. Consequently $D' \simeq B \oplus (1 - P)D'$.

Proof. Identify B with a norm-closed subspace of D' and consider the $\sigma(D'', D')$-closed subspace $B^\perp = \{a \in D'' \mid \forall_{\varphi \in B} \langle \varphi, a \rangle = 0\}$. Since B is invariant under the maps R_a and L_a (defined on D' by $\langle R_a\varphi, b\rangle = \langle\varphi, ba\rangle$ and $\langle L_a\varphi, b\rangle = \langle\varphi, ab\rangle$ for $\varphi \in D'$, $a, b \in D$) $B^{\perp}$ is closed under left and right multiplication by elements of D. By the $\sigma(D'', D')$-continuity of multiplication, $B^{\perp}$ is a $\sigma(D'', D')$-closed two sided ideal in D'' and hence of the form $B^{\perp} = ZD'' = D''Z$ for a unique central projection $Z \in D''$. Define $\langle P\varphi, a\rangle = \langle\varphi, (1-Z)a\rangle = \langle\varphi, (1-Z)\,a(1-Z)\rangle$ for $\varphi \in D'$, $a \in D$. Clearly P and $1-P$ are positive and $PD = B^{\perp\perp}$ is the bipolar of B in $\langle D', D''\rangle$. Since B was norm-closed, we have $B^{\perp\perp} = B$. ∎

We may also express this result by saying that the "old" states $K \subset \mathscr{E}$ and the "new" states $\bar{K}\backslash K \subset \mathscr{D}_q'$ can be combined only by statistical mixtures but not by "coherent superpositions" of any kind. As in the classical case ($B = \mathscr{L}^1(\Omega, \mu)$, $D = \mathscr{C}(\Omega)$) P can be understood as splitting each state into a "singular" and an "absolutely continuous" part. In the classical case a different splitting is also suggestive, namely into a "finite" part supported by $X \subset \hat{X}$ and a state "at infinity" supported by $\hat{X}\backslash X$. In this case the "finite" part may contain singular constituents like point measures on X. The second part of the following proposition shows that this cannot hapen in the quantum case: the states at infinity are then precisely the singular states. Since $\mathscr{E} * \mathscr{D}_c \subset \mathscr{D}_q$ and $\mathscr{E} * \mathscr{D}_q \subset \mathscr{D}_c$, we shall extend the convolution operation to $\mathscr{E} \times \mathscr{D}_q' \to \mathscr{D}_c'$ and $\mathscr{E} \times \mathscr{D}_c' \to \mathscr{D}_q'$ by setting $\langle T * \phi, A\rangle := \langle\phi, (\beta_- T) * A\rangle$ for $T \in \mathscr{E}$ and $\phi \in \mathscr{D}_q'$, $A \in \mathscr{D}_c$ or $\phi \in \mathscr{D}_c'$, $A \in \mathscr{D}_q$.

Proposition 6.2. Let $\mathscr{D}_c$, $\mathscr{D}_q$, and i: $X \to \hat{X}$ be as above. Then:

(1) The following conditions are equivalent: (a) i is a compactification, i.e., a homeomorphism of X onto $i(X)$. (b) i is injective and $\hat{X}\backslash i(X)$ is closed. (c) The C^*-algebra generated by $\mathscr{D}_c$ contains one nonzero function (and hence all functions) vanishing at infinity.

(2) Suppose the equivalent conditions $\mathscr{K}_c \subset \mathscr{D}_c$ and $\mathscr{K}_q \subset \mathscr{D}_q$ are satisfied. Then for $i = c$ and $i = q$ there are projections P_i: $\mathscr{D}_i' \to \mathscr{D}_i'$ such that P_i and $1 - P_i$ are positive, $P_i\mathscr{D}_i' = \mathscr{K}_i'$ and $(1-P_i)\mathscr{D}_i' = \mathscr{K}_i^{\perp} \simeq (\mathscr{D}_i/\mathscr{K}_i)'$. For any $T \in \mathscr{E}$: $T * (P_c\phi) = P_q(T * \phi)$ for $\phi \in \mathscr{D}_c'$ and $T * (P_q\phi) = P_c(T * \phi)$ for $\phi \in \mathscr{D}_q'$. If $\mathscr{D}_q$ is a C^*-algebra, P_q is the projection of the Lebesgue decomposition with respect to $\mathscr{K}_q' = \mathscr{E}$.

