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Realism, Operationalism, and Quantum Mechanics

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A comprehensive formal system is developed that amalgamates the operational and the realistic approaches to quantum mechanics. In this formalism, for example, a sharp distinction is made between events, operational propositions, and the properties of physical systems.

1. INTRODUCTION

The purpose of this paper is to introduce a comprehensive mathematical formalism that has been developed as a consequence of the authors' recent collaboration at Amherst and Geneva. This formalism consolidates the operational approach to quantum mechanics, developed by C. Randall and D. Foulis,[(6,7,19)] and the realistic approach developed by C. Piron.[(17,18)] In the present paper, we concentrate on the basic mathematical structure of our formalism; a more expository article is forthcoming.

It is our contention that the realistic view implicit in classical physics need not be abandoned to accomodate the contemporary conceptions of quantum physics. All that must be abandoned is the presumption that each set of experiments possesses a common refinement (that is, the experiments are compatible). As we shall argue, this in no way excludes the notion of physical systems existing exterior to an observer, nor does it imply that the properties of such systems depend on the knowledge of the observer.

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2. QUASIMANUALS

The familiar notion that an experiment, measurement, or operation can be represented by a corresponding *sample space* E of mutually exclusive and exhaustive possible *outcomes* is routinely used in contemporary lectures on probability and statistics. In what follows, we shall also adopt this representation. Consequently, a set of (possibly incompatible) experiments can be represented by a set $\mathcal{A}$ of sample spaces,

$$\mathcal{A} = \{E, F, G, \ldots\}$$

Since distinct experiments can share common outcomes, we do allow the sets E, F, G,... to overlap. This motivates the following definition.

Definition 1. A *quasimanual* is a nonempty set of nonempty sets. Henceforth, the symbol $\mathcal{A}$ will denote a quasimanual.

Definition 2.

(i) A set $E \in \mathcal{A}$ is called an *$\mathcal{A}$-operation.*

(ii) $X = X(\mathcal{A}) = \bigcup \{E \mid E \in \mathcal{A}\}$ is called the set of *$\mathcal{A}$-outcomes.*

(iii) A subset A of X is called an *$\mathcal{A}$-event* if there exists an $E \in \mathcal{A}$ with $A \subseteq E$.

(iv) $\mathscr{E} = \mathscr{E}(\mathcal{A})$ denotes the set of all *$\mathcal{A}$-events.*

As usual, if operation $E \in \mathcal{A}$ is executed and an outcome $x \in E$ is secured as a consequence, then an event $A \subseteq E$ is said to have *occurred* (*nonoccurred*) if $x \in A$ $(x \notin A)$. Accordingly, if $A \in \mathscr{E}$, then an operation $E \in \mathcal{A}$ for which $A \subseteq E$ is called a *test operation* for A. Notice that an event A can occur or nonoccur only as a consequence of an execution of a test operation for A. Furthermore, it is essential to understand that an occurrence or nonoccurrence of A is to be regarded as being independent of the test operation for A that was actually executed.

Definition 3.

(i) A family of events (A_i) is said to be *compatible* if there exists an operation $E \in \mathcal{A}$ such that $\bigcup_i A_i \subseteq E$.

(ii) A family of events (A_i) is said to be *jointly orthogonal* if it is compatible and the sets (A_i) are pairwise disjoined.

(iii) If the events A and B form a jointly orthogonal family, we say that A and B are *orthogonal* and write $A \perp B$. If x and y are $\mathcal{A}$-outcomes and

$\{x\} \perp \{y\}$, we simply write $x \perp y$. If M is a set of outcomes (not necessarily an event) we define

$$M^{\perp} = \{x \in X \mid x \perp y \text{ for every } y \in M\}$$

(iv) If A and C are orthogonal events for which $A \cup C = E \in \mathcal{O}\!\mathcal{l}$, then A and C are said to be *operational complements*, and we write $A \;\underline{\text{oc}}\; C$.

(v) If A and B are events for which there exists an event C with $A \;\underline{\text{oc}}\; C$ and $C \;\underline{\text{oc}}\; B$, we say that A and B are *operationally perspective* and we write $A \;\underline{\text{op}}\; B$. Here we refer to the event C as the *axis* of the perspectivity.

Most quasimanuals that arise in practice satisfy certain reasonable normative conditions, among which are the following:

Definition 4.

(i) The quasimanual $\mathcal{O}\!\mathcal{l}$ is said to be *irredundant* if $E, F \in \mathcal{O}\!\mathcal{l}$, and $E \subseteq F$ implies $E = F$.

(ii) The quasimanual $\mathcal{O}\!\mathcal{l}$ is said to be *coherent* if $A, B \in \mathscr{E}$, and $A \subseteq B^{\perp}$ implies $A \perp B$.

(iii) The quasimanual $\mathcal{O}\!\mathcal{l}$ is said to be *orthocoherent* if every triple of pairwise orthogonal events are jointly orthogonal.

(iv) The quasimanual $\mathscr{A}$ is said to be *regular* if $A, B \in \mathscr{E}$, and $A \;\underline{\text{op}}\; B$ implies $A^{\perp} = B^{\perp}$.

(v) The quasimanual $\mathcal{O}\!\mathcal{l}$ is said to be a *manual* if $A, B, C \in \mathscr{E}$, $A \;\underline{\text{op}}\; B$, and $A \perp C$ implies $B \perp C$.

It is not difficult to show that every manual is irredundant, as is every regular quasimanual, and that every coherent manual is orthocoherent. Moreover, a coherent quasimanual is regular if and only if it is a manual.

If $\mathcal{O}\!\mathcal{l}$ is a manual, then $\underline{\text{op}}$ is an equivalence relation on $\mathscr{E}$. Thus, it is natural to introduce the following definition:

Definition 5. Let $\mathcal{O}\!\mathcal{l}$ be a manual.

(i) For every $A \in \mathscr{E}$, we define $p(A) = \{B \in \mathscr{E} \mid A \;\underline{\text{op}}\; B\}$.

(ii) $\pi(\mathcal{O}\!\mathcal{l}) = \{p(A) \mid A \in \mathscr{E}\}$.

It can be shown that the relations introduced in the following definition are well-defined.

Definition 6. Let $\mathcal{O}\!\mathcal{l}$ be a manual and let A, B, and $C \in \mathscr{E}$.

(i) $p(A) \perp p(B)$ if and only if $A \perp B$.

(ii) If $p(A) \perp p(B)$, we define $p(A) \oplus P(B) = p(A \cup B)$.

(iii) If $A \subseteq E \in \mathfrak{A}$, we define $p(A)' = p(E - A)$.

(iv) $p(A) \leqslant p(B)$ if and only if there exists $p(C)$ with $p(A) \perp p(C)$ and $p(A) \oplus p(C) = p(B)$.

If $\mathfrak{A}$ is a manual and $A \in \mathscr{E}$, we call $p(A)$ an *operational proposition* and we say that $p(A)$ is *confirmed* if an event $B \in p(A)$ occurs. If an event $C \in p(A)'$ occurs, we say that $p(A)$ is *refuted*. We refer to $\pi(\mathfrak{A})$, equipped with the relations of Definition 6, as the *operational logic* of the manual $\mathfrak{A}$. Note that $(\pi(\mathfrak{A}), \leqslant)$ is orthocomplemented by $p(A) \mapsto p(A)'$, but it need not be a lattice. Indeed, $p(A) \oplus p(B)$ need not be the least upper bound of $p(A)$ and $p(B)$, although it is a minimal upper bound. In fact, the manual $\mathfrak{A}$ is orthocoherent if and only if $p(A) \oplus p(B)$ is always the least upper bound of $p(A)$ and $p(B)$, in which case, $\pi(\mathfrak{A})$ is an orthomodular poset (OMP).

Definition 7. Let $\mathfrak{A}$ be a manual and let $E, F \in \mathfrak{A}$. We say that the operation F *refines* the operation E if for every $A \subseteq E$, there exists $B \subseteq F$ such that $p(A) = p(B)$. If $\mathfrak{A}$ is an orthocoherent manual in which every pair of operations has a common refinement, then $\pi(\mathfrak{A})$ is a Boolean algebra. When we abandon the classical supposition that operations are always compatible (possess a common refinement), then, in general, we lose the classical (Boolean) characteristics of the operational logic $\pi(\mathfrak{A})$.

In this paper, we are primarily concerned with the realistic approach to quantum mechanics. In other places (for example, Refs. 8, 9, 20, and 21) we have considered a statistical approach. In the latter, the basic tool is the notion of a "global stochastic model" or weight function.

Definition 8. Let $\mathfrak{A}$ be a quasimanual with outcome set X.

(i) An $\mathfrak{A}$*-weight* is a function $\omega: X \to [0, 1]$ such that, for every $E \in \mathfrak{A}$, $\sum_{x \in E} \omega(x) = 1$, (where the summation is understood in the unordered sense).

(ii) If ω is an $\mathfrak{A}$-weight, we define the *support* of ω, in symbols, $\text{supp}(\omega)$, by $\text{supp}(\omega) = \{x \in X \mid \omega(x) \neq 0\}$.

(iii) If ω is an $\mathfrak{A}$-weight and A is an $\mathfrak{A}$-event, we define $\omega(A) = \sum_{x \in A} \omega(x)$.

If $A \in \mathscr{E}$, then $\omega(A)$ is interpreted as the long-run relative frequency with which A occurs when tested (according to the model ω). If A and B are operationally perspective events, it is easy to see that $\omega(A) = \omega(B)$; hence, if $\mathfrak{A}$ is a manual, we can, and do, define $\omega(p(A)) = \omega(A)$. Naturally, we interpret $\omega(p(A))$ as the long-run relative frequency with which $p(A)$ is confirmed when tested.

At the end of this paper is a bibliography in which the details of these matters can be found.

3. EXAMPLES

In this section we shall introduce a number of pertinent examples of manuals.

Example 9. $\mathfrak{A} = \{E\}$, where E is a nonempty set. We refer to such an $\mathfrak{A}$ as a *classical* manual. Here the events are arbitrary subsets of E. The operational logic $\pi(\mathfrak{A})$ is isomorphic to the Boolean algebra of all subsets of E.

Example 10. $\mathfrak{A} = \{E_i \mid i \in I\}$, where $(E_i)_{i \in I}$ is a family of pairwise disjoint nonempty sets. We refer to such an $\mathfrak{A}$ as a *semiclassical* manual. Here the operational logic is a complete atomic orthomodular lattice, but is Boolean only if $\mathfrak{A}$ is classical.

