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ON THE ϱ-SYSTEMS OF CIRCLES

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Let $\{C_i\}$ be a packing of cirles in the Euclidean plane. A circle C is said to be a *supporting circle* of the circle system $\{C_i\}$ if it has no common interior point with $\{C_i\}$ and touches at least three circles of $\{C_i\}$. If ϱ is the greatest lower bound of the radii ϱ^* of all supporting circles of $\{C_i\}$ and if $\varrho=\inf \varrho^*>0$, then $\{C_i\}$ is called a *ϱ-system of circles.*

The *density of a circle system* $\{C_i\}$ *with respect to the Euclidean plane* is defined by

$$\delta = \overline{\lim_{R\to\infty}} \frac{\sum\limits_i (C_i\cap C(R))}{C(R)}\ ^{1}$$

where $C(R)$ is a circle of radius R centred at a fixed point O of the plane.[2]

Subsequent to the investigations of MOLNÁR ([10], [11]), concerning ϱ-systems of circles, we prove the following

THEOREM.[3] *If d denotes the density of a packing in the Euclidean plane by a ϱ-system of circles of radii contained in the interval* $[\varepsilon, 1]$, *where* $\varepsilon>0$, *then*

$$d \leqq \frac{\arccos \frac{1}{1+\varrho}}{\sqrt{2\varrho+\varrho^2}}.$$

Equality holds if $\varrho=\frac{2\sqrt{3}}{3}-1$, $\sqrt{2}-1$ *and* 1 *and the ϱ-system consists only of unit circles.*[4]

Consider three circles of radii 1, 1, ϱ and centres A, B, C mutually touching one another (Fig. 4). Then $d(\varrho)=\frac{\arccos \frac{1}{1+\varrho}}{\sqrt{2\varrho+\varrho^2}}$ is the density of the unit circles in the triangle ABC, namely the ratio of the area of the part of the triangle ABC covered by the unit circles to the area of the whole triangle.

[1] We denote a domain and its area by the same symbol.

[2] It is easy to see that δ does not depend on the choice of O; see FEJES TÓTH [1].

[3] Attention should also be drawn to the quadrilateral tessellation and the lemmas employed in the proof of this theorem which may be useful for future density investigations. Lemmas 5 and 6 are due to Hárs and Florian respectively, the remaining part of the article is the work of Molnár.

[4] See Fig. 1, 2 and 3.

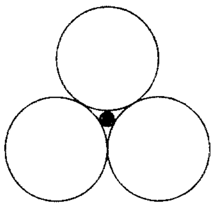

Fig. 1

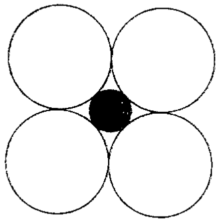

Fig. 2

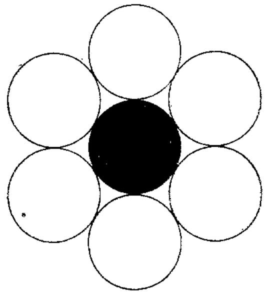

Fig. 3

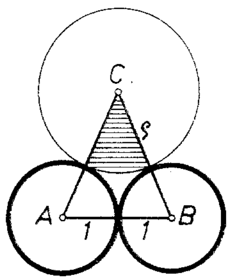

Fig. 4

Without loss of generality we may suppose that the packing of the ϱ-system of circles is saturated. We shall construct a tessellation with quadrilateral faces, the vertices of which are alternatively centres $O_1, O_2, \ldots$ of the circles $C_1, C_2, \ldots$ and centres $V_1, V_2, \ldots$ of the supporting circles of $\{C_i\}$. In order to prove our assertion we shall show that in each quadrangle of the tessellation the density of $\{C_i\}$ does not exceed $d(\varrho)$.

We introduce the notion of the (algebraic) *distance* $d(P, C)=\overline{OP}-r$ of a point P from a circle C of radius r centred at O. Let us associate with any circle C_i the set S_i of all points P lying "nearer" to C_i than to any other circle C_j, i.e. $d(P, C_i)<d(P, C_j)$ $(j\neq i)$.[5] It is not difficult to show that S_i is a star region with respect to the pole O_i (Fig. 5). The star regions $\{S_i\}$ are bounded by arcs of hyperbolae and segments of straight lines.

