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On the summability of Fourier series with the method of lacunary arithmetic means

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Let f be a 2π-periodic summable function with Fourier series

$$f(x) \sim \sum_k c_k e^{ikx},$$

and let $S_n(f;\, x)$ denote the partial sum of order n. It is known (see e.g. [12]) that the Fejér means

$$\frac{1}{N+1} \sum_{k=0}^{N} S_k(f;\, x)$$

converge at every Lebesgue point, and, if f is continuous, they converge even uniformly.

We are concerned with the following problem: for which monotone sequences $\{n_k\}$ do the means

$$\frac{1}{N} \sum_{k=1}^{N} S_{n_k}(f;\, x) \tag{1}$$

converge to f? Starting with the paper [11] of Zalcwasser, several mathematicians dealt with this question [2—7, 10]. A complete answer has been obtained only for the case of strong summability. Namely, for the uniform convergence

$$\frac{1}{N} \sum_{k=1}^{N} |S_{n_k}(f;\, x) - f(x)| \to 0 \tag{2}$$

for every continuous f it is necessary (Salem [7]) and, if $\{n_k\}$ is convex, also sufficient (Trigub—Zagorodniĭ [10]) that

$$\sup_k k^{-1/2} \ln n_k < \infty.$$

The same result was independently obtained by Carleson (announced in [4]).

As for the uniform convergence of (1) on the class of continuous functions, it is clear that the above condition remains sufficient. About necessity the following is known. It is easy to check that, if $n_{k+1}/n_k \to \infty$, then there is a continuous function

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for which we do not have uniform convergence. NEWMAN [6] showed that uniform convergence fails also for the sequence $n_k=2^k$. Thus, for the model sequence of the given problem $n_k=[2^{k^\beta}]$ we have: if $0<\beta\leqq 1/2$ then (1) converges uniformly for every continuous function; on the other hand, if $\beta\geqq 1$ then, in general, there is no convergence.

In the present paper we are going to prove a general statement, which implies in particular that there is no convergence in the case $1/2<\beta<1$, either (cf. Corollary).

Let L_N be the linear operator on the space of continuous functions defined by

$$L_N\colon f\to\frac{1}{N}\sum_{k=1}^{N}S_{n_k}(f).$$

If the sequence $\{n_k\}$ is convex, we have according to [10]

$$\|L_N\|_{C(-\pi,\pi]\to C(-\pi,\pi]}\leqq A_0(N^{-1/2}\ln n_N+1). \tag{3}$$

Here and in the sequel A_i denotes different positive constants. Similar lower estimation cannot be obtained in the general case; put e.g. $n_k=2^{2^k}$. Using the Fejér polynomials it is not difficult to show that

$$\|L_N\|_{C(-\pi,\pi]\to C(-\pi,\pi]}\asymp N^{-1}2^N.$$

The following assertion holds true.

Theorem. *Suppose that the convex sequence of positive integers $\{n_k\}$ satisfies the weak lacunarity condition $n_{k+1}/n_k>1+ck^{-\alpha}$, c is a constant, $0\leqq\alpha\leqq 1/2$. Then there exists a positive integer S depending on c and α such that*

$$\|L_N\|_{C(-\pi,\pi]\to C(-\pi,\pi]}\geqq A_1N^{-1/2}\ln n_{[NS^{-1}]}.$$

Proof. Using the Dirichlet formula, simple transformations give

$$\|L_{N+M}\|_{C(-\pi,\pi]\to C(-\pi,\pi]}=\sup_{\|f\|_C\leqq 1}\left|\frac{1}{N+M}\sum_{k=1}^{N+M}S_{n_k}(f;\,x)\right|=$$

$$=\sup_{\|f\|_C\leqq 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\frac{1}{N+M}\sum_{k=1}^{N+M}\frac{\sin(n_k+\frac{1}{2})x}{2\sin x/2}\,dx\geqq$$

$$\geqq\frac{1}{\pi}\int_0^{\pi}\left|\sum_{k=1}^{N+M}\sin n_kx\right|\frac{dx}{x}-O\left(\frac{1}{\sqrt{N+M}}\right).$$