Example 11. Let $(S, \mathscr{M})$ be a Borel space; that is, a nonempty set S and a σ-field $\mathscr{M}$ of subsets of S. Let $\mathscr{B}(S, \mathscr{M})$ denote the set of all countable partitions of S into nonempty sets in $\mathscr{M}$. We refer to $\mathscr{B}(S, \mathscr{M})$ as a *Borel manual.* Here the events are countable families of pairwise disjoint nonempty sets in $\mathscr{M}$. The operational logic $\pi(\mathscr{B}(S, \mathscr{M}))$ is isomorphic to $\mathscr{M}$, and consequently is a σ-complete Boolean algebra. There is a natural 1–1 correspondence between $\mathscr{B}(S, \mathscr{M})$-weights and probability measures on the Borel space $(S, \mathscr{M})$.

Example 12. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{F}(\mathscr{H})$ denote the collection of all maximal orthonormal subsets of $\mathscr{H}$. We refer to $\mathscr{F}(\mathscr{H})$ as a *frame manual.* Here the events are the orthonormal subsets of $\mathscr{H}$ and the operational logic $\pi(\mathscr{F}(\mathscr{H}))$ is isomorphic to the complete atomic orthomodular lattice of all projection operators on $\mathscr{H}$. If $\mathscr{H}$ has three-dimensions or more, there is a natural 1–1 correspondence between the $\mathscr{F}(\mathscr{H})$-weights and the von Neumann density operators (statistical operators) on $\mathscr{H}$.

Example 13. Let $\mathbb{A}$ be a von Neumann algebra and let $\mathscr{P}(\mathbb{A})$ denote the collection of all maximal sets of pairwise orthogonal nonzero projections in $\mathbb{A}$. We refer to $\mathscr{P}(\mathbb{A})$ as the *projection manual* of $\mathbb{A}$. The operational logic $\pi(\mathscr{P}(\mathbb{A}))$ is isomorphic to the complete orthomodular lattice of all projections in $\mathbb{A}$ (the projection geometry of $\mathbb{A}$). In the favorable cases covered by Lodkin's theorem,[13] the $\mathscr{P}(\mathbb{A})$-weights are again in a natural 1–1 correspondence with statistical operators.

Our remaining examples, which are all finite, provide simple counterexamples to a variety of conjectures. They are so-called Greechie manuals.[11]

Definition 14. A quasimanual $\mathfrak{A}$ is called a *Greechie* manual if it satisfies the following conditions:

(i) If E and F are distinct $\mathfrak{A}$-operations, then $E \cap F$ consists of at most one outcome.

(ii) If E, F, and G are three distinct $\mathfrak{A}$-operations and F consists of exactly two outcomes, then $F \nsubseteq E \cup G$.

(iii) $\mathfrak{A}$ is irredundant.

It can be shown that any Greechie manual is in fact a manual.

Example 15. The *Wright Triangle*:

$$\mathfrak{A} = \{\{a, z, b\}, \{b, x, c\}, \{a, y, c\}\}$$

This is an example of a manual that is regular, but not orthocoherent. The operational logic $\pi(\mathfrak{A})$ is not an orthomodular poset. The set of all $\mathfrak{A}$-weights forms a three-dimensional polytope.

Example 16. The *Window Manual*:

$$\mathfrak{A} = \{\{a, b, c, d\}, \{e, f, g, h\}, \{j, k, l, m\}, \{a, e, j\}, \{b, f, k\}, \{c, g, l\}, \{d, h, m\}\}$$

This is a coherent manual, and consequently, its operational logic $\pi(\mathfrak{A})$ is an orthomodular poset (which is not a lattice). Here there are no $\mathfrak{A}$-weights. Nevertheless, as we shall see, it is possible to define realistic states for this manual.

Example 17. *The Collar Manual*:

$$\mathfrak{A} = \{\{a, b, c, d\}, \{a, e, j\}, \{b, f, j\}, \{c, g, k\}, \{d, h, k\}\}$$

This is a regular manual, but it is not orthocoherent.

3. SUPPORTS

In their famous 1935 paper, Einstein, Podolsky, and Rosen[4] wrote:

"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

We shall also adopt this reasonable criterion and introduce an appropriate mathematical representation for such elements of physical reality. Thus, let $\mathcal{O}$ be a quasimanual with outcome set X. Suppose that $\mathscr{C} \subseteq \mathscr{E}(\mathcal{O})$ consists of all of the events that are certain to occur (in the sense of EPR) if tested. Then, an event is certain to nonoccur if and only if it is $\underline{oc}$ to some event in $\mathscr{C}$. Let I be the set-theoretic union of all events certain to nonoccur. Clearly I is the set of impossible outcomes and $P = X - I$ is the set of possible outcomes. We claim that P should satisfy the condition in the following definition:

Definition 18. A set P of $\mathcal{O}$-outcomes is called an *$\mathcal{O}$-support* provided that it satisfies the following *exchange condition*: if $E, F \in \mathcal{O}$ and $E \cap P \subseteq F$, then $F \cap P \subseteq E$. The collection of all $\mathcal{O}$-supports is denoted by $\mathscr{S}(\mathcal{O})$.

The argument that the set P of possible outcomes satisfies the exchange condition is as follows: Suppose $E, F \in \mathcal{O}$ and $E \cap P \subseteq F$. Let $A = E \cap P$. Since all outcomes in $E - A$ are impossible, A is certain to occur if the test operation E is executed. By hypothesis, $A \subseteq F$, and consequently, A is certain to occur if F is executed. Thus, $F - A$ consists of impossible outcomes, and it follows that $(F - A) \cap P$ is empty. Therefore, $P \cap F \subseteq E$.

The following definition formalizes the notions of certainty of occurrence and nonoccurrence utilized in the argument above.

Definition 19. Let $P \in \mathscr{S}(\mathcal{O})$ and $A \in \mathscr{E}(\mathcal{O})$.

(i) A is *P-true* if there exists $E \in \mathcal{O}$ such that $P \cap E \subseteq A \subseteq E$.

(ii) A is *P-false* if $P \cap A$ is empty.

Because of the exchange condition, if F is any test operation for an event A, then A is P-true if and only if $P \cap F \subseteq A$. Also, if C is an operational complement of A, then A is P-true if and only if C is P-false. As a consequence, if A and B are operationally perspective events, then A is P-true if and only if B is P-true, and A is P-false if and only if B is P-false. The following Lemma is obvious:

Lemma 20. $\mathscr{S}(\mathcal{O})$ is closed under arbitrary set-theoretic unions, and therefore forms a complete lattice with respect to set-theoretic inclusion. It is evident that the empty set is always an $\mathcal{O}$-support. It is equally clear that the set X of all $\mathcal{O}$-outcomes is a support if and only if the quasimanual $\mathcal{O}$ is irredundant.

As the following lemma shows, the condition of regularity provides a lavish supply of supports:

Lemma 21. The following conditions are equivalent:

(i) $\mathcal{U}$ is regular.

(ii) For every $\mathcal{U}$-outcome x, $X - x^{\perp} \in \mathcal{S}(\mathcal{U})$.

Proof. Suppose $\mathcal{U}$ is regular and that x is an $\mathcal{U}$-outcome. That $X - x^{\perp}$ satisfies the exchange condition is equivalent to $E - x^{\perp} \subseteq F$ implies $F - x^{\perp} \subseteq E$ for $E, F \in \mathcal{U}$. Thus, suppose $E - x^{\perp} \subseteq F$, and let $A = E \cap x^{\perp}$, $C = E - x^{\perp}$, $B = F - C$. Then $A \underline{\text{op}} B$ with axis C, so, since $\mathcal{U}$ is regular, $A^{\perp} = B^{\perp}$. Because $x \in A^{\perp}$, it follows that $x \in B^{\perp}$. To prove that $F - x^{\perp} \subseteq E$, select any outcome $y \in F - x^{\perp}$. Then $y \notin x^{\perp}$, so, because $x \in B^{\perp}$, we have $y \notin B$. Thus, $y \in F - B = C \subseteq E$. This completes the proof that (i) implies (ii). Conversely, suppose that condition (ii) holds and assume that A and B are operationally perspective events with axis C. Let $E = A \cup C$, $F = B \cup C$. We need to prove $A^{\perp} = B^{\perp}$. By symmetry, it will suffice to prove $A^{\perp} \subseteq B^{\perp}$. To this end, select $x \in A^{\perp}$, and suppose that $x \notin B^{\perp}$. Then, there exists an outcome $b \in B$ with $b \notin x^{\perp}$. Hence, $b \in F - x^{\perp}$. Now, since $x \in A^{\perp}$, we have $A \subseteq x^{\perp}$, and it follows that $E - x^{\perp} \subseteq C \subseteq F$. By condition (ii), we therefore have $F - x^{\perp} \subseteq E$. Since $b \in F - x^{\perp}$, it follows that $b \in E$. Thus, either $b \in A$ or else $b \in C$. Because $A \subseteq x^{\perp}$ and $b \notin x^{\perp}$, we cannot have $b \in A$; hence, we must have $b \in C$. Therefore, $b \in B \cap C$. But $B \cap C$ is empty, and this contradiction completes the proof. ∎

Corollary 22. If $\mathcal{U}$ is regular, then, for every set of outcomes $M \subseteq X$, $X - M^{\perp} \in \mathcal{S}(\mathcal{U})$.

The following lemma establishes the connection between stochastic models and supports.

Lemma 23.

(i) If ω is any $\mathcal{U}$-weight, then $\text{supp}(\omega) \in \mathcal{S}(\mathcal{U})$.

(ii) If an $\mathcal{U}$-weight ω is a proper convex combination of a family of $\mathcal{U}$-weights (ω_i), then $\text{supp}(\omega) = \bigcup_i \text{supp}(\omega_i)$.

Notice that, although the manual in Example 16 carries no statistical weights, it is regular, and thus, according to Corollary 22, it actually has many supports.

A *deterministic* or *dispersion free* $\mathcal{U}$-weight is one that takes on only the values 0 and 1. The support P of such a deterministic weight is characterized by the fact that it meets every operation in exactly one outcome. Evidently, for such a *deterministic support* P, every $\mathcal{U}$-event is either P-true or P-false.

Clearly, for a semicalssical manual, every support is a union of deterministic supports. However, for a Hilbert space $\mathscr{H}$ of dimension three or more, the frame manual $\mathscr{F}(\mathscr{H})$ admits no deterministic supports.

4. ENTITIES

We are now able to provide a mathematical representation for a physical system, object, or entity. Following Aerts,[1] we prefer to use the neutral term "entity."

Definition 24. An *entity* is a pair $(\mathfrak{A}, \Sigma)$ such that

(i) $\mathfrak{A}$ is a quasimanual with outcome set X.