Obviously the star regions $S_1, S_2, \ldots$ constitute a tessellation S. Joining the centre O_i $(i=1, 2, \ldots)$ with the vertices $V_1, V_2, \ldots$ of the corresponding star region S_i, we obtain a new tessellation T with quadrilateral faces (Fig. 6).

We proceed to show that in each quadrilateral face (quadrangle) of T the density of $\{C_i\}$ is $\leqq d(\varrho)$.

To prove this statement we need a certain number of lemmas.

[5] See FEJES TÓTH—MOLNÁR [2].

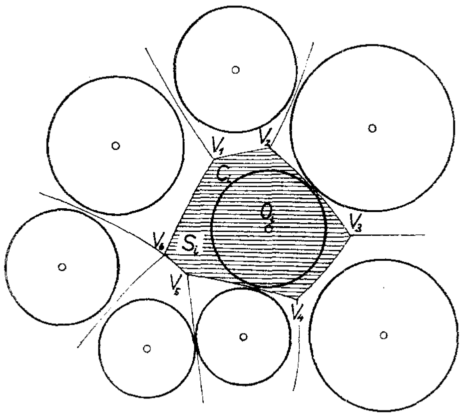

Fig. 5

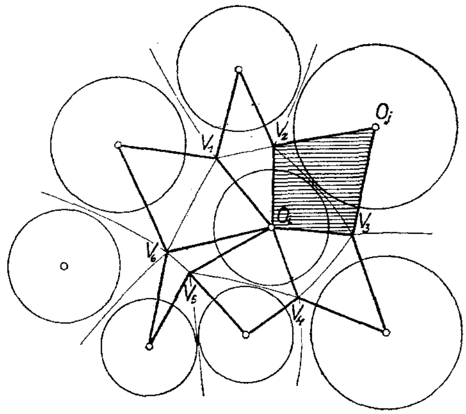

Fig. 6

LEMMA 1. *Let AOB be a triangle of $\sphericalangle OAB \geqq \frac{\pi}{2}$. If M is the midpoint of the side AB then $\sphericalangle AOM > \sphericalangle BOM$.*

PROOF. The condition $\sphericalangle OAB \geqq \frac{\pi}{2}$ implies $\overline{OB} > \overline{OA}$ (Fig. 7). Let O^* be the mirror point of O with respect to M. Considering the triangle OO^*B, we have $\sphericalangle BOM < \sphericalangle BO^*M = \sphericalangle AOM$ in consequence of $\overline{O^*B} = \overline{OA} < \overline{OB}$.

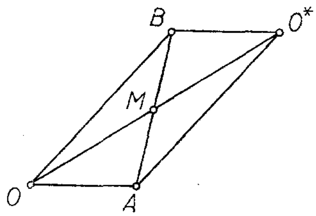

Fig. 7

Let O_1AO_2B be a simple quadrangle and let C_1, C_2 be two circles of centres O_1, O_2. Denote by $C_1(AO_1B)$ and $C_2(AO_2B)$ the sectors of the circles C_1 and C_2 corresponding to the angles $\sphericalangle AO_1B$ and $\sphericalangle AO_2B$ of O_1AO_2B. We define the *density of the circles C_1, C_2 with respect to the quadrangle O_1AO_2B* by

$$d_{12}(AB) = \frac{C_1(AO_1B) + C_2(AO_2B)}{O_1AO_2B}. \tag{1}$$

LEMMA 2. *In the Euclidean plane, consider two circles C_1, C_2 of centres O_1, O_2 and of radii r_1, r_2 $(r_1 < r_2)$, resp. Let A, B be two different points, both at the same distance from C_1 and C_2 and on the same side of the straight line O_1O_2. If $\sphericalangle BAO_1 \geqq \frac{\pi}{2}$ (hence $\overline{O_1A} < \overline{O_1B}$), then for any point P of the segment AB we have $d_{12}(AP) \geqq d_{12}(PB)$.*

PROOF.[6] We first remark that A, B are points on the same branch of the hyperbola H of foci O_1, O_2, the length of the transverse axis is r_2-r_1. Since the line AB is a secant of H and $\sphericalangle BAO_1 \geqq \frac{\pi}{2}$ we have also $\sphericalangle BAO_2 \geqq \frac{\pi}{2}$ (Fig. 8).