Proof. Proposition 6.2.1. was already noted in Section 3, except for the statement implicit in Proposition 6.2.1(c) that $\mathscr{K}_c$ is minimal in the lattice of α_x-invariant C^*-subalgebras of $\mathscr{L}^{\infty}$. This was proven by Werner.[12] In Proposition 6.2.2 the crucial observation is that in both cases $\mathscr{K}_i'$ may be identified in a canonical way with a subspace of $\mathscr{D}_i'$: $\mathscr{K}_c'$ is the space of finite measures on the locally compact space X and $\mathscr{K}_q'$ is the trace class.

Thus we may define $P_i\phi$ as the element of $\mathscr{D}_i'$ identified with the restriction of ϕ to $\mathscr{K}_i$. With this construction the relations stated in the proposition are straightforward with the exception of $1-P_i \geqslant 0$. Let $\phi \in \mathscr{D}_c'$ be positive. $\mathscr{D}_c$ is an order unit subspace of $\tilde{\mathscr{D}} \simeq \mathscr{C}(\hat{X})$, the C^*-algebra generated by $\mathscr{D}_c$. Hence ϕ has a positive extension $\tilde{\phi} \in \tilde{\mathscr{D}}'$. We may apply our construction to $\tilde{\mathscr{D}}$, obtaining a projection $\tilde{P}$ on $\tilde{\mathscr{D}}$. Since ϕ and $\tilde{\phi}$ are equal on $\mathscr{K}_c$, the restrictions of $\tilde{P}\tilde{\phi}$ and $(1-\tilde{P})\phi$ to $\mathscr{D}_c$ are equal to $P\phi$ and $(1-P)\phi$, and we only need to show that $(1-\tilde{P})$ is positive. This is clear since $(1-\tilde{P})$ acts on the measures in $\tilde{\mathscr{D}}' \simeq \mathscr{C}(\hat{X})'$ like multiplication with the characteristic function of the compact $G_\delta \hat{X}\backslash X$. In the quantum case we may argue similarly, using an extension $\tilde{\phi}$ of ϕ to $\mathscr{B}(\mathscr{H})$. Since the singular states on $\mathscr{B}(\mathscr{H})$ are precisely those annihilating $\mathscr{K}_q$, the projection $\tilde{P}$ obtained as above coincides with the Lebesgue decomposition and $1-\tilde{P}$ is positive. ∎

Intuitively the conditions $\mathscr{K}_i \subset \mathscr{D}_i$ mean that each point of X can be distinguished from very distant points: The measurements described by the elements of $\mathscr{K}_i \oplus \mathbb{C}1$ are practically insensitive to systems of exceedingly high momentum as well as to systems which are very far away spatially. An important example of a continuous subspace $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$ violating the conditions $\mathscr{K}_q \subset \mathscr{D}_q$ is the CCR-algebra, i.e., the C^*-algebra generated by $\{E(x) \mid x \in X\}$. Its corresponding space $\mathscr{D}_c \subset \mathscr{L}^\infty$ is the space of almost periodic functions[12] on X and the $\mathscr{D}_c$-completion i: $X \to \hat{X}$ is distinguished by the property that $\hat{X}$ is itself a topological group, and i is an injective homomorphism. $\hat{X}$ is sometimes called the Bohr compactification[13] of X, but it is plain by the above result that $(i, \hat{X})$ is not a compactification of X. The topology induced on X by the almost periodic functions is indeed rather strange: each neighborhood of each point contains infinitely many points outside of an arbitrarily large sphere. Almost periodicity is hardly to be expected of the response function of a real apparatus, so that in spite of its usefulness for discussing representations we shall have to discard the CCR-algebra as a set of "realistic" observables inducing a physical uniformity on K.