(ii) Σ is a set of nonempty $\mathfrak{A}$-supports.

(iii) $\bigcup \mathfrak{A} = X = \bigcup \Sigma$.

We refer to Σ as the set of *states* of the entity. It is to be understood that, at any given moment, the entity is *in* exactly one state S in Σ. This, of course, entails that the S-true $\mathfrak{A}$-events will occur with certainty if tested and, likewise, the S-false $\mathfrak{A}$-events will surely nonoccur if tested. Unless S is deterministic, there will exist $\mathfrak{A}$-events whose occurrence, if tested, is possible, but not certain. The uncertainty associated with such events must be regarded as intrinsic or ontological—and not a consequence of a lack of knowledge. In other words, by definition, the state provides the realistic description of the entity in so far as the operations in $\mathfrak{A}$ are concerned. Naturally, the state of the entity can change, either spontaneously, under a dynamic law, or as a result of an observation.

Definition 25. Let $(\mathfrak{A}, \Sigma)$ be an entity.

(i) By a *property* of the entity $(\mathfrak{A}, \Sigma)$, we mean a set P of $\mathfrak{A}$-outcomes such that P is a set-theoretic union of states in Σ.

(ii) $\mathscr{L} = \mathscr{L}(\mathfrak{A}, \Sigma)$ is defined to be the set of all properties of the entity $(\mathfrak{A}, \Sigma)$.

Since every state is an $\mathfrak{A}$-support, it follows from Lemma 20 that every property is an $\mathfrak{A}$-support. Evidently, the set-theoretic union of properties is again a property; consequently, $\mathscr{L}$ is a complete lattice with respect to set inclusion. Thus, we refer to $\mathscr{L}$ as the *property lattice* of the entity $(\mathfrak{A}, \Sigma)$. If (P_i) is a family of elements of $\mathscr{L}$, we denote the infimum (greatest lower bound) of the family by $\bigwedge_i P_i$ and the supremum (least upper bound) of the family by $\bigvee_i P_i$. Of course, we have $\bigvee_i P_i = \bigcup_i P_i$.

A nonempty set Γ of states can be regarded as representing an assertion that the entity is in some state $S \in \Gamma$, although we may not know which one. The property $P = \bigcup \Gamma$ then consists of all the outcomes that could be possible under these circumstances. Thus, whereas the states can only reflect ontological uncertainties, the properties may also describe epistemic uncertainties. A property P is said to be *actual* if the entity is in a state S such that $S \subseteq P$; otherwise P is said to be *potential*. Notice that, if P and Q are properties, then $P \subseteq Q$ holds if and only if whenever P is actual, so is Q. As the state of the entity changes, some of the actual properties may become potential, and some of the potential properties may become actual.

Of course, every state is a property and, if the entity is in state S, then S, as a property, is actual. However, one must be careful because a state S, regarded as a property, may be actual even though the entity is not in state S, but is in some state properly contained in S. If desired, this can be ruled out by invoking the following condition:

Definition 26. The set of states Σ of an entity $(\mathcal{O}, \Sigma)$ is said to be *irredundant* if, for S, $T \in \Sigma$, and $T \subseteq S$ implies $T = S$. If Σ is irredundant, then $\mathcal{L}$ is an atomistic (fully atomic) lattice, with Σ as its set of atoms.

The following lemma is easy to verify:

Lemma 27. Let $(\mathcal{O}, \Sigma)$ be an entity.

(i) If $P \in \mathcal{L}$, then $P = \bigvee \{S \in \Sigma \mid S \subseteq P\}$.

(ii) If $S \in \Sigma$, then $S = \bigwedge \{P \in \mathcal{L} \mid S \subseteq P\}$.

Suppose (P_i) is a family of properties and $P = \bigwedge_i P_i$. Since $P \subseteq P_i$ for every i, it follows that, if P is actual, so is every P_i. Conversely, suppose every P_i is actual. Then the entity is in a state S with $S \subseteq P_i$ for every i. Because P is the greatest lower bound of all the properties P_i, it follows that $S \subseteq P$, so P is actual. In brief, P is actual if and only if every P_i is actual. In this sense, the infimum in $\mathcal{L}$ serves as a logical conjunction. In particular then, part (ii) of Lemma 27 is an assertion of the classical Aristotelian principle that the state of an entity corresponds to the collection of all its actual properties.

Again suppose (P_i) is a family of properties, but now let $Q = \bigvee_i P_i$. Evidently, Q is the smallest property that is actual when at least one of the P_i is actual. However, it can happen that Q is actual, whereas no P_i is actual. Thus, in general, the supremum in $\mathcal{L}$ cannot be interpreted as a logical disjunction. As a consequence, following Aerts,[1] we introduce the following definition:

Definition 28. A family (P_i) of properties is said to be *separated by a superselection rule* (ssr) if, whenever $S \in \Sigma$ is such that $S \subseteq \bigvee_i P_i$, it follows that $S \subseteq P_i$ for some i. In other words, (P_i) is ssr if and only if $\bigvee_i P_i$ serves as a logical disjunction of the properties in the family (P_i).

Definition 29. Let $\Gamma \subseteq \Sigma$.

(i) The *superposition closure* Γ^{sp} of Γ is $\Gamma^{\mathrm{sp}} = \{S \in \Sigma \mid S \subseteq \bigcup \Gamma\}$.

(ii) S is said to be a *superposition* of states in Γ if $S \in \Gamma^{\mathrm{sp}}$. If $S \in \Gamma^{\mathrm{sp}}$, but $S \notin \Gamma$, we call S a *proper* superposition of states in Γ.

(iii) Γ is sp-*closed* if $\Gamma = \Gamma^{\mathrm{sp}}$.

(iv) Let $\mathscr{C}(\Sigma, \mathrm{sp})$ denote the set of all sp-closed subsets of Σ.

It is easy to see that the set-theoretic intersection of sp-closed sets is again sp-closed, hence, $\mathscr{C}(\Sigma, \mathrm{sp})$ is a complete lattice under set-theoretic inclusion.

The customary quantum mechanical entity in the following example in part justifies the superposition terminology in Definition 29.

Example 30. Let $\mathscr{F}(\mathscr{H})$ be a Hilbert space frame manual as in Example 12. For each normalized vector ψ in $\mathscr{H}$, let $S_\psi = \{\phi \in \mathscr{H} \mid \|\phi\| = 1$ and $|\langle\psi \mid \phi\rangle^2 \neq 0\}$. Such an S_ψ is an $\mathscr{F}(\mathscr{H})$-support. Let $\Sigma = \{S_\psi \mid \psi \in \mathscr{H}$ and $\|\psi\| = 1\}$. Then Σ is an irredundant set of states for the quantum mechanical entity $(\mathscr{F}(\mathscr{H}), \Sigma)$, and the complete atomistic property lattice $\mathscr{L}$ is naturally isomorphic to the lattice of closed linear subspaces of $\mathscr{H}$ under the correspondence

$$P \leftrightarrow \text{closed linear span of } \{\psi \mid S_\psi \subseteq P\}$$

Therefore, if ψ, ϕ, and ξ are normalized vectors in $\mathscr{H}$, then S_ψ is a superposition of states S_ϕ and S_ξ if and only if ψ is a superposition of ϕ and ξ in the usual sense.

It is interesting to constrast the quantum mechanical entity in Example 30 with the following more classical example:

Example 31. Let $\mathbb{B}$ be a Boolean algebra (a complemented distributive lattice). Each element $b \in \mathbb{B}$ has a unique complement b'. If $a \leqslant b'$, say that a is *orthogonal* to b. Let $\mathscr{B}$ denote the collection of all maximal finite sets of pairwise orthogonal nonzero elements of $\mathscr{B}$. Then $\mathscr{B}$ is a coherent manual for which $\pi(\mathscr{B})$ is isomorphic to $\mathbb{B}$. Let Σ be the set of all deterministic supports for $\mathscr{B}$. Again, Σ forms an irredundant set of states for the entity $(\mathscr{B}, \Sigma)$. In this case, the complete atomistic property lattice $\mathscr{L}$ coincides with $\mathscr{S}(\mathscr{B})$, and in fact, is a dual Brouwerian lattice.[3] Of course, Σ can be identified with the usual Stone space dual to $\mathbb{B}$[12]; in fact, the closed sets of the Stone topology are precisely the sp-closed sets of Σ. Thus, no finite set of states admits a proper superposition, although—even in this classical situation—there do exist proper superpositions of infinite sets of states.

The simple connection between properties and sp-closed sets of states is effected by the so-called Cartan map.

Definition 32. For $P \in \mathscr{L}$, define $\Sigma_P = \{S \in \Sigma \mid S \subseteq P\}$. We refer to $P \mapsto \Sigma_P$ as the *Cartan map.*

Lemma 33. The Cartan map $P \mapsto \Sigma_P$ establishes an isomorphism of the property lattice $\mathscr{L}$ onto the lattice $\mathscr{C}(\Sigma, \text{sp})$ of sp-closed sets of states.

Corollary 34.

(i) Let (P_i) be any family of properties and let (Γ_i) be the corresponding family of sp-closed sets of states, so that $\Gamma_i = \Sigma_{P_i}$. Then (P_i) is ssr if and only if $\bigcup_i \Gamma_i$ is sp-closed.

(ii) If every pair of properties is ssr, then $\mathscr{L}$ is a dual Brouwerian lattice.

(iii) If Σ is irredundant and every family of properties is ssr, then $\mathscr{L}$ is isomorphic to the complete atomistic Boolean algebra of all subsets of Σ.

In general, the lattice $\mathscr{L}$ of properties of an entity enjoys no special features—other than being complete and containing at least two elements. In fact, we have the following:

Lemma 35. Given any complete lattice L with at least two elements, there exists an entity $(\mathscr{O}, \Sigma)$ whose property lattice $\mathscr{L}$ is isomorphic to L.

Proof. Let $\mathscr{O} = \{L\}$. For $x \in L$, $x \neq 0$, define $S_x = \{y \in L \mid x \not\leq y\} \cup \{1\}$. Let $\Sigma = \{S_x \mid x \in L \text{ and } x \neq 0\}$. Then the property lattice $\mathscr{L}$ of the entity $(\mathscr{O}, \Sigma)$ is given by $\mathscr{L} = \Sigma \cup \{\phi\}$. The desired isomorphism $L \to \mathscr{L}$ is $x \mapsto \phi$ if $x = 0$, $x \mapsto S_x$ if $x \neq 0$. ∎

A natural connection between events and properties is established by the following concepts:

Definition 36. Let $(\mathscr{O}, \Sigma)$ be an entity with property lattice $\mathscr{L}$, and let $A \in \mathscr{E}(\mathscr{O})$.