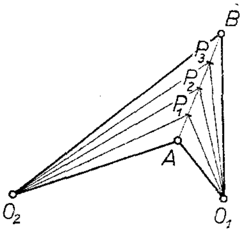

Fig. 8

Let $P_1, P_2, \dots, P_n$ be equidistant points on AB, i.e. $\overline{AP_1}=\overline{P_1P_2}=\dots=\overline{P_{n-1}P_n}=\overline{P_nB}$. In view of Lemma 1, the angles $\sphericalangle AO_1P_1=\alpha_1$, $\sphericalangle P_1O_1P_2=\alpha_2, \dots$, $\sphericalangle P_nO_1B=\alpha_{n+1}$ and $\sphericalangle AO_2P_1=\beta_1$, $\sphericalangle P_1O_2P_2=\beta_2, \dots$, $\sphericalangle P_nO_2B=\beta_{n+1}$ form two decreasing sequences. On the other hand, the quadrangles $O_1AO_2P_1$, $O_1P_1P_2P_2$, $\dots$, $O_1P_nO_2B$ have all the same area. Therefore, employing the notation introduced in (1), we see that the sequence $d_{12}(AP_1)$, $d_{12}(P_1P_2)$, $\dots$, $d_{12}(P_nB)$ decreases monotonically. Consequently we get

$$d_{12}(AP_i) \geqq d_{12}(P_{i-1}P_i) > d_{12}(P_iP_{i+1}) \geqq d_{12}(P_iB) \quad (i=1, \dots, n).$$

But the inequality $d_{12}(AP_i)>d_{12}(P_iB)$ is true for any n and $i=1, \dots, n$. This concludes the proof of Lemma 2.

Obviously, $d_{12}(AP) \geqq d_{12}(PB)$ implies $d_{12}(AP) \geqq d_{12}(AB)$.

LEMMA 3. *Let H be a hyperbola branch and F the focus lying in the convex domain bounded by H. Let us denote by H^* one of the half branches of H determined by the transverse axis of H. The circle of diameter FP, where P is a point of H^*, has at most one further common point with H^*.*

PROOF. Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the equation of the hyperbola H and let H^* be the half branch of H lying in the first quadrant of the coordinate system (Fig. 9). Let $F(c, 0)$ be the corresponding focus of H and $P(\lambda, \mu)$ a point of H^*. The equation of the circle C with diameter FP is

$$x^2+y^2-(\lambda+c)x-\mu y+\lambda c=0.$$

The abscissae of the common points of C and H satisfy the equation

$$f(x) \equiv \frac{c^2}{a^2}x^2-(\lambda+c)x-\frac{a}{b}\mu\sqrt{x^2-a^2}+\lambda c-b^2=0.$$

[6] A different proof which does not make use of Lemma 1, was given later by A. Florian.

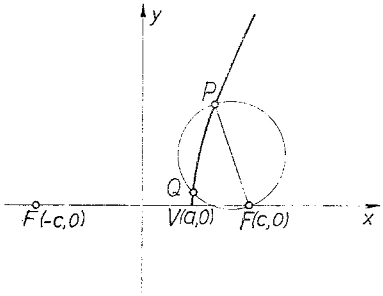

Fig. 9

Obviously, $f(x)$ is a strictly convex function for $x \geqq a$ which vanishes at $x=\lambda$. Now we distinguish two cases:

(i) $\lambda > a$ implies

$$f'(\lambda) = \lambda \frac{c^2}{a^2} - c > \frac{c}{a}(c-a) > 0$$

and $f(a)=(\lambda-a)(c-a)>0$. Therefore, the function $f(x)$ has precisely two zeros in the interval $x>a$, the greater of which is λ. Let $V=(a, 0)$ be the vertex of H^*. If $\lambda>a$ and consequently $P\neq V$ then C intersects H^* in exactly two points, namely P and $Q\neq P$, where Q lies between P and V. From this we deduce immediately that any point P^* on the open arc PQ of H^* has the property $\sphericalangle PP^*F > \frac{\pi}{2}$.

(ii) For $\lambda=a$ the function $f(x)$ vanishes only at $x=a$ and $x=\frac{a(ac-b^2)}{c^2}$; but $ac-b^2<c^2$. It follows that, if $P=V$, the circle C and the hyperbola half branch H^* touch each other at V and do not have any other point in common.