It is easy to see that no state in $K \subset \mathscr{T} = \mathscr{K}_q'$ or in $\mathscr{K}_c'$ can be invariant under any phase space translation. In constrast, $\mathscr{D}_q'$ and $\mathscr{D}_c'$ always contain invariant states, which are related in the following simple manner:

Proposition 6.3. For $i=c$ or $i=q$, let $S_i := \{\phi \in \mathscr{D}_i' \mid \phi \geqslant 0, \langle \phi, 1 \rangle = 1, \forall_{x\in X} \alpha_x \phi = \phi\}$. Then S_i is a nonempty $\sigma(\mathscr{D}_i', \mathscr{D}_i)$-compact convex set. There is a canonical isomorphism j: $S_c \to S_q$ given by $j\phi_c = T * \phi_c$ and $j^{-1}\phi_q = T * \phi_q$ for all $\phi_i \in S_i$ and $T \in \mathscr{T}$ with $\operatorname{tr} T = 1$. S_c and S_q are simplices.

Proof. S_i is nonempty, since for any $T \in \mathscr{T}$ with $T \geqslant 0$ and $\operatorname{tr} T = 1$, and every invariant mean η on X the functional $\phi \in \mathscr{D}_i'$ given by $\langle \phi, A \rangle := \eta(\langle T, \alpha.A \rangle)$ is in S_i, Let $\phi \in S_i$, $A \in \mathscr{D}_i$, and $\rho \in \mathscr{L}^1$. Since α_x is strongly continuous on $\mathscr{D}_i$, $\beta_-\rho * A$ can be approximated in norm by linear combinations of translates of A, so that $\langle \rho * \phi, A \rangle := \langle \phi, (\beta_-\rho) * A \rangle = \langle \phi, A \rangle \cdot \int dx\, \rho(x)$. Hence for $T_1, T_2 \in \mathscr{T}$: $T_1 * T_2 * \phi = \phi \cdot \int dx (T_1 * T_2)(x) = \phi \cdot \operatorname{tr} T_1 \cdot \operatorname{tr} T_2$. Thus j and j^{-1} are inverses of each other and independent of T. S_c is a simplex by a standard result of classical ergodic theory. ∎

Note that in the sense of the decomposition 6.2 invariant states are necessarily "states at infinity". Thus we may expect a complete characterization of $\bar{K}$ in the case that, conversely, all states at infinity are invariant. The following considerations are intended to make this assumption plausible on physical grounds.

Suppose that $A \in \mathscr{C}_q \subset \mathscr{B}(\mathscr{H})$ describes a measurement apparatus, which has a regular behavior at infinity in the sense that for $T \in \mathscr{T}$ and certain sequences $\{x_n\} \subset X$ with $|x_n| \to \infty$, the limit of expectation values $\lim_{n\to\infty} \langle T, \alpha_{x_n} A \rangle$ exists. $\{x_n\}$ describes a sequence of operations on A, taking the apparatus further and further away from the preparing apparatus described by T or to larger and larger velocities relative to it. We may expect that for large n the expectation values will not change significantly if A is shifted by another small amount, say $x \in X$, and that the limits of $\langle T, \alpha_{x_n+x} A \rangle$ and $\langle T, \alpha_{x_n} A \rangle$ will be equal. Thus $\lim_n \langle T, \alpha_{x_n}(A - \alpha_x A) \rangle = 0$ and if this holds for all such sequences $\{x_n\}$, $y \to \langle T, \alpha_y(A - \alpha_x A) \rangle$ will be a continuous function vanishing at infinity. Then by the Tauberian Theorem 5.6.3, $A - \alpha_x A$ is a compact operator. This motivates the following:

Definition 6.4. Let $A \in \mathscr{B}(\mathscr{H})$ (respectively, $A \in \mathscr{L}^\infty$). Then A is called *invariant at infinity*, if for all $x \in X$: $(A - \alpha_x A) \in \mathscr{K}_q$ (respectively, $(A - \alpha_x A) \in \mathscr{K}_c$). The space of all such elements will be denoted by $\mathscr{E}_q$ (respectively, $\mathscr{E}_c$).