(i) $\Sigma_A = \{S \in \Sigma \mid A \text{ is } S\text{-true}\}$.

(ii) $[A] = \bigcup \Sigma_A \in \mathscr{L}$.

(iii) A property of the form $[A]$ is called a *principal property.*

(iv) $(\mathscr{O}, \Sigma)$ is a *principal entity* if every property $P \in \mathscr{L}$ is principal.

The quantum mechanical entity in Example 30 is always principal, whereas the entity in Example 31 is principle if and only if the Boolean algebra $\mathbb{B}$ is finite.

It is easy to see that $\Sigma_A = \Sigma_{[A]}$, since $A \subseteq E \in \mathfrak{A}$ implies $[A] \cap E \subseteq A$. One might expect the latter inclusion to be an equality; however, this is so only if the following condition holds:

Definition 37. The entity $(\mathfrak{A}, \Sigma)$ is *unital* if, for every $\mathfrak{A}$-outcome x, $[x] \neq \phi$. Notice that $(\mathscr{A}, \Sigma)$ is unital if and only if $A \subseteq [A]$ holds for every $A \in \mathscr{E}(\mathfrak{A})$.

Whereas the states are join dense in $\mathscr{L}$ (every property is a least upper bound of states), the following lemma shows that the principal properties are meet dense in $\mathscr{L}$ (every property is a greatest lower bound of principal properties).

Lemma 38. If $P \in \mathscr{L}$, then $P = \bigwedge \{[E \cap P] \mid E \in \mathfrak{A}\}$.

Proof. We begin by showing that P is a lower bound for the set of all properties of the form $[E \cap P]$, $E \in \mathfrak{A}$. Thus, if $S \in \Sigma$ with $S \subseteq P$, then $S \cap E \subseteq E \cap P$, and so $S \subseteq [E \cap P]$. Since S was arbitrary, $P \subseteq [E \cap P]$. Conversely, to show that P is the greatest lower bound for all properties of the form $[E \cap P]$, $E \in \mathfrak{A}$, suppose that Q is a lower bound in $\mathscr{L}$ for all such properties, so that $Q \subseteq [E \cap P]$ holds for all $E \in \mathfrak{A}$. We must prove that $Q \subseteq P$. To this end, let $S \in \Sigma$ with $S \subseteq Q$. Then $S \subseteq [E \cap P]$ holds for all $E \in \mathfrak{A}$. It follows that $E \cap S \subseteq E \cap P$; hence that

$$S = \bigcup \{E \cap S \mid E \in \mathfrak{A}\} \subseteq \bigcup \{E \cap P \mid E \in \mathfrak{A}\} = P$$

Since S was arbitrary, $Q \subseteq P$, and the proof is complete. ∎

Corollary 39. If $A \in \mathscr{E}$, then $[A] = \bigcap \{[[A] \cap E] \mid E \in \mathfrak{A}\}$.

Proof. Since the greatest lower bound of a family of properties is always contained in its set-theoretic intersection, $[A] \subseteq \bigcap \{[[A] \cap E] \mid E \in \mathfrak{A}\}$ follows from Lemma 38. The opposite inclusion follows from the fact that, if we select $E \in \mathfrak{A}$ with $A \subseteq E$, then $[A] \cap E \subseteq A$, so that $[[A] \cap E] \subseteq [A]$. ∎

Finally, suppose $\mathfrak{A}$ is a manual. Since $A \underline{\text{op}} B$ implies that A is S-true if and only if B is S-true, $[\cdot]$ can be lifted to $\pi(\mathfrak{A})$ by defining $[p(A)] = [A]$. As the following obvious lemma asserts, this *canonical map* $[\cdot]: \pi(\mathfrak{A}) \to \mathscr{L}$ is order preserving.

Lemma 40. If $\mathfrak{A}$ is a manual and $p(A) \leqslant p(B)$ in $\pi(\mathfrak{A})$, then $[p(A)] \subseteq [p(B)]$.

5. ORTHOCOMPLEMENTATION

Many of the "property lattices" that have been proposed are not orthocomplemented—for example, the complete lattice of faces of a statistical figure (convex set) introduced by Mielnik.[14–16, 5] In fact, we have seen that $\mathscr{L}$ need not be orthocomplemented for the essentially classical entity in Example 31. Nevertheless, for an orthodox quantum mechanical entity (Example 30), $\mathscr{L}$ is not only orthocomplemented, it is even orthomodular. This suggests that we seek conditions on a more general entity $(\mathcal{O}\!\ell, \Sigma)$ that will provide $\mathscr{L}$ with a natural orthocomplementation.

It is tempting to begin by considering an entity $(\mathcal{O}\!\ell, \Sigma)$ for which $\mathcal{O}\!\ell$ is a manual; in this case, $\pi(\mathcal{O}\!\ell)$ is orthocomplemented and the canonical map $[\cdot]$ might be used to transfer this orthocomplementation to $\mathscr{L}$. In other words, we are looking for an orthocomplementation $': \mathscr{L} \to \mathscr{L}$ such that $[p(A)'] = [p(A)]'$. Thus, if $A \subseteq E \in \mathcal{O}\!\ell$, we must require that $[A]' = [E - A]$. Since an orthocomplementation must satisfy the De Morgan laws, there is now only one possible extension of $'$ from the meet dense principal properties to all of $\mathscr{L}$.

Definition 41. If $P \in \mathscr{L}$, let $P' = \bigcup \{[E - P] \mid E \in \mathcal{O}\!\ell\}$. Notice that this definition does not require $\mathcal{O}\!\ell$ to be a manual; hence, for the remainder of this section, we assume only that $(\mathcal{O}\!\ell, \Sigma)$ is an entity.

The following easily verified lemmas are encouraging:

Lemma 42. If $P \subseteq Q$ in $\mathscr{L}$, then $Q' \subseteq P'$.

Proof. If $P \subseteq Q$, then $E - Q \subseteq E - P$ holds for every $E \in \mathcal{O}\!\ell$; hence, $[E - Q] \subseteq [E - P]$. Therefore, $Q' \subseteq P'$. ∎

Lemma 43. If $A \subseteq E \in \mathcal{O}\!\ell$, then $[E - A] \subseteq [A]'$.

Proof. Let $S \in \Sigma$ with $S \subseteq [E - A]$. Then $S \cap A = \phi$ and, since $[A] \cap E \subseteq A$, we have $S \cap [A] \cap E = \phi$. Thus, $S \subseteq [E - [A]] \subseteq [A]'$. Because S was arbitrary, it follows that $[E - A] \subseteq [A]'$. ∎

In general, there will be events $A \subseteq E \in \mathcal{O}\!\ell$ for which $[A]' \not\subseteq [E - A]$. In fact, this is even possible when $': \mathscr{L} \to \mathscr{L}$ is an orthocomplementation. Thus, if we wish $'$ to behave properly on the principal supports, we must impose the following condition:

Definition 44. An entity $(\mathcal{O}\!\ell, \Sigma)$ is said to be *strongly unital* (SU) if, for every $A \subseteq E \in \mathcal{O}\!\ell$, $[A]' \subseteq [E - A]$. As we shall see, SU does indeed imply unital.

Theorem 45. Let $(\mathscr{A}, \Sigma)$ be an entity. Then, the following conditions are equivalent:

(i) $(\mathscr{A}, \Sigma)$ is SU.

(ii) If $A \subseteq E \in \mathscr{A}$, $B \subseteq F \in \mathscr{A}$, and $[A] \subseteq [B]$, then $[F - B] \subseteq [E - A]$.

(iii) if $S \in \Sigma$, $F \in \mathscr{A}$, $A \in \mathscr{E}$, and $S \cap [A] \cap F = \phi$, then $S \cap A = \phi$.

Proof.

(i) implies (ii): Assume (i) and the hypothesis of (ii). Since $[A] \subseteq [B]$, we have $[B]' \subseteq [A]'$ by Lemma 42. By (i) and Lemma 43, $[F - B] = [B]' \subseteq [A]' = [E - A]$.

(ii) implies (iii): Assume (ii) and the hypothesis of (iii). By Lemma 38, $[A] \subseteq [[A] \cap F]$. Let $A \subseteq E \in \mathscr{A}$. Then, by (ii), $[F - [A]] \subseteq [E - A]$. Since $S \cap [A] \cap F = \phi$, we have $S \subseteq [F - [A]]$, and so $S \subseteq [E - A]$, from which it follows that $S \cap A = \phi$.

(iii) implies (i): Assume (iii) and let $A \subseteq E \in \mathscr{A}$. Suppose that $x \in [A]'$. Then $x \in [F - [A]]$ for some $F \in \mathscr{A}$. Thus, for some $S \in \Sigma$, $x \in S$ and $S \cap F \subseteq F - [A]$, so $S \cap [A] \cap F = \phi$. By (iii), it follows that $S \cap A = \phi$; hence, $x \in S \subseteq [E - A]$.

Corollary 46. If $(\mathscr{A}, \Sigma)$ is SU, then it is unital.

Proof. Assume $(\mathscr{A}, \Sigma)$ is SU and let $x \in X$. Because $\bigcup \Sigma = X$, there exists $S \in \Sigma$ with $x \in S$. By part (iii) of Theorem 45, $S \cap [x] \neq \phi$; hence, $[x] \neq \phi$. ▮

Moreover, we have the following results:

Lemma 47. If $(\mathscr{A}, \Sigma)$ is unital and $P \in \mathscr{L}$, then $P \cup P' = X$.

Proof. Let $x \in X$ and suppose $x \notin P$. Select $E \in \mathscr{A}$ with $x \in E$. Thus, $x \in E - P$, which implies $x \in [E - P]$ because $(\mathscr{A}, \Sigma)$ is unital. Consequently, $x \in P'$. ▮

Lemma 48. If $(\mathscr{A}, \Sigma)$ is SU, then, for $P \in \mathscr{L}$, $P'' \subseteq P$.

Proof. Let $x \in P''$. Then, there exists $F \in \mathscr{A}$ such that $x \in [F - P']$; Hence, there exists $S \in \Sigma$ with $x \in S$ and $S \cap F \cap P' = \phi$. Since $P' = \bigcup \{[E - P] \mid E \in \mathscr{A}\}$, it follows that $S \cap F \cap [E - P] = \phi$ for every $E \in \mathscr{A}$. Hence, by part (iii) of Theorem 45, $S \cap (E - P) = \phi$ holds for all $E \in \mathscr{A}$. Because $S \subseteq X = \bigcup \mathscr{A}$, it follows that $S \subseteq P$; hence, $x \in P$. ▮

In the presence of SU, it is now clear that $'\colon \mathscr{L} \to \mathscr{L}$ is an orthocomplementation if and only if, for all $P \in \mathscr{L}$, $P \subseteq P''$. Unfortunately, this is seldom the case, and we are therefore led to the following condition:

Definition 49. An entity $(\mathcal{O}, \Sigma)$ is said to be *symmetric* if for every $P, Q \in \mathcal{L}$, $P \subseteq Q'$ implies $Q \subseteq P'$.