LEMMA 4. *In the Euclidean plane, consider two non-overlapping circles C_1, C_2 of centres O_1, O_2 and radii r_1, r_2 $(r_1<r_2)$. Let A, B be two different points, both equidistant from C_1 and C_2 and on the same side of the straight line O_1O_2. Let H be that branch of the hyperbola of foci O_1, O_2 having the length r_2-r_1 of the transverse axis which contains A, B. If $\overline{O_1A}<\overline{O_1B}$, then for any interior point P of the arc AB of H we have, using the notation* (1),

$$d_{12}(AP) \geqq d_{12}(PB). \tag{2}$$

PROOF. It suffices to prove the lemma under the assumption that A is not the vertex V of H, carrying out the limiting process $A\to V$ in the other case.

We can find, on the basis of Lemma 3, on the open arc AB of H a sequence of points $P_1, P_2, \ldots, P_n$, so that the angles $\sphericalangle P_1AO_1$, $\sphericalangle P_2P_1O_1, \ldots, \sphericalangle BP_nO_1$ are obtuse. A sequence of this property we call admissible. Since the tangent at any point P of H is the bisecting line of $\sphericalangle O_1PO_2$, the angles $\sphericalangle P_1AO_2$, $\sphericalangle P_2P_1O_2$, $\ldots, BP_nO_2$ are obtuse too (Fig. 10).

We shall first prove the inequality

$$\lim_{M\to P_i} d_{12}(MP_i) > \lim_{N\to P_i} d_{12}(P_iN) \quad (i=1, \dots, n)$$

where M and N are pointson the segments $P_{i-1}P_i$ and P_iP_{i+1} respectively (Fig. 11).

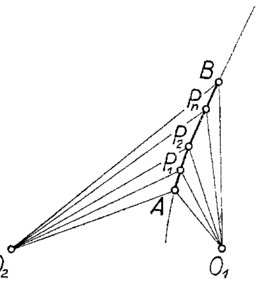

Fig. 10

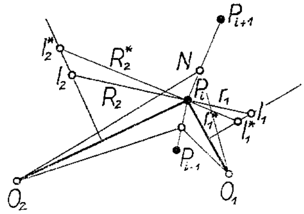

Fig. 11

Write $\sphericalangle MO_1P_i=\varepsilon_1$, $\sphericalangle MO_2P_i=\varepsilon_2$, $\sphericalangle NO_1P_i=\varepsilon_1^*$, $\sphericalangle NO_2P_i=\varepsilon_2^*$ and denote by I_1, I_1^* and I_2, I_2^* the intersections of the perpendicular bisector of the segment O_1P_i and O_2P_i respectively, with the straight lines perpendicular to $P_{i-1}P_i$ and P_iP_{i+1} at P_i.

Taking into account that $\overline{P_iI_1}=R_1>\overline{P_iI_1^*}=R_1^*$, $\overline{P_iI_2}=R_2<\overline{P_iI_2^*}=R_2^*$ and that

$$\frac{r_2^2}{r_1^2} > \frac{\overline{O_2P_i}^2}{\overline{O_1P_i}^2} = \frac{(r_2+\varrho)^2}{(r_1+\varrho)^2}$$

it is easy to see that

$$\lim_{M\to P_i} d_{12}(MP_i) \geqq \lim_{\varepsilon_1\to 0} \frac{\frac{\varepsilon_1}{\varepsilon_2} r_1^2+r_2^2}{\frac{\varepsilon_1}{\varepsilon_2}\overline{O_1P_i}^2+\overline{O_2P_i}^2} = \frac{\frac{R_2}{R_1} r_1^2+r_2^2}{\frac{R_2}{R_1}\overline{O_1P_i}^2+\overline{O_2P_i}^2} >$$

$$> \frac{\frac{R_2^*}{R_1^*} r_1^2+r_2^2}{\frac{R_2^*}{R_1^*}\overline{O_1P_i}^2+\overline{O_2P_i}^2} = \lim_{\varepsilon_1^*\to 0} \frac{\frac{\varepsilon_1^*}{\varepsilon_2^*} r_1^2+r_2^2}{\frac{\varepsilon_1^*}{\varepsilon_2^*}\overline{O_1P_i}^2+\overline{O_2P_i}^2} \geqq \lim_{N\to P_i} d_{12}(P_iN).$$