Proposition 6.5: (1) $\mathscr{E}_c$ and $\mathscr{E}_q$ are corresponding continuous subspaces. Both are nonseparable C^*-algebras.

(2) There are canonical maps π_i: $\mathscr{E}_i \to \mathscr{C}(\hat{X}^{\mathscr{E}} \backslash X)$ $(i = c, q)$ with the following properties: (a) π_i is a surjective $*$-homomorphism with kernel $\mathscr{K}_i$, (b) For $T \in \mathscr{T}$, $\operatorname{tr} T = 1$, $A \in \mathscr{E}_q$, $f \in \mathscr{E}_c$: $\pi_c(T * A) = \pi_q(A)$ and $\pi_q(T * f) = \pi_c(f)$.

(3) Spaces $\mathscr{D}_c$ and $\mathscr{D}_q$ with $\mathscr{K}_i \subset \mathscr{D}_i \subset \mathscr{E}_i$ are corresponding iff $\pi_c(\mathscr{D}_c) = \pi_q(\mathscr{D}_q)$. $\mathscr{D}_i$ is a C^*-algebra iff $\pi_i(\mathscr{D}_i)$ is a C^*-algebra, and in this case Proposition 6.5.2 is valid with $\mathscr{E}$ being replaced by $\mathscr{D}$. Moreover the decomposition 6.2 takes the form $\mathscr{D}_c' \simeq \mathscr{K}_c' \oplus \mathscr{C}(\hat{X}^{\mathscr{D}} \backslash X)'$ and $\mathscr{D}_q' \simeq \mathscr{K}_q' \oplus \mathscr{C}(\hat{X}^{\mathscr{D}} \backslash X)'$.

Proof. (1) Since for $A \in \mathscr{E}_i$: $\{\alpha_x A \mid x \in X\} \subset A + \mathscr{K}_i$, and $\mathscr{K}_i$ is separable, Lemma 4.1 implies that $\mathscr{E}_i$ is a continuous subspace. For $A, B \in \mathscr{E}_i$ we have $AB - \alpha_x(AB) = A(B - \alpha_x B) + (A - \alpha_x A)\,\alpha_x B \in \mathscr{K}_i$ since $\mathscr{K}_q$ is a two sided ideal in $\mathscr{B}(\mathscr{H})$, and $\mathscr{K}_c$ is an ideal in $\mathscr{C}_c$. Hence $\mathscr{E}_i$ is a C^*-algebra. $\mathscr{E}_c$ and $\mathscr{E}_q$ are corresponding since $\mathscr{K}_c$ and $\mathscr{K}_q$ are corresponding, thus $T \in \mathscr{T}$ and $A \in \mathscr{E}_c \cup \mathscr{E}_q$ imply $T * A - \alpha_x(T * A) = T * (A - \alpha_x A) \in \mathscr{K}_c \cup \mathscr{K}_q$. We shall show that $\mathscr{E}_i$ is not separable only in the case $\dim X = 2$. Consider the function $f(p, q) := \tanh(q^3(1 + p^2 + q^2)^{-1})$. Then $f \in \mathscr{E}_c$ and for $(p, q) \neq 0$: $\lim_{\lambda \to \infty} f(\lambda p, \lambda q) = \operatorname{sign}(q)$. Thus if $R \subset G$ denotes the group of rotations in phase space, $g \in R \to \beta_g f$ is not norm continuous and hence $\{\beta_g f \mid g \in R\} \subset \mathscr{E}_c$ is not separable by Lemma 4.1.