Lemma 50. Let $(\mathcal{O}, \Sigma)$ be an entity. Then, the following conditions are equivalent:

(i) $(\mathcal{O}, \Sigma)$ is symmetric.

(ii) For every $S \in \Sigma$, $S \subseteq S''$.

(iii) For every $P \in \mathcal{L}$, $P \subseteq P''$.

In summary, we have the following theorem:

Theorem 51. An entity $(\mathcal{O}, \Sigma)$ is SU and symmetric if and only if $'$: $\mathcal{L} \to \mathcal{L}$ is an orthocomplementation for which the condition $[A]' = [E - A]$ holds whenever $A \subseteq E \in \mathcal{O}$.

The conditions SU, and symmetry in particular, are quite strong and indeed may prove to be far too restrictive for many physical situations. In fact we have:

Theorem 52. Let $(\mathcal{O}, \Sigma)$ be an SU entity.

(i) If every state is principal, then $(\mathcal{O}, \Sigma)$ is symmetric.

(ii) If $(\mathcal{O}, \Sigma)$ is symmetric and Σ is irredundant, then every state in Σ is of the form $[x]$ for some $x \in X$.

Proof.

(i) If $S = [A]$ for $A \subseteq E \in \mathcal{O}$, then $S'' = [E - A]' = [A] = S$. Therefore, if every state is principal, $(\mathcal{O}, \Sigma)$ is symmetric by part (ii) of Lemma 50.

(ii) Suppose $(\mathcal{O}, \Sigma)$ is symmetric. Then, by Theorem 51, $'$: $\mathcal{L} \to \mathcal{L}$ is an orthocomplementation. Let $S \in \Sigma$. By Lemma 38, $S' = \bigwedge \{[E \cap S'] \mid E\partial \in \mathcal{O}\}$, and it follows that

$$S = S'' = \bigvee \{[E \cap S']' \mid E \in \mathcal{O}\} = \bigvee \{[E - S'] \mid E \in \mathcal{O}\}$$

Because $S \neq \phi$, there exists $E \in \mathcal{O}$ such that $\phi \neq [E - S'] \subseteq S$. Select $x \in E - S'$. By Corollary 46, $(\mathcal{O}, \Sigma)$ is unital; hence, $\phi \neq [x] \subseteq [E - S'] \subseteq S$. Now, if Σ is irredundant, we have $[x] = S$. ∎

According to the Wigner–von Neumann projection postulate [Ref. 2, p. 77], if a measurement of the first kind is executed twice in succession, the same outcome will be secured each time. If an outcome $x \in X$ is secured in the first execution of such a measurement, then the entity must be left in a state S for which $S \subseteq [x]$. Now, if $[x]$ is a state, and in addition, if Σ is irredundant, then $S = [x]$. In this sense, $[x]$ is an "observable" state. Thus, the preceding theorem asserts that all states are observable for an SU symmetric entity with a irredundant set of states (provided, of course, that every outcome x for which $[x]$ is a state is contained in some operation E of the first kind).

6. ORTHOGONALITY

It is always possible to introduce reasonable orthogonality relations on $\mathscr{L}$, even when orthocomplementation is not possible. Some of the candidates are as follows:

Definition 53. Let $P, Q \in \mathscr{L}$.

(i) If there exists an operation $E \in \mathscr{O}$ for which $P \cap Q \cap E = \phi$, we say that P and Q are *uniformly orthogonal* and write $P \underline{\text{uo}}\, Q$.

(ii) If, for every $S \in \Sigma_P$ and every $T \in \Sigma_Q$, $S \underline{\text{uo}}\, T$, we say that P is *orthogonal* to Q and write $P \perp Q$.

(iii) If $P \subseteq Q'$, we say that P is *weakly orthogonal* to Q and write $P \underline{\text{wo}}\, Q$.

The fact that the same symbol $\perp$ is used for orthogonality of events on the one hand and orthogonality of properties on the other causes no confusion—one can always tell from the context what is intended. Notice that $\perp$ agrees with uniform orthogonality on Σ. Whereas uniform orthogonality and $\perp$ are automatically symmetric, weak orthogonality is symmetric if and only if the entity is symmetric. Although a nonempty property cannot be orthogonal or uniformly orthogonal to itself, it can be weakly orthogonal to itself. Uniformly orthogonal properties are automatically both orthogonal and weakly orthogonal.

The following two lemmas are useful and easily verified:

Lemma 54. Let $P \in \mathscr{L}$, $A, B \in \mathscr{E}(\mathscr{O})$, and $S \in \Sigma$. Then:

(i) $P' = \bigcup \{T \in \Sigma \mid T \underline{\text{uo}}\, P\}$.

(ii) $A \perp B$ implies that $[A] \underline{\text{uo}}\, [B]$.

(iii) $A \cap S = \phi$ implies that $[A] \underline{\text{uo}}\, S$.

Lemma 55. For an entity $(\mathscr{O}, \Sigma)$, the following conditions are equivalent:

(i) $(\mathscr{O}, \Sigma)$ is SU.

(ii) If $A \in \mathscr{E}(\mathscr{O})$ and $S \in \Sigma$ with $[A] \underline{\text{uo}}\, S$, then $A \cap S = \phi$.

(iii) If $x \in X$ and $S \in \Sigma$ with $[x] \underline{\text{uo}}\, S$, then $x \notin S$.

If $P \underline{\underline{\text{uo}}}\, Q$ in $\mathscr{L}$, then there exists an operation $E \in \mathfrak{A}$ for which the event $P \cap E$ is orthogonal to the event $Q \cap E$. Because $P \cap E$ is P-true and $Q \cap E$ is Q-true, E "separates" the properties P and Q. Indeed, if we knew that either P or Q is actual, then we could tell which by executing E and observing whether $P \cap E$ or $Q \cap E$ occurs. However, when $P \perp Q$, there may not exist a single operation E that uniformly separates P and Q; although, for each $S \in \Sigma_P$ and each $T \in \Sigma_Q$, there will exist an operation E (depending on S and T) that separates S and T.

For an orthogonality space such as $(\Sigma, \perp)$, a complete orthocomplemented lattice is always naturally available. For $\Gamma \subseteq \Sigma$, let $\Gamma^{\perp} = \{T \in \Sigma \mid T \perp S \text{ for all } S \in \Gamma\}$ and define $\Gamma^{\perp\perp} = (\Gamma^{\perp})^{\perp}$. The set $\mathscr{C}(\Sigma, \perp) = \{\Gamma \subseteq \Sigma \mid \Gamma = \Gamma^{\perp\perp}\}$ of all $\perp$-closed subsets of Σ is a complete lattice with respect to the order relation of set inclusion and it is orthocomplemented by the map $\Gamma \mapsto \Gamma^{\perp}$. For the quantum mechanical entity of Example 30, $S_\phi \perp S_\psi$ in Σ if and only if $\phi \perp \psi$ in $\mathscr{H}$ (that is, $\langle \phi \mid \psi \rangle = 0$). Moreover, $\mathscr{C}(\Sigma, \perp)$ is isomorphic to the complete orthomodular lattice of closed linear subspaces of $\mathscr{H}$ and therefore it is isomorphic to the property lattice $\mathscr{L}$. This is a consequence of Lemma 33 and the fact that, in this case, $\mathscr{C}(\Sigma, \text{sp}) = \mathscr{C}(\Sigma, \perp)$.

In general, $\mathscr{C}(\Sigma, \text{sp}) \neq \mathscr{C}(\Sigma, \perp)$, and it seems reasonable to seek conditions under which the equality holds. In Ref. 1, Aerts considers the following physically plausible axioms and finds that they do force the desired equality.

Definition 56. Let $(\mathfrak{A}, \Sigma)$ be an entity.

(i) $(\mathfrak{A}, \Sigma)$ satisfies $A1$ if $S \in \Sigma$, $A \in \mathscr{E}(\mathfrak{A})$, and $S \cap A \neq \phi$ implies that there exists $T \in \Sigma_A$ such that T is not orthogonal to S.

(ii) $(\mathfrak{A}, \Sigma)$ satisfies $A2$ if, for every $S \in \Sigma$, there exists $P \in \mathscr{L}$ such that $\{T \in \Sigma \mid T \perp S\} = \Sigma_P$.

In condition $A2$, the $P \in \mathscr{L}$ is in fact equal to S'. Moreover, we have a number of conditions equivalent to $A1$:

Lemma 57. Let $(\mathfrak{A}, \Sigma)$ be an entity. Then, the following conditions are equivalent:

(i) $(\mathfrak{A}, \Sigma)$ satisfies $A1$.

(ii) For $A \in \mathscr{E}(\mathfrak{A})$ and $S \in \Sigma$, $[A] \perp S$ implies $S \cap A = \phi$.

(iii) For $x \in X = \bigcup \mathfrak{A}$ and $S \in \Sigma$, $[x] \perp S$ implies $x \notin S$.

(iv) For $x \in X$ and $S \in \Sigma$, $x \in S$ implies that there exists $T \in \Sigma_x$ for which T is not orthogonal to S.

(v) If $A \subseteq E \in \mathfrak{A}$, $(\Sigma_A)^{\perp} \subseteq \Sigma_{E-A}$.

Proof.

(i) implies (ii): Evidently, condition (ii) is just the contrapositive of condition (i).

(ii) implies (iii): Put $A = \{x\}$ in (ii).

(iii) implies (iv): Evidently, condition (iv) is just the contrapositive of condition (iii).

(iv) implies (i): Assume (iv) and suppose $S \in \Sigma$, $A \in \mathscr{E}(\mathcal{O})$, $S \cap A \neq \phi$. Now let $x \in S \cap A$. By (iv), there exists $T \in \Sigma_x$ such that T is not orthogonal to S. But $x \in A$, so $T \in \Sigma_x \subseteq \Sigma_A$, and (i) holds.

(ii) implies (v): Suppose $S \in (\Sigma_A)^\perp$. Then $S \perp [A]$, so $S \cap A = \phi$ by (ii), and it follows that $S \in \Sigma_{E-A}$.