By the triangle inequality we have

$$\frac{1}{N+M}\int_0^\pi \Big|\sum_{k=1}^{N+M}\sin n_k x\Big|\frac{dx}{x} \geqq$$

$$\geqq \frac{1}{N+M}\int_0^\pi \Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|\frac{dx}{x} - \frac{1}{N+M}\int_0^\pi \Big|\sum_{k=1}^{M}\sin n_k x\Big|\frac{dx}{x}.$$

In virtue of (3), the second integral does not exceed

$$A_2 M(N+M)^{-1}(M^{-1/2}\ln n_M+1).$$

In order to obtain a lower bound for the first integral, we represent it in the form

$$\frac{1}{N+M}\sum_{p=0}^{\infty}\int_{\pi 2^{-p-1}}^{\pi 2^{-p}}\Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|\frac{dx}{x} \geqq \frac{1}{N+M}\sum_{p=0}^{Q}\int_{\pi 2^{-p-1}}^{\pi 2^{-p}}\Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|\frac{dx}{x} \geqq$$

(4)

$$\geqq \frac{1}{N+M}\sum_{p=0}^{Q}\frac{2^p}{\pi}\int_{\pi 2^{-p-1}}^{\pi 2^{-p}}\Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|\,dx,$$

where Q is a positive integer to be defined later. Further, we use the inequality

$$\|f\|_{L^1} \geqq \|f\|_{L^2}^3\cdot\|f\|_{L^4}^{-2}$$

which can be easily obtained from the Hölder inequality. We get

$$\int_{\pi 2^{-p-1}}^{\pi 2^{-p}}\Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|\,dx \geqq$$

$$\geqq \left\{\int_{\pi 2^{-p-1}}^{\pi 2^{-p}}\Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|^2 dx\right\}^{3/2}\left\{\int_{\pi 2^{-p-1}}^{\pi 2^{-p}}\Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|^4 dx\right\}^{-1/2}.$$

To estimate the integral of the square of a polynomial from below we make use of the following result due to WIENER (cf. [12], p. 355):

Wiener's Theorem. *Consider the finite sum*

$$P(\vartheta) = \sum_{k=-N}^{N} c_k e^{in_k\vartheta} \quad (n_{-k} = -n_k)$$

where

$$n_{k+1}-n_k \geqq q > 0 \quad (k = 0, 1, \dots),$$

and let I be an interval of length $|I|>2\pi(1+\delta)q^{-1}$, $\delta>0$. Then there is a positive constant A_δ depending only on δ such that

$$\sum_k |c_k|^2 \leqq A_\delta \frac{1}{|I|}\int_I |P(\vartheta)|^2 d\vartheta.$$

To obtain an upper bund for the integral of the polynomial of fourth degree we shall use the following lemma (to be proved later):

Lemma. *Suppose that the convex sequence of positive integers* $\{n_k\}$ *satisfies the weak lacunarity condition with an exponent* $\alpha\in[0, 1/2]$, *i.e.* $n_{k+1}/n_k > 1+ck^{-\alpha}$. *Let us assume, furthermore, that*

$$n_{k+1}-n_k > q > 0 \quad (k = M+1, \ldots),\ n_M > q,$$

and let us consider an interval I of length $|I|>2\pi q^{-1}$. *Then*

$$\frac{1}{|I|}\int_I \Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big|^4 d\vartheta \leqq A_2 N(N+M)\max_k |c_k|^4,$$

where A_2 *depends only on* c.