(2) $\mathscr{E}_c$ is isomorphic to $\mathscr{C}(\hat{X}^{\mathscr{E}})$ and we shall define π_c: $\mathscr{E}_c \to \mathscr{C}(\hat{X}^{\mathscr{E}} \setminus X)$ to be the restriction map. Then Proposition 6.5.2(a) for $i = c$ is trivial. Moreover, the evaluation functionals $\langle \phi_{\hat{x}}, f \rangle := (\pi_0 f)(\hat{x})$ for $\hat{x} \in \hat{X}^{\mathscr{E}} \setminus X$ are invariant states on $\mathscr{E}_c$. We now define $\pi_q(A) := \pi_c(T * A)$ for some $T \in \mathscr{T}$ with $\operatorname{tr} T = 1$. Then Proposition 6.3 shows that this definition is independent of the choice of T and that $\pi_c(T * f) = \pi_q(f)$. If $\pi_q(A) = 0$ then for any regular $T \in \mathscr{T}$ $\pi_c(T * A) = 0$, hence $T * A \in \mathscr{K}_c$ and $A \in \mathscr{K}_q$ by Theorem 5.6.3. It remains to be shown that π_q is a homomorphism. For this we shall use the fact that $A \to T * A$ for $T \in \mathscr{T}$, $T \geqslant 0$, $\operatorname{tr} T = 1$ is a completely positive map on $\mathscr{L}^\infty$ and on $\mathscr{B}(\mathscr{H})$ so that Kadison's inequality $T * |A|^2 \geqslant |T * A|^2$ is valid.[12] Let $f \in \mathscr{E}_c$ and $T, S \in \mathscr{T}$ be positive and normalized. Then $\pi_q(|T * f|^2) \leqslant \pi_q(T * |f|^2) = \pi_c(|f|^2) = |\pi_c f|^2 = |\pi_q(T * f)|^2$ and on the other hand, $\pi_q(|T * f|^2) = \pi_c(S * |T * f|^2) \geqslant \pi_c(|S * T * f|^2) = |\pi_c(S * T * f)|^2 = |\pi_q(T * f)|^2$. Since $T * \mathscr{E}_c$ is dense in $\mathscr{E}_q$ we conclude that $\pi_q(|A|^2) = |\pi_q(A)|^2$ and by polarization $\pi_q(AB) = \pi_q(A)\,\pi_q(B)$.

(3) is trivial. ∎

For a typical application of this result consider the ball compactification of X discussed at the end of Section 3. Take $\mathscr{D}_c$ to be the set of functions with continuous extensions to this compactification and $\mathscr{D}_q$ its unique corresponding space. Combining our results to this point, we conclude that $\mathscr{D}_q \subset \mathscr{B}(\mathscr{H})$ is a separable C^*-algebra on which the whole affine symplectic group acts as a strongly continuous symmetry group.

All states on $\mathscr{D}_q$ are of the form $\phi = W \oplus \mu$ where $W \in \mathscr{T}$, and $W \geqslant 0$, μ is a positive measure on the sphere $\hat{X}^{\mathscr{D}_c} \setminus X = S^{2N-1}$ at infinity and $\operatorname{tr} W + \int \mu = 1$. The above proposition defines an exact sequence $0 \to \mathscr{K}_q \to \mathscr{D}_q \xrightarrow{\pi_q} \mathscr{C}(S^{2N-1}) \to 0$ of C^*-algebras, so that $\mathscr{D}_q$ is an *extension* of $\mathscr{K}_q$ by $\mathscr{C}(S^{2N-1})$. There is an extensive classification theory of such extensions[14], which in the case $N = 1$ ($\dim X = 2$) suggests the following independent characterization of the algebra $\mathscr{D}_q$: Let $\{|n\rangle\}_{n \geqslant 0}$ be the eigenbasis of the oscillator Hamiltonian $\frac{1}{2}(P^2 + Q^2)$ and consider the partial isometry V on $\mathscr{H}$ given by $V|n\rangle = |n+1\rangle$. Then for $T_0 = |0\rangle\langle 0|$, $T_0 * V$ can be calculated explicitly and, in polar coordinates (r, φ) for X, is of the form $(T_0 * V)(r, \varphi) = f(r)e^{i\varphi}$ with $\lim_{r\to\infty} f(r) = 1$. Consequently $T_0 * V \in \mathscr{D}_c$ and $V \in \mathscr{D}_q$ with $\pi_q(V)(\varphi) = e^{i\varphi}$. Evidently, $\mathscr{D}_q$ is generated by $\mathscr{K}_q$ and V. (Since the index of V is equal to -1, this extension of $\mathscr{K}_q$ is nontrivial.) However, since $\mathscr{K}_q$ is generated by the elements $V^n(V^*V - VV^*)V^{*m}$ $(n, m \geqslant 0)$, $\mathscr{D}_q$ is a already generated by the single element V satisfying $V^*V = 1 \neq VV^*$.