(v) implies (ii); Assume (v). Then $[A] \perp S$ implies $S \in (\Sigma_A)^\perp \subseteq \Sigma_{E-A}$, from which it follows that $S \cap A = \phi$. ∎

Notice that $\Sigma_{E-A} \subseteq (\Sigma_A)^\perp$ whenever $A \subseteq E \in \mathcal{O}$; thus (v) asserts that $\Sigma_{E-A} = (\Sigma_A)^\perp$.

Lemma 58. Let $(\mathcal{O}, \Sigma)$ be an entity.

(i) $(\mathcal{O}, \Sigma)$ satisfies $A2$ if and only if $\mathscr{C}(\Sigma, \perp) \subseteq \mathscr{C}(\Sigma, \text{sp})$.

(ii) If $(\mathcal{O}, \Sigma)$ satisfies $A1$, then $\mathscr{C}(\Sigma, \text{sp}) \subseteq \mathscr{C}(\Sigma, \perp)$.

Proof.

(i) Suppose $A2$ is satisfied. If $\Gamma \in \mathscr{C}(\Sigma, \perp)$, then for some $\Lambda \subseteq \Sigma$. we have $\Gamma = \Lambda^\perp$. Thus,

$$\Gamma = \Lambda^\perp = \bigcap \{\{S\}^\perp \mid S \in \Lambda\} = \bigcap \{\Sigma_{S'} \mid S \in \Lambda\}$$

by $A2$. But, by Lemma 33,

$$\bigcap \{\Sigma_{S'} \mid S \in \Lambda\} = \Sigma_P \in \mathscr{C}(\Sigma, \text{sp})$$

where $P = \bigwedge \{S' \mid S \in \Lambda\}$, and it follows that $\Gamma \in \mathscr{C}(\Sigma, \text{sp})$. Conversely, suppose $\mathscr{C}(\Sigma, \perp) \subseteq \mathscr{C}(\Sigma, \text{sp})$. Then, if $S \in \Sigma$, it follows that $\{S\}^\perp \in \mathscr{C}(\Sigma, \perp)$, so $\{S\}^\perp \in \mathscr{C}(\Sigma, \text{sp})$. Thus, by Lemma 33, $\{S\}^\perp = \Sigma_P$ for some $P \in \mathscr{L}$.

(ii) Suppose $A1$ is satisfied and $\Gamma \in \mathscr{C}(\Sigma, \text{sp})$. Then Lemma 33 implies that $\Gamma = \Sigma_P$ for some $P \in \mathscr{L}$. Since $P = \bigwedge \{[E \cap P] \mid E \in \mathcal{O}\}$ by Lemma 38, it follows from Lemma 33 and $A1$ that

$$\Sigma_P = \bigcap \{\Sigma_{E \cap P} \mid E \in \mathcal{O}\} = \bigcap \{(\Sigma_{E-P})^\perp \mid E \in \mathcal{O}\} \in \mathscr{C}(\Sigma, \perp)$$ ∎

Theorem 59. If an entity $(\mathcal{O}, \Sigma)$ satisfies $A1$ and $A2$, then the Cartan map $\mathscr{L} \to \mathscr{C}(\Sigma, \perp)$ is a lattice isomorphism for which $\Sigma_{P'} = (\Sigma_P)^\perp$ for all $P \in \mathscr{L}$.

Proof. By Lemma 58, $\mathscr{C}(\Sigma, \mathrm{sp}) = \mathscr{C}(\Sigma, \perp)$. By Lemma 33, the Cartan map is a lattice isomorphism. By Definition 41, $P' = \bigcup \{[E-P] \mid E \in \mathcal{O}\}$. By $A1$ and Lemma 33, this implies that

$$\Sigma_{P'} = \bigvee \{\Sigma_{E-P} \mid E \in \mathcal{O}\} = \bigvee \{(\Sigma_{E \cap P})^\perp \mid E \in \mathcal{O}\}$$
$$= \left(\bigcap \{\Sigma_{E \cap P} \mid E \in \mathcal{O}\}\right)^\perp = (\Sigma_P)^\perp \qquad \blacksquare$$

In the presence of $A1$ and $A2$, the orthocomplementation imposed on $\mathscr{L}$ by the Cartan map is, according to Theorem 59, precisely the map $': \mathscr{L} \to \mathscr{L}$ provided by Definition 41. As a consequence, the next two lemmas should come as no surprise.

Lemma 60. Let $(\mathcal{O}, \Sigma)$ be an entity.

(i) If $(\mathcal{O}, \Sigma)$ satisfies $A1$, it is SU.

(ii) If $(\mathcal{O}, \Sigma)$ satisfies $A1$ and $A2$, it is symmetric.

Proof.

(i) Compare Lemmas 55(ii) and 57(ii) and note that uniform orthogonality implies orthogonality.

(ii) Symmetry follows immediately from the fact that, for $P \in \mathscr{L}$, $\Sigma_{P'} = (\Sigma_P)^\perp$. $\blacksquare$

Lemma 61. Let $(\mathcal{O}, \Sigma)$ be an entity.

(i) If $(\mathcal{O}, \Sigma)$ is SU and symmetric, it satisfies $A1$.

(ii) If $(\mathcal{O}, \Sigma)$ is SU and symmetric and Σ is irredundant, is satisfies $A2$.

Proof.

(i) Suppose $(\mathcal{O}, \Sigma)$ is SU and symmetric, let $S \in \Sigma$, $x \in E \in \mathcal{O}$, and assume that $S \perp [x]$. By Lemma 55(iii) and Lemma 57(iii), it suffices to show that $S \ \underline{\mathrm{uo}}\ [x]$. Because $S \perp [x]$, it follows that $S \perp T$ for every $T \in \Sigma_x$. But this implies by Lemma 54(i) that $T \subseteq S'$ for every $T \in \Sigma_x$. Therefore, $[x] \subseteq S'$. As a consequence of symmetry and SU, it follows that $S \subseteq [x]' = [E-x]$, so $x \notin S$ and, by Lemma 54(iii), $S \ \underline{\mathrm{uo}}\ [x]$.

(ii) Assume that $(\mathcal{O}, \Sigma)$ is SU and symmetric and that Σ is irredundant. By part (i), $(\mathcal{O}, \Sigma)$ satisfies $A1$. Let $S \in \Sigma$. By Theorem 52(ii), there exists $x \in E \in \mathfrak{A}$ such that $S = [x]$. Hence, by Lemma 57(v), $\{S\}^{\perp} = (\Sigma_x)^{-} = \Sigma_{E-x}$. ∎

Although it is rather attractive on physical grounds, the orthocomplementation $'$: $\mathcal{L} \to \mathcal{L}$ allowed by $A1$ and $A2$ (or by SU and symmetry) is not the only possibility. An entity $(\mathfrak{A}, \Sigma)$ can possess a more or less naturally orthocomplemented property lattice $\mathcal{L}$ and still satisfy none of the above conditions. Just let $\mathfrak{A}$ be the regular collar manual of Example 17 and let $\Sigma = \{X - x^{\perp} \mid x \in X\}$. By Lemma 21, Σ is a set of $\mathfrak{A}$-supports, and Σ can be shown to be irredundant. Moreover, $(\mathfrak{A}, \Sigma)$ is an entity that satisfies neither $A1$ nor $A2$ and which is therefore neither SU nor symmetric. Nevertheless, the complete orthocomplemented lattice $\mathcal{C}(X, \perp) = \{M \subseteq X \mid M = M^{\perp\perp}\}$ is isomorphic to $\mathcal{L}$ under the map $M \mapsto X - M^{\perp}$.

7. QUESTIONS AND MORPHISMS

Property lattices (propositional systems) are developed by one of us in Ref. 17, in terms of yes-no questions. In our present formalism, this can be achieved simply by letting $\mathfrak{A}$ be a semi-classical manual (Example 10) in which every operation consists of exactly two outcomes. Thus, if x is an $\mathfrak{A}$-outcome, there exists exactly one operation $E_x \in \mathfrak{A}$ such that $x \in E_x$. Here we regard the event $\{x\}$ as corresponding to the following question:

> "If the apparatus for E_x is assembled and the operation E_x is executed, is the outcome $x \in E_x$ secured?"

The inverse of $\{x\}$ is then given by $\{x\}^{\sim} = E_x - x$. More generally, suppose α is a nonempty set of $\mathfrak{A}$-outcomes such that $\alpha \cap E$ contains at most one outcome for every $E \in \mathfrak{A}$. Then α corresponds in an obvious way to the product of questions $\pi_{x \in \alpha}\{x\}$, and $\alpha^{\sim} = \bigcup \{E_x - x \mid x \in \alpha\}$ corresponds to $(\pi_{x \in \alpha}\{x\})^{\sim}$.

Now suppose that the questions introduced above concern a particular physical system or entity. Then the states, as introduced in Ref. 17, correspond in an obvious way to certain supports $S \in \mathcal{S}(\mathfrak{A})$ and the set Σ of all such supports is the state space of the entity $(\mathfrak{A}, \Sigma)$.

In this section, we are going to show that there is a close connection between general entities $(\mathfrak{B}, \Sigma)$ on the one hand, and entities $(\mathfrak{A}, \Sigma)$ for which $\mathfrak{A}$ is a semiclassical manual of dichotomies, on the other. The following notion of a *morphism* is the tool used to effect this connection.

Definition 62. Let $\mathfrak{A}$ and $\mathfrak{B}$ be quasimanuals with outcome sets X

and Y, respectively. A *morphism* $\mathcal{A} \to^{\phi} \mathcal{B}$ is a map ϕ: $X \to \mathcal{E}(\mathcal{B})$ satisfying the following conditions:

(i) For all $A \in \mathcal{E}(\mathcal{A})$, $\phi(A) =_{\text{def}} \bigcup \{\phi(x) \mid x \in A\} \in \mathcal{E}(\mathcal{B})$.

(ii) $A \underline{\text{op}} B$ in $\mathcal{E}(\mathcal{A})$ implies $\phi(A) \underline{\text{op}} \phi(B)$ in $\mathcal{E}(\mathcal{B})$.

In case $\mathcal{A}$ and $\mathcal{B}$ are manuals, the conditions in Definition 62 are precisely what is required to guarantee that

$$\phi(p(A)) =_{\text{def}} p(\phi(A))$$

defines an order preserving map ϕ: $\pi(\mathcal{A}) \to \pi(\mathcal{B})$ on the operational logics.

Definition 63. Let $\mathcal{A} \to^{\phi} \mathcal{B}$ be a morphism, $X = \bigcup \mathcal{A}$, and $Y = \bigcup \mathcal{B}$. If $N \subseteq Y$, define

$$\phi^+(N) = \{x \in X \mid \phi(x) \cap N \neq \phi\}$$

Theorem 64. Let $\mathcal{A} \to^{\phi} \mathcal{B}$ be a morphism. Then:

(i) If $Q \in \mathcal{S}(\mathcal{B})$, then $\phi^+(Q) \in \mathcal{S}(\mathcal{A})$.