Now choose Q in (4) such that the conditions of the above assertions be fulfilled. In consequence of convexity and weak lacunarity we have

$$n_{k+1}-n_k \geqq n_{M+1}-n_M = n_M(n_{M+1}/n_M-1) \geqq n_M(1+cM^{-\alpha}-1) = cn_M M^{-\alpha}.$$

Hence the condition

$$\pi 2^{-Q-1} > 4\pi M^\alpha c^{-1} n_M^{-1}$$

is fulfilled, i.e. one can choose $Q<\ln n_M-\alpha\ln M-A$; $Q\leqq A_4\ln n_M$. An application of Wiener's theorem and the lemma yields

$$\frac{1}{N+M}\sum_{p=0}^{Q} 2^p \int_{\pi 2^{-p-1}}^{\pi 2^{-p}} \Big|\sum_{k=M+1}^{N+M}\sin n_k x\Big|\,dx \geqq \frac{A_5}{N+M}\sum_{p=0}^{A_4\ln n_M} 2^p\left(\frac{N}{2^p}\right)^{3/2}\left(\frac{N(N+M)}{2^p}\right)^{-1/2} =$$

$$= A_6 N(N+M)^{-3/2}\ln n_M.$$

Thus, we conclude

$$\|L_{N+M}\|_{C(-\pi,\pi]\to C(-\pi,\pi]} \geqq$$

$$\geqq A_6 N(N+M)^{-3/2}\ln n_M + A_2 M^{1/2}(N+M)^{-1}\ln n_M - M(N+M)^{-1} + O\left(\frac{1}{(N+M)^{1/2}}\right),$$

whence

$$\|L_N\|_{C(-\pi,\pi]\to C(-\pi,\pi]} = \|L_{(N-[N/S])+[N/S]}\|_{C(-\pi,\pi]\to C(-\pi,\pi]} \geqq$$

$$\geqq N^{-1}\ln n_{[N/S]}(A_6 N^{-1/2}(N-[N/S])-A_2[N/S]^{1/2})-[N/S]\cdot N^{-1}+O(1/N) \geqq$$

$$\geqq N^{-1/2}\ln n_{[N/S]}(A_6(1-S^{-1})-A_2 S^{-1/2})-S^{-1}+O(1/N)$$

for arbitrary N. Choosing S such that the right-hand side be positive, we obtain

$$\|L_N\|_{C(-\pi,\pi]\to C(-\pi,\pi]} \geqq A_7 N^{-1/2}\ln n_{[N/S]},$$

which completes the proof.

Corollary. *If the convex sequence $\{n_k\}$ satisfies the weak lacunarity condition with an exponent $\alpha\in[0, 1/2)$ (in particular, if $n_k=[2^{k^\beta}]$, $1/2<\beta\leqq 1$) then there are continuous functions for which the means* (1) *do not converge.*

Proof. Putting $N=SM$ in the Theorem, we conclude

$$\|L_N\| \geqq$$

$$\geqq \frac{\ln n_M}{\sqrt{SM}} = \frac{1}{\sqrt{SM}} \ln \frac{n_M}{n_{M-1}} \frac{n_{M-1}}{n_{M-2}} \cdots \frac{n_2}{n_1} n_1 \geqq \frac{1}{\sqrt{SM}} \sum_{k=1}^{M-1} \ln\left(1+\frac{c}{k^\alpha}\right) \geqq A_8 M^{1/2-\alpha}.$$

Applying the Banach—Steinhaus theorem, we obtain the assertion.

Proof of the Lemma. In the proof we shall make use of the reasoning in [1, Lemma 5.2], where the case $I=[-\pi, \pi]$ is considered.

We can suppose that I is symmetric with respect to 0. Choose p satisfying the condition $\pi 2^{-p-1}\leqq|I|<\pi 2^{-p}$. Then

$$\frac{1}{|I|}\int_I \Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big|^4 d\vartheta \leqq \frac{2^{p+1}}{\pi} \int_{-\pi 2^{-p-1}}^{\pi 2^{-p-1}} \Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big|^4 d\vartheta.$$

Using the inequality $\frac{2}{\pi}x\leqq\sin x\leqq x$ for $0\leqq x\leqq\frac{\pi}{2}$ we have

$$\frac{2^{p+1}}{\pi} \int_{-\pi 2^{-p-1}}^{\pi 2^{-p-1}} \Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big|^4 d\vartheta \leqq$$

$$\leqq \left(\frac{\pi}{2}\right)^2 \frac{2^{p+1}}{\pi} \int_{-\pi 2^{-p-1}}^{2\pi^{-p-1}} \Big|\sum_{k=M+1}^{N+M} c_k e^{in_k x}\Big| \left[\frac{\sin 2^{p-1}\vartheta}{2^{p-1}\sin\vartheta}\right]^2 d\vartheta \leqq$$

$$\leqq \pi 2^{p-1} \int_{-\pi}^{\pi} \Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big| \left[\frac{\sin 2^{p-1}\vartheta}{2^{p-1}\sin\vartheta}\right]^2 d\vartheta.$$