Therefore, and in view of Lemma 2, we obtain

$$d_{12}(AP_1) > d_{12}(P_1P_2) > \dots > d_{12}(P_nB),$$

whence

$$d_{12}(AP_i) \geqq d_{12}(P_{i-1}P_i) > d_{12}(P_iP_{i+1}) \geqq d_{12}(P_iB) \quad (i=1, \dots, n).$$

Let P be an arbitrary interpolating point on the open arc $P_{i-1}P_i$ of H. Recalling the property $\sphericalangle P_iP_{i-1}O_1>\frac{\pi}{2}$, we note that $\sphericalangle PP_{i-1}Q_1>\frac{\pi}{2}$ and, according to Lemma 3, $\sphericalangle P_iPO_1>\frac{\pi}{2}$. Thus the sequence $P_1, \dots, P_{i-1}, P, P_i, \dots, P_n$ is also admissible and the inequality (2) is shown.

Let ABC be a triangle where the lengths of the sides AC and BC are supposed to be fixed. The notation is chosen so that $\overline{AC} \leqq \overline{BC}$. We draw attention to the density

$$\delta(x) = \frac{\lambda\alpha + \mu\beta + \nu x}{\frac{1}{2}\overline{AC} \cdot \overline{BC} \cdot \sin x}$$

where x indicates the angle enclosed by AC and BC (Fig. 12). Herein λ, μ, ν denote non-negative constants, not all of which are zero. $\delta(x)$ represents the ratio of a weighted sum of the angles to the area of the triangle ABC.

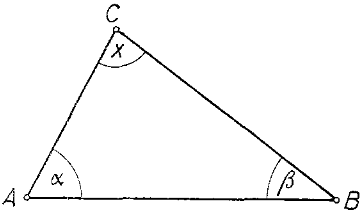

Fig. 12

LEMMA 5.[7] *Let us vary the angle x of the triangle ABC, so that $0 \leqq x \leqq \pi$. Then, in any subinterval of $(0, \pi)$, $\delta(x)$ attains its maximum at one of the endpoints.*

Functions having this property we shall call in the following *quasiconvex*. For the sake of simplicity let us consider, instead of $\delta(x)$, its constant multiple

$$S(x) = \frac{\lambda\alpha + \mu\beta + \nu x}{\sin x}. \tag{3}$$

We remark that $S(x)$ is continuous in $(0, \pi)$.

Making use of the cosine theorem and introducing the notation $\overline{AC}/\overline{BC} = p$ $(p \leqq 1)$, we have

$$\cos\alpha = \frac{p - \cos x}{\sqrt{1 + p^2 - 2p\cos x}}. \tag{4}$$

Differentiation yields

$$\alpha' = \frac{d\alpha}{dx} = \frac{p\cos x - 1}{1 + p^2 - 2p\cos x}, \tag{5}$$

and, in view of $\alpha + \beta + x = \pi$,

$$\beta' = \frac{d\beta}{dx} = -1 - \alpha' = \frac{p\cos x - p^2}{1 + p^2 - 2p\cos x}. \tag{6}$$

[7] See L. HÁRS [4]. In a previous paper [3] A. FLORIAN proved a more special result in a similar way.

Putting, for brevity's sake, $\lambda\alpha+\mu\beta+\nu x=y$, we obtain

$$y'=\frac{-[(\mu-\nu)p^2+\lambda-\nu]+p\cos x[\lambda+\mu-2\nu]}{1+p^2-2p\cos x} \tag{7}$$

and, with the notation

$$A=(\mu-\nu)p^2+\lambda-\nu;\quad B=p(\lambda+\mu-2\nu), \tag{8}$$

$$y'=\frac{-A+B\cos x}{1+p^2-2p\cos x}. \tag{9}$$

Differentiating once again, we have

$$y''=\frac{p(p^2-1)(\mu-\lambda)\sin x}{(1+p^2-2p\cos x)^2} \tag{10}$$

and, owing to (3),

$$S'=\frac{y'\sin x-y\cos x}{\sin^2 x}=\frac{P}{\sin^2 x}. \tag{11}$$

We observe that S' has the same sign as P. The function

$$Q=\frac{y'\sin x-y\cos x}{\cos x}=\frac{P}{\cos x}=y'\operatorname{tg} x-y \tag{12}$$

is continuous on the set $[0,\pi/2)\cup(\pi/2,\pi]$ and has the values

$$Q(0)=-y(0)\leqq 0,\quad Q(\pi)=-y(\pi)\leqq 0. \tag{13}$$

Since $\cos x>0$ in $[0,\pi/2)$ and $\cos x<0$ in $(\pi/2,\pi]$ we can state that

a1) for $x<\pi/2$ S is increasing if $Q>0$,
a2) S is decreasing if $Q<0$,
b1) for $\pi/2<x\leqq\pi$ S is decreasing if $Q>0$,
b2) S is increasing if $Q<0$.