There is not enough space here to apply the general program outlined above to more concrete quantum systems. For systems composed of many particles it is natural to use compactifications of the form $\hat{X} = \prod_\nu \hat{X}_\nu$, with $\hat{X}_\nu$ being a suitable compactification of the phase space of the νth particle, and the associated spaces $\mathscr{D}_c$ and $\mathscr{D}_q$. The proper (anti-)symmetrizations introduce nontrivial problems here, since there will no longer be a natural action of the entire group $X = \prod X_\nu$. Further problems with much of the flavor of geometric scattering theory arise from the introduction of nontrivial interactions and the corresponding continuity requirements.

I shall conclude by giving an alternate motivation for the ball (or rather the circle) compactification for one of the simplest quantum systems: the free particle in one-dimensional space. We shall adopt the view that the only possible observations of our system are measurements of position, performed at different times. Thus momenta are measured only indirectly by means of flight time detectors. Classical position measurements at time zero are described by functions of the form $f(p, q) = \tilde{f}(q)$, where $\tilde{f}$ is a continuous function on $\mathbb{R}$ such that $\tilde{f}(+\infty) := \lim_{q\to+\infty} \tilde{f}(q)$ and $\tilde{f}(-\infty) := \lim_{q\to-\infty} \tilde{f}(q)$ exist. The corresponding space of quantum mechanical position measurements consists of all operators $\tilde{f}(Q)$, where Q denotes the position operator.[12] These subspaces, say $\mathscr{D}_{1c}$ and $\mathscr{D}_{1q}$, are isomorphic, although the isomorphism is not given by any of the maps $A \to T * A$ $(T \in \mathscr{T})$. The evaluation functions $\tilde{f} \to \tilde{f}(q_0)$ can be interpreted as states with sharp position and can also be extended by the Hahn–Banach theorem to any space $\mathscr{D}_q \supset \mathscr{D}_{1q}$. Clearly such states cannot be given by trace class operators, so that in the sense of Proposition 6.2 they must be "states at infinity", in this case states of infinite momentum in accordance with the uncertainty relation.

Position measurements at other times are described by the set $\mathscr{D}_{2c}$ of functions $\beta_t f$ with $f \in \mathscr{D}_{1c}$, i.e., by functions of the form $f(p, q) = \tilde{f}(pt + q)$. A subbase for the uniformity $\sigma(X, \mathscr{D}_{2c})$ is given by the entourages $V_t = \{((p, q), (p', q')) \mid (pt + q, p't + q') \in V\}$ with $t \in \mathbb{R}$ and V being an entourage of the two-point compactification of $\mathbb{R}$. We shall now calculate explicitly the set of points in the induced compactification $\hat{X}^{\mathscr{D}_{2c}}$ of X: Let $\{(p_\nu, q_\nu)\}$ be a Cauchy net for $\sigma(X, \mathscr{D}_{2c})$. Then for any $t \in \mathbb{R}$, $\{p_\nu t + q_\nu\}$