Since $(\sin 2^{p-1}\vartheta)^2(2^{p-1}\sin\vartheta)^{-2}$ is a trigonometric polynomial of degree not higher than 2^p, it will be sufficient to estimate the number of those terms in the sum $\Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big|^4$ for which

$$|n_{j_1}\pm n_{j_2}\pm n_{j_3}\pm n_{j_4}| \leqq 2^p. \tag{5}$$

First we give an evaluation to the number of the members of the sequence $\{n_k\}$ in a given interval. If n_z and n_s are the largest and the least numbers in the interval $[a, b]$, then $n_z/n_s\leqq b/a$. On the other hand, from the condition of weak lacunarity and in virtue of the inequality $1+x\geqq\exp(x/2)$, valid for $0\leqq x\leqq 1$, we have

$$\frac{n_z}{n_s} \geqq \prod_{j=s}^{z-1}\left(1+\frac{c}{j^\alpha}\right) \geqq \prod_{j=s}^{z-1}\left(1+\frac{c}{(N+M)^\alpha}\right) = \left(1+\frac{c}{(N+M)^\alpha}\right)^{z-s} \geqq \exp\left(\frac{c(z-s)}{2(N+M)^\alpha}\right).$$

Hence

$$\exp\left[\frac{c(z-s)}{2(N+M)^\alpha}\right] \leqq b/a; \quad z-s+1 \leqq \frac{2(N+M)^\alpha}{c}\ln b/a+1.$$

Clearly, it will suffice to find an estimate for the number of solutions of inequality (5) satisfying the condition

$$j_1 \geqq j_2 \geqq j_3 \geqq j_4.$$

If $j_1=j_2$ then, since $q>2^{p+1}$, (5) can be satisfied only if $j_3=j_4$. Consequently, there are at most N^2 solutions.

Suppose $j_1>j_2$, and evaluate the number of solutions of (5) such that

$$(*) \qquad 2^k \leqq n_{j_2}/n_{j_3} < 2^{k+1}$$

for a fixed k. We have N different possibilities for n_{j_1}. Since

$$n_{j_2}+n_{j_3}+n_{j_4}+2^p \leqq n_{j_2}+3n_{j_3} \leqq n_{j_2}(1+3\cdot 2^{-k}),$$

inequality (5) is solvable if

$$n_{j_1} \leqq n_{j_2}(1+3\cdot 2^{-k}).$$

Thus, n_{j_2} lies in the interval $[n_{j_1}(1+3\cdot 2^{-k})^{-1}, n_{j_1}]$; consequently, it can take at most

$$\frac{2(N+M)^\alpha}{c}\ln(1+3\cdot 2^{-k})+1 < \frac{2(N+M)^\alpha}{c}3\cdot 2^{-k}+1$$

different values. If

$$2c^{-1}(N+M)^\alpha 3\cdot 2^{-k} < 1,$$

then n_{j_2} can be chosen uniquely. Therefore $n_{j_2}=n_{j_1}$ holds, but this case has been excluded. Hence

$$6c^{-1}2^{-k}(N+M)^\alpha > 1,$$

i.e.

$$k \leqq A_8 \ln(N+M).$$

Using again $(*)$, we can see that n_{j_3} belongs to the interval $[n_{j_2}2^{-k-1}, n_{j_2}]$, whence it can be chosen in at most

$$2c^{-1}(N+M)^\alpha \ln 2^{k+1}+1 = 2c^{-1}(N+M)^\alpha(k+1)+1$$

different ways. Finally, the choice of $n_{j_1}, n_{j_2}, n_{j_3}$ being made, for the value of n_{j_4} there are left no more than four possibilities (differing in the choice of the signs).