To examine the sign of Q it will be useful to see whether Q is increasing or decreasing in a given interval. For this purpose we shall need its derivative

$$Q'=\frac{p(p^2-1)(\mu-\lambda)\sin^2 x}{(1+p^2-2p\cos x)^2\cos x}+\frac{(-A+B\cos x)\sin^2 x}{(1+p^2-2p\cos x)\cos^2 x}.$$

Since

$$\operatorname{sgn} Q'=\operatorname{sgn}\left[\frac{\cos^2 x}{\sin^2 x}(1+p^2-2p\cos x)^2 Q'\right]$$

in $(0,\pi/2)$ and $(\pi/2,\pi)$ we have to consider the function

$$R(x)=\frac{\cos^2 x}{\sin^2 x}(1+p^2-2p\cos x)^2 Q'=$$

$$=-2pB\cos^2 x+[(p^2-1)(\mu-\lambda)p+2Ap+B(1+p^2)]\cos x-A(1+p^2).$$

But, by (8), the coefficient of $\cos x$ is $4pA$, so that we finally have

(14) $$R(x) = -2pB\cos^2 x + 4pA\cos x - A(1+p^2).$$

Obviously, $R(x)$ is a polynomial in $\cos x$ of degree $\leqq 2$. Denoting it by F

$$F(z) = -2pBz^2 + 4pAz - A(1+p^2)$$

then $R(x) = F(\cos x)$. The discriminant of F is

(15) $$D = 8Ap^2(p^2-1)(\mu-\lambda).$$

In proving S to be quasiconvex, we have to distinguish several cases and subcases.

I. $B=0$. If also $A=0$, then from (9) it follows that $y'=0$ and from (11) that $\operatorname{sgn} S' = -\operatorname{sgn}(\cos x)$. Therefore, S is decreasing for $x<\pi/2$ and increasing for $x>\pi/2$, which means that S is quasiconvex.

If, however, $A\neq 0$, then $F(z) = A[4pz-(p^2+1)]$ is a linear polynomial in z having the root $\dfrac{p^2+1}{4p}>0$.

I.1. $A>0$. For $\cos x = z \leqq 0$ we have $R(x)<0$ by (14). More generally, if $R(x)\leqq 0$ for $x>\pi/2$ (Q is decreasing) or $R(x)<0$ for $\pi/2<x<x_1$ and $R(x)>0$ in $x_1<x<\pi$ with any $x_1\in(\pi/2, \pi)$ (Q is decreasing in $(\pi/2, x_1)$ and increasing in (x_1, π)), we shall refer to it as *case* c). Since $Q(\pi)\leqq 0$ by (13), in this case Q is either negative or positive in the whole interval $(\pi/2, \pi)$, or positive in a certain subinterval $(\pi/2, x_0)$ and negative in (x_0, π). Then we can state that:

in the first case (case b2)) S is increasing,

in the second case (case b1)) S is decreasing and

in the third case (case b1) in $(\pi/2, x_0)$ and case b2) in (x_0, π)) S is decreasing in $(\pi/2, x_0)$ and increasing in (x_0, π).

If we can show, moreover, that for $x<\pi/2$ we have $S'(x)\leqq 0$ (this will be supposed to hold in case c)) then S follows to be quasiconvex.

In fact, for $0<x<\pi/2$ $\cos x$ and $\sin x$ are positive, so that, in view of (11), it will be sufficient to verify the inequality $y'<0$. But this is trivial by (9) and $B=0$, $A>0$.