(ii) The map ϕ^+: $\mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{A})$ preserves arbitrary set-theoretic unions.

Proof.

(i) Let $Q \in \mathcal{S}(\mathcal{B})$ and E, $F \in \mathcal{A}$. We must show that $\phi^+(Q) \cap E \subseteq F$ implies that $\phi^+(Q) \cap F \subseteq E$; and this is equivalent to showing that $(E - F) \cap \phi^+(Q) = \phi$ implies that $(F - E) \cap \phi^+(Q) = \phi$. This, in turn, is equivalent to showing that $\phi(E - F) \cap Q = \phi$ implies that $\phi(F - E) \cap Q = \phi$. In other words, we have to show that, if $\phi(E - F)$ is Q-false, then $\phi(F - E)$ is Q-false. But the events $E - F$ and $F - E$ are operationally perspective with axis $E \cap F$; hence, $\phi(E - F) \underline{\text{op}} \phi(F - E)$. By the remarks following Definition 19, $\phi(E - F)$ is Q-false if and only if $\phi(F - E)$ is Q-false.

(ii) $x \in \phi^+(\bigcup_i Q_i)$ if and only if $\phi(x) \cap \bigcup_i Q_i \neq \phi$
if and only if $\bigcup_i (\phi(x) \cap Q_i) \neq \phi$
if and only if $x \in \bigcup_i \phi^+(Q_i)$. ∎

Definition 65. A morphism $\mathcal{A} \to^{\phi} \mathcal{B}$ is said to be:

(i) *positive* if $\phi(x) \neq \phi$ for all $x \in X = \bigcup \mathcal{A}$.

(ii) *operation preserving* if $E \in \mathcal{A}$ implies $\phi(E) \in \mathcal{B}$.

(iii) a *conditioning* if $A \perp B$ in $\mathcal{E}(\mathcal{A})$ implies that $\phi(A) \perp \phi(B)$ in $\mathcal{E}(\mathcal{B})$.

(iv) an *interpretation* if it is an operation preserving conditioning.

(v) *outcome-faithful* if, for every $y \in Y = \bigcup \mathscr{B}$, there exists $x \in X = \bigcup \mathscr{A}$ such that $\phi(x) = \{y\}$.

(vi) $\underline{\text{oc}}$*-faithful* if $C \ \underline{\text{oc}} \ D$ in $\mathscr{E}(\mathscr{B})$ implies that there exists $A \ \underline{\text{oc}} \ B$ in $\mathscr{E}(\mathscr{A})$ for which $\phi(A) = C$ and $\phi(B) = D$.

It is not difficult to prove the following technically useful lemma:

Lemma 66. Let $\mathscr{A} \to^{\phi} \mathscr{B}$ be an $\underline{\text{oc}}$-faithful morphism. Then:

(i) ϕ is outcome-faithful.

(ii) If $\mathscr{B}$ is irredundant, then ϕ is operation preserving.

It is worth noting that the interpretations are precisely the morphisms that "pull back" our global stochastic models or weights. Specifically, if ω is a $\mathscr{B}$-weight and $\mathscr{A} \to^{\phi} \mathscr{B}$ is an interpretation, then $(\phi^{+}\omega)(x) =_{\text{def}} \omega(\phi(x))$ determines an $\mathscr{A}$-weight $\phi^{+}\omega$. Moreover,

$$\operatorname{supp}(\phi^{+}\omega) = \phi^{+}(\operatorname{supp} \omega)$$

Observe that $\phi^{+}(\operatorname{supp} \omega)$ is an $\mathscr{A}$-support even if the morphism $\mathscr{A} \to^{\phi} \mathscr{B}$ is not an interpretation.

Lemma 67. Let $(\mathscr{B}, \Sigma)$ be an entity and let $\mathscr{A} \to^{\phi} \mathscr{B}$ be a morphism. Define $\phi^{+}\Sigma = \{\phi^{+}(T) \mid T \in \Sigma\}$. Then $(\mathscr{A}, \phi^{+}\Sigma)$ is an entity if and only if ϕ is positive.

Proof.

$$\bigcup \phi^{+}\Sigma = \bigcup \{\phi^{+}(T) \mid T \in \Sigma\} = \phi^{+}\left(\bigcup \Sigma\right) = \phi^{+}\left(\bigcup \mathscr{B}\right) = \phi^{+}(Y)$$

$$= \{x \in X \mid \phi(x) \cap Y \neq \phi\} = \{x \in X \mid \phi(x) \neq \phi\}.$$

There, $\bigcup \phi^{+}\Sigma = X$ if and only if ϕ is positive. ∎

For the remainder of this section, we assume that $(\mathscr{B}, \Sigma)$ is an entity and $\mathscr{A} \to^{\phi} \mathscr{B}$ is a positive morphism. The following results exhibit some of the relationships between the two entities $(\mathscr{A}, \phi^{+}\Sigma)$ and $(\mathscr{B}, \Sigma)$.

Lemma 68. If $\mathscr{A} \to^{\phi} \mathscr{B}$ is a positive morphism, then $\phi^{+}\colon \mathscr{L}(\mathscr{B}, \Sigma) \to \mathscr{L}(\mathscr{A}, \phi^{+}\Sigma)$ is a surjective union preserving map.

Proof. Theorem 64. ∎

Lemma 69. If $\mathscr{A} \to^{\phi} \mathscr{B}$ is a positive outcome-faithful morphism, then $\phi^{+}\colon \mathscr{L}(\mathscr{B}` \Sigma) \to \mathscr{L}(\mathscr{A}, \phi^{+}\Sigma)$ is a lattice isomorphism.

Proof. Given Lemma 68, all we need to prove is that $\phi^+(P) \subseteq \phi^+(Q)$ implies that $P \subseteq Q$ for P, $Q \in \mathscr{L}(\mathscr{B}, \Sigma)$. Thus, assume $\phi^+(P) \subseteq \phi^+(Q)$. Hence, $\phi(x) \cap P \neq \phi$ implies that $\phi(x) \cap Q \neq \phi$. Let $y \in P$. Since ϕ is outcome-faithful, there exists $x \in X$ such that $\phi(x) = \{y\}$, and consequently, $\phi(x) \cap P \neq \phi$. It follows that $\phi(x) \cap Q \neq \phi$; hence, $y \in Q$. ∎

Lemma 70. Let $\mathscr{A} \to^{\phi} \mathscr{B}$ be a positive morphism, $T \in \Sigma$, and $A \subseteq E \in \mathscr{A}$. Then:

(i) If, in addition, ϕ is operation preserving, then $\phi^+(T) \subseteq [A]$ implies that $T \subseteq [\phi(A)]$.

(ii) If, in addition, ϕ is an interpretation, then $T \subseteq [\phi(A)]$ implies that $\phi^+(T) \subseteq [A]$.

Proof. First notice that, if ϕ is operation preserving, then $T \subseteq [\phi(A)]$ holds if and only if $T \cap \phi(E) \subseteq \phi(A)$.

(i) Suppose that ϕ is operation preserving and $\phi^+(T) \subseteq [A]$. Then $\phi^+(T) \cap E \subseteq A$. Let $y \in T \cap \phi(E)$. By the remark above, it will be enough to prove that $y \in \phi(A)$. Since $y \in \phi(E)$, there exists $x \in E$ such that $y \in \phi(x)$. Therefore, $y \in \phi(x) \cap T$, so $\phi(x) \cap T \neq \phi$, and it follows that $x \in \phi^+(T) \cap E$. Consequently, $x \in A$, so $y \in \phi(x) \subseteq \phi(A)$.

(ii) Suppose that ϕ is an interpretation and $T \subseteq [\phi(A)]$. Again by our initial remark, $T \cap \phi(E) \subseteq \phi(A)$. To prove that $\phi^+(T) \subseteq [A]$, it suffices to show that $\phi^+(T) \cap E \subseteq A$. To this end, select $x \in \phi^+(T) \cap E$. We must prove that $x \in A$. Because $x \in \phi^+(T)$, there exists $y \in T \cap \phi(x) \subseteq T \cap \phi(E) \subseteq \phi(A)$. Since $y \in \phi(x) \cap \phi(A)$, we cannot have $\phi(x) \perp \phi(A)$; hence, because ϕ is a conditioning, we cannot have $x \perp A$. But $x \in E$ and $A \subseteq E$. It follows that $x \in A$. ∎

Corollary 71. If $\mathscr{A} \to^{\phi} \mathscr{B}$ is a positive interpretation, then, for every $A \in \mathscr{E}(\mathscr{A})$, $\phi^+[\phi(A)] = [A]$.

Proof. By Lemma 70,

$$\phi^+[\phi(A)] = \phi^+\left(\bigcup \{T \in \Sigma \mid T \subseteq [\phi(A)]\}\right)$$
$$= \bigcup \{\phi^+(T) \mid T \in \Sigma, \phi^+(T) \subseteq [A]\} = [A]$$ ∎

Lemma 72. Let $\mathscr{A} \to^{\phi} \mathscr{B}$ be a positive morphism and suppose that P, $Q \in \mathscr{L}(\mathscr{B}' \Sigma)$. Then:

(i) If, in addition, ϕ is operation preserving, then $\phi^+(P) \,\underline{\text{uo}}\, \phi^+(Q)$ implies $P \,\underline{\text{uo}}\, Q$.

(ii) If, in addition, ϕ is $\underline{\text{oc}}$-faithful, then $P\,\underline{\text{uo}}\,Q$ implies $\phi^+(P)\,\underline{\text{uo}}\,\phi^+(Q)$.

Proof.

(i) Suppose that $\phi^+(P)\cap\phi^+(Q)\cap E=\phi$, where $E\in\mathcal{O}\!\!\!\mathcal{A}$. It will be sufficient to prove that $P\cap Q\cap\phi(E)=\phi$. Suppose, on the contrary, that $y\in P\cap Q$ and $y\in\phi(x)$ for some $x\in E$. Then $\phi(x)\cap P\neq\phi$ and $\phi(x)\cap Q\neq\phi$, so $x\in\phi^+(P)\cap\phi^+(Q)\cap E=\phi$, a contradiction.