We have obtained that inequality (5) has at most

$$N[c^{-1}3(N+M)^\alpha 2^{-k+1}+1]\,[2c^{-1}(N+M)^\alpha(k+1)+1]4$$

solutions under the condition $(*)$. Summing up for all $k \leqq A_8 \ln(N+M)$ we obtain that the full number of solutions does not exceed

$$A_9 N(N+M)^{2\alpha}+N^2 \leqq A_9 N(N+M).$$

Furthermore, squaring the sum

$$\Big|\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big|^2 = \Big[\sum_{k=M+1}^{N+M} c_k e^{in_k\vartheta}\Big]\Big[\sum_{k=M+1}^{N+M} \bar{c}_k e^{-in_k\vartheta}\Big]$$

and interchanging the order of summation and integration, we have

$$\sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} c_{j_1}\bar{c}_{j_2}c_{j_3}\bar{c}_{j_4}\int_{-\pi}^{\pi} e^{i(n_{j_1}\pm n_{j_2}\pm n_{j_3}\pm n_{j_4})\vartheta}\left[\frac{\sin 2^{p-1}\vartheta}{2^{p-1}\sin\vartheta}\right]^2 d\vartheta.$$

In virtue of what has been just shown, all summands but at most $N(N+M)$ ones are equal to 0, and for the rest we have the upper bound

$$\max_k |c_k|^4 \int_{-\pi}^{\pi}\left|\frac{\sin 2^{p-1}\vartheta}{2^{p-1}\sin\vartheta}\right|^2 d\vartheta = \frac{\pi}{2^{p-1}}\max_k |c_k|^4.$$

This proves the Lemma.

Applying a method used in [8], one can prove an even stronger form of our Lemma.

Lemma A. *Consider the finite sum*

$$P(\vartheta) = \sum_{k=1}^{N} c_k e^{in_k\vartheta}$$

where

$$n_{k+1}-n_k \geqq q > 0, \quad n_1 > q, \quad n_{k+1}/n_k > 1+c/k^\alpha, \qquad 0 \leqq \alpha < 1,$$

and let I be an interval of length $|I|>2\pi q^{-1}$. *If*

$$|c_k| \leqq \frac{\tilde{c}}{k^\alpha}\Big(\sum_{m=1}^{k}|c_m|^2\Big)^{1/2} \quad (k = 1, 2, \dots), \tag{6}$$

then

$$\left\{\frac{1}{|I|}\int_I\Big|\sum_{k=1}^{N} c_k e^{in_k\vartheta}\Big|^4 d\vartheta\right\}^{1/4} \leqq A_{10}\Big(\sum_{k=1}^{N}|c_k|^2\Big)^{1/2},$$

where A_{10} depends only on $\alpha, q, c, \tilde{c}$.

Note that, as it was shown in [9], condition (6) cannot be dropped even for $I=[-\pi,\pi]$. For every $\alpha\in(0,1)$ there is a weakly lacunar (with exponent α) convex sequence $\{n_k\}$ and a function φ with Fourier series $\sum_k c_k e^{in_k\vartheta}$ such that $\varphi\in L^2(-\pi,\pi]$, but $\varphi\notin L^4(-\pi,\pi]$.

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О суммируемости рядов Фурье методом средних арифметических с пропусками

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В работе устанавливается следующий результат. Пусть f—непрерывная функция, $S_n(f)$—частные суммы порядка n ее ряда Фурье. Для каждой выпуклой последовательности $\{n_k\}$, удовлетворяющей условию $n_{k+1}/n_k > 1 + ck^{-\alpha}$, c—положительная постоянная, $0 \leqq \alpha < 1/2$, существует такая непрерывная функция f_0, что средние $\frac{1}{N} \sum_{k=1}^{N} S_{n_k}(f_0)$ не сходятся равномерно.

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ПРОСПЕКТ ПАНФИЛОВА 1
ГЛАВНЫЙ ВЫЧИСЛИТЕЛЬНЫЙ ЦЕНТР МУП УССР