must be Cauchy in $\mathbb{R} \cup \{+\infty, -\infty\}$ and we may define $\Theta_{\pm} := \{t \in \mathbb{R} \mid p_\nu t + q_\nu \to \pm\infty\}$ and $\Theta_0 := \{t \in \mathbb{R} \mid p_\nu t + q_\nu \to q \in \mathbb{R}\}$. It is easy to see that if Θ_0 contains at least two different times, $\Theta_0 = \mathbb{R}$ and the nets $\{p_\nu\}$ and $\{q_\nu\}$ converge separately. The resulting limit points in $\hat{X}^{\mathscr{D}_{2c}}$ will be denoted by $(p, q) := (\lim p_\nu, \lim q_\nu) \in X$. The points for which $\Theta_+ = \mathbb{R}$ or $\Theta_- = \mathbb{R}$ will be denoted by $(0, +\infty)$ and $(0, -\infty)$, and correspond to systems located at $\pm\infty$ and staying there forever. In all other cases Θ_+ and Θ_- are nonempty convex sets, and since $\Theta_0 = \mathbb{R} \backslash (\Theta_+ \cup \Theta_-)$ consists of at most one point, define a Dedekind section of a unique $\tau \in \mathbb{R}$. The resulting limit points in $\hat{X}^{\mathscr{D}_{2c}}$ describe systems of infinite momentum ($\pm\infty$ depending on whether Θ_+ is to the right or left of τ), which traverse configuration space in no time at all, and at the very moment τ of passage hit some point $q := \lim_\nu (p_\nu \tau + q_\nu) \in \mathbb{R} \cup \{+\infty, -\infty\}$. These points will be denoted by triples $(\tau, \pm\infty, q)$.

The space $\hat{X}^{\mathscr{D}_{2c}}$ is not metrizable and the action of the time translations on $\hat{X}^{\mathscr{D}_{2c}}$ is not continuous. Hence β_t is not strongly continuous on the algebra generated by $\mathscr{D}_{2c}$. This discontinuity is easily traced back to the idealization that measurements of position can be carried out at one precise instant of time: the function $t \to \beta_t f$ is not norm continuous for any (nonconstant) function f depending on q alone. Thus it is natural to consider the space $\mathscr{D}_{3c}$ of those functions in the C^*-algebra generated by $\mathscr{D}_{2c}$ for which $t \to \beta_t f$ is norm continuous. Then all nets $\{(p_\nu, q_\nu)\}$ convergent in $\hat{X}^{\mathscr{D}_{2c}}$ will remain convergent in $\hat{X}^{\mathscr{D}_{3c}}$, but in addition the nets $\{(p_\nu, q_\nu + t_\nu p_\nu)\}$ with $t_\nu \to 0$ will also be convergent. Consider the sequence $p_\nu = \nu \cdot p_1$; $q_\nu = \nu q_1 + q_0$ $(\nu \in \mathbb{N})$ with $p_1 \neq 0$. Then $\lim_\nu (p_\nu, q_\nu) = (-q_1/p_1, \ \infty \cdot \text{sign } p_1, q_0)$ and $\lim(p_\nu, q_\nu \pm \nu^{-1/2} p_\nu) = (-q_1/p_1, \ \infty \cdot \text{sign } p_1, \ \pm\infty \cdot \text{sign } p_1)$. In $\hat{X}^{\mathscr{D}_{3c}}$, these limits must be equal so that the points $(\tau, \pm\infty, q)$ for different q will coincide. It is easy to see that $\hat{X}^{\mathscr{D}_{3c}}$ is indeed precisely the circle compactification of X, where the asymtotic ratios $-q/p$ for points on the circle at infinity can now be interpreted as the passage times of systems with infinite momentum. This interpretation can be taken over immediately to the states on $\mathscr{D}_{3q}$, the algebra corresponding to $\mathscr{D}_{3c}$. Of course it would have been possible to motivate the choice of $\mathscr{D}_{3q}$ as a natural algebra of quantum observables by an analogous construction, carried out entirely in the quantum mechanical setting. But this would be difficult without invoking the above correspondence results and the derivation would certainly lose much of the descriptiveness of phase space geometry.

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