(ii) Suppose $P\cap Q\cap F=\phi$, where $F\in\mathcal{B}$. Since $P\cap F\,\underline{\text{oc}}\,F-P$ and ϕ is $\underline{\text{oc}}$-faithful, there exist A, $B\in\mathcal{E}(\mathcal{O}\!\!\!\mathcal{A})$ with $A\,\underline{\text{oc}}\,B$, $\phi(A)=P\cap F$, and $\phi(B)=F-P$. Let $E=A\cup B$, noting that $E\in\mathcal{O}\!\!\!\mathcal{A}$, so that it will suffice to prove that $\phi^+(P)\cap\phi^+(Q)\cap E=\phi$. Suppose, on the contrary, that $x\in E=A\cup B$, $\phi(x)\cap P\neq\phi$, and $\phi(x)\cap Q\neq\phi$. If $x\in A$, then $\phi\neq\phi(x)\cap Q\subseteq\phi(A)\cap Q=P\cap F\cap Q=\phi$, a contradiction. Therefore, $x\in B$, and we have $\phi\neq\phi(x)\cap P\subseteq\phi(B)\cap P=(F-P)\cap P=\phi$, another contradiction. ∎

Corollary 73. If $\mathcal{O}\!\!\!\mathcal{A}\rightarrow^{\phi}\mathcal{B}$ is a positive $\underline{\text{oc}}$-faithful morphism and P, $Q\in\mathcal{L}(\mathcal{B},\Sigma)$, then $P\perp Q$ if and only if $\phi^+(P)\perp\phi^+(Q)$.

Theorem 74. Suppose that $\mathcal{O}\!\!\!\mathcal{A}\rightarrow^{\phi}\mathcal{B}$ is a positive $\underline{\text{oc}}$-faithful morphism. Then:

(i) $\phi^+(Q')=(\phi^+(Q))'$ for all $Q\in\mathcal{L}(\mathcal{B},\Sigma)$.

(ii) $(\mathcal{O}\!\!\!\mathcal{A},\phi^+\Sigma)$ is symmetric if and only if $(\mathcal{B},\Sigma)$ is symmetric.

(iii) If ϕ is also an interpretation, then $(\mathcal{O}\!\!\!\mathcal{A},\phi^+\Sigma)$ is SU if and only if $(\mathcal{B},\Sigma)$ is SU.

Proof.

(i) By Lemma 54(i), $\phi^+(Q')=\phi^+(\bigcup\{T\in\Sigma\mid T\,\underline{\text{uo}}\,Q\})$. Hence, by Lemma 72,

$$\phi^+(Q')=\bigcup\{\phi^+(T)\mid T\in\Sigma,\phi^+(T)\,\underline{\text{uo}}\,\phi^+(Q)\}=(\phi^+(Q))'$$

(ii) This follows immediately from part (i) and the fact that ϕ^+: $\mathcal{L}(\mathcal{B},\Sigma)\rightarrow\mathcal{L}(\mathcal{O}\!\!\!\mathcal{A},\phi^+\Sigma)$ is a lattice isomorphism (Lemma 69).

(iii) Suppose that $(\mathcal{O}\!\!\!\mathcal{A},\phi^+\Sigma)$ is *SU*, $T\in\Sigma$, $y\in Y=\bigcup\mathcal{B}$, and $T\,\underline{\text{uo}}\,[y]$. We must prove that $y\notin T$. By Lemma 72(ii), $\phi^+(T)\,\underline{\text{uo}}\,\phi^+([y])$. By Lemma 66, $\{y\}=\phi(x)$ for some $x\in X=\bigcup\mathcal{O}\!\!\!\mathcal{A}$; hence, $\phi^+([y])=\phi^+([\phi(x)])=[x]$ by Corollary 71. Therefore, $\phi^+(T)\,\underline{\text{uo}}\,[x]$, so $x\notin\phi^+(T)$ because $(\mathcal{O}\!\!\!\mathcal{A},\phi^+\Sigma)$ is SU. It follows that $\phi(x)\cap T=\phi$; hence, $y\notin T$. Conversely, suppose that $(\mathcal{B},\Sigma)$ is SU, $T\in\Sigma$, $x\in X$

and $\phi^+(T)\ \underline{\text{uo}}\ [x]$. We must prove that $x \notin \phi^+(T)$; that is, $\phi(x) \cap T = \phi$. By Corollary 71, $[x] = \phi^+([\phi(x)])$, so $\phi^+(T)\ \underline{\text{uo}}\ \phi^+([\phi(x)])$. By Lemma 72, it follows that $T\ \underline{\text{uo}}\ [\phi(x)]$. Since $(\mathscr{B}, \Sigma)$ is SU, we conclude that $\phi(x) \cap T = \phi$, as desired. ∎

If $\mathfrak{A} \to^\phi \mathscr{B}$ is a positive $\underline{\text{oc}}$-faithful interpretation and Σ is irredundant, it follows from Theorem 74 and Lemma 61 that $(\mathfrak{A}, \phi^+\Sigma)$ satisfies $A1$ and $A2$ if and only if $(\mathscr{B}, \Sigma)$ does. In fact, with some additional effort, we could dispense with the irredundancy of Σ and prove:

Theorem 75. If $\mathfrak{A} \to^\phi \mathscr{B}$ is a positive $\underline{\text{oc}}$-faithful interpretation, then:

(i) $(\mathfrak{A}, \phi^+\Sigma)$ satisfies $A1$ if and only if $(\mathscr{B}, \Sigma)$ satisfies $A1$.

(ii) $(\mathfrak{A}, \phi^+\Sigma)$ satisfies $A2$ if and only if $(\mathscr{B}, \Sigma)$ satisfies $A2$.

With the preceding results, it is now easy to establish the promised close connection between general entities $(\mathscr{B}, \Sigma)$ on the one hand, and entities $(\mathfrak{A}, \Sigma^+)$ for which $\mathfrak{A}$ is a semiclassical manual of dichotomies, on the other. Thus, suppose $(\mathscr{B}, \Sigma)$ is a given entity. As a matter of technical convenience, we assume that every operation $F \in \mathscr{B}$ contains at least two outcomes. (A tedious argument, which we omit here, shows that one can always reduce to such a case without loss of generality.)

Let

$$\mathfrak{A} = \{\{(A, E), (E - A, E)\} \mid A \subseteq E \in \mathscr{B}, \phi \neq A \neq E\}$$

Naturally, to execute the $\mathfrak{A}$-operation $\{(A, E), (E - A, E)\}$, one executes the $\mathscr{B}$-operation E and records the outcome (A, E) if the $\mathscr{B}$-event A occurs, and the outcome $(E - A, E)$ otherwise. It is easy to check that the map

$$\phi((A, E)) = A$$

defines an $\underline{\text{oc}}$-faithful positive interpretation $\mathfrak{A} \to^\phi \mathscr{B}$. Thus, by Lemma 67, since $(\mathscr{B}, \Sigma)$ is an entity, so is $(\mathfrak{A}, \phi^+\Sigma)$. By Lemma 69, the property lattices $\mathscr{L}(\mathfrak{A}, \phi^+\Sigma)$ and $\mathscr{L}(\mathscr{B}, \Sigma)$ are isomorphic; furthermore, by Theorem 74(i), Lemma 72, and Corollary 73, the isomorphism preserves $'$, uniform orthogonality, and $\perp$. Moreover, according to Theorems 74 and 75, these entities are indistinguishable with respect to SU, symmetry, $A1$, and $A2$. However, one must be careful—although it is clear that $(\mathfrak{A}, \phi^+\Sigma)$ and $(\mathscr{B}, \Sigma)$ are essentially indistinguishable with respect to states and properties—as we shall show in future papers, *they combine differently*. A significant hint of this state of affairs appears in the paper of Aerts[1] where operations containing more than two outcomes must be introduced in order to compose two entities.

8. CONCLUDING REMARKS

The comprehensive formalism developed above makes a sharp distinction between the event calculus $\mathscr{E}(\mathfrak{A})$—or, in the case of a manual, the operational logic $\pi(\mathfrak{A})$—and the property lattice $\mathscr{L}$. To the best of our knowledge, no other formalism is able to discriminate between operational propositions and properties. Indeed, it is our contention that such distinctions are critical in the study of the foundations of quantum physics.

As we have seen, $\mathscr{L}$ is a complete lattice, whereas, in general, $\pi(\mathfrak{A})$ may contain (even orthogonal) pairs of elements with no least upper bound. However, $\pi(\mathfrak{A})$ is always orthocomplemented—and even in a sense orthomodular—but, in general, $\mathscr{L}$ is not.

Furthermore, the $\mathfrak{A}$-weights introduced in Definition 8 naturally provide the event calculus $\mathscr{E}(\mathfrak{A})$—and, in the case of a manual, the operational logic $\pi(\mathfrak{A})$—of an entity $(\mathfrak{A}, \Sigma)$ with a convex set of probability models. In general, there is no such collection that naturally arises for the property lattice $\mathscr{L}(\mathfrak{A}, \Sigma)$. Nevertheless, it is reasonable to inquire about the probability of certain events if an entity is known to be in a state $S \in \Sigma$. What we do know is that every event in

$$S^1 \underset{\text{def}}{=} \{A \in \mathscr{E}(\mathfrak{A}) \mid S \in \Sigma_A\}$$

will occur with certainty if tested and every event in

$$S^0 \underset{\text{def}}{=} \{A \in \mathscr{E}(\mathfrak{A}) \mid S \cap A = \phi\}$$

will not occur if tested. Thus, S defines a function $\omega_S\colon S^0 \cup S^1 \to [0, 1]$ for which $\omega_S = 0$ on S^0 and $\omega_S = 1$ on S^1. It is natural to ask when ω_S admits an extension to an $\mathfrak{A}$-weight defined on all of $\mathscr{E}(\mathfrak{A})$. Of course, when S is deterministic, then $S^0 \cup S^1 = \mathscr{E}(\mathfrak{A})$ and ω_S—an $\mathfrak{A}$-weight—is its own (unique) extension. In general, there may be many extensions of ω_S—or none at all.

In Hilbert space, a "miracle" occurs! In Example 30, to begin with, the canonical map $[\cdot]\colon \pi(\mathscr{F}(\mathscr{H})) \to \mathscr{L}$ is a lattice isomorphism. It follows that

$$\pi(\mathscr{F}(\mathscr{H})) \simeq \mathscr{L} \simeq \mathscr{C}(\Sigma, \perp) \simeq \mathscr{C}(\Sigma, \text{sp}) \simeq \mathscr{C}(X, \perp)$$

and all of these are isomorphic to the lattice of closed linear subspaces of $\mathscr{H}$. Moreover, no S in Σ is deterministic; nevertheless, the partial map ω_S: $S^0 \cup S^1 \to [0, 1]$ always admits an extension to an $\mathscr{F}(\mathscr{H})$-weight, which by

Gleason's theorem[10] is unique. Who is to say that physical entities should enjoy all of these remarkable properties?

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