I.2. $A<0$. Then we have $R(\pi/2)>0$. More generally, if $R(x)\geqq 0$ for $0<x<\pi/2$ (Q is increasing) or $R(x)<0$ in $(0, x_2)$ and $R(x)>0$ in $(x_2, \pi/2)$ with any $x_2\in(0, \pi/2)$ (Q is decreasing in $(0, x_2)$ and increasing in $(x_2, \pi/2)$), we shall refer to it s *case* d). Since $Q(0)\leqq 0$ by (13) Q is, in this case, either negative or positive in the whole interval $(0, \pi/2)$, or negative in a certain subinterval $(0, x_0)$ and positive in $(x_0, \pi/2)$. Therefore, we again have to distinguish three cases here:

In the first case (case a2)) S is decreasing

in the second case (case a1)) S is increasing and

in the third case S is decreasing in $(0, x_0)$ and increasing in $(x_0, \pi/2)$.

It is easy to see that if for $x>\pi/2$ we have $S'(x)\geqq 0$ (this is supposed to be valid in case d)), then S is proved to be quasiconvex.

But now $\cos x<0$, $\sin x>0$ and, by (11), we have only to show that $y'>0$. This inequality follows from (9) in view of $A<0$, $B=0$.

From now on we can suppose that $B \neq 0$.

II. $D \leqq 0$ (see (15)). In this case the polynomial F does not change its sign.

II.1. $R \leqq 0$ in $(0, \pi)$. Combining $Q'(x) \leqq 0$ in $(0, \pi/2)$ with $Q(0) \leqq 0$ by (13), we find that $Q(x) \leqq 0$ and, by (12) and (11), that also $S'(x) \leqq 0$ for $x \leqq \pi/2$. Since case c) is realized here the function $S(x)$ turns out to be quasiconvex.

II.2. $R \geqq 0$ in $(0, \pi)$. Combining $Q'(x) \geqq 0$ in $(\pi/2, \pi)$ with $Q(\pi) \leqq 0$ by (13) we get $Q(x) \leqq 0$ and, owing to (12) and (11), $S'(x) \geqq 0$ for $x \geqq \pi/2$. Since the conditions of case d) are fulfilled, the function $S(x)$ is quasiconvex.

Consequently, in the following we shall confine ourselves to the more complicated case $D > 0$.

III. $D > 0$. This assumption ensures that $p < 1$ and

$$A(\mu-\lambda) < 0, \tag{16}$$

as can be seen from (15). The quadratic equation $F(z) = 0$ has exactly two different real roots

$$z_{+,-} = \frac{-2A \pm \sqrt{2A(p^2-1)(\mu-\lambda)}}{-2B}. \tag{17}$$

Obviously, they have the same sign if and only if

$$AB > 0. \tag{18}$$

Consequently, we have to study four subcases corresponding to the signs of A and B.

III.1. $A > 0$, $B > 0$. The graph of $F(z)$ is exhibited in Fig. 13a. Since $z_+ + z_- = 2\frac{A}{B}$ we obtain $z_+, z_- > 0$.

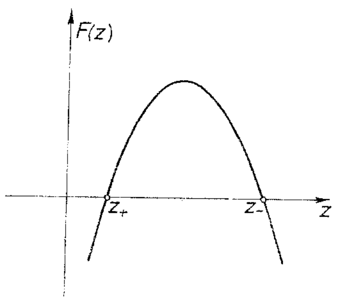

Fig. 13a

We proceed to show that the conditions of case c) are satisfied. For $x \geqq \pi/2$ is $z = \cos x \leqq 0$, hence $F(z) < 0$ and $R(x) < 0$. Now let $x < \pi/2$, then by (11) $S'(x) \leqq 0$, provided $y' < 0$ or $-A + B\cos x < 0$. But

$$-A+B\cos x < -A+B = (p-1)[-(\mu-\nu)p+\lambda-\nu] \tag{19}$$

where the first factor is negative. As $A>0$ we deduce from (16) $\lambda>\mu$, whence $\lambda-\nu>\mu-\nu$. Since $B=p(\lambda+\mu-2\nu)>0$, we have $(\lambda-\nu)+(\mu-\nu)>0$ and therefore $\lambda-\nu>0$. Consequently, we obtain $\lambda-\nu>p(\mu-\nu)$, so that the second factor in (19) is positive and $-A+B\cos x<0$, according to our assertion.

III.2. $A>0$, $B<0$ (see Fig. 13b). Then $z_-<0<z_+$. Observing that trivially $-A+B\cos x<0$ or $S'(x)<0$ for $x\leqq\pi/2$, we state that there is case c) again.

III.3. $A<0$, $B>0$ (see Fig. 13c). Then $z_+<0<z_-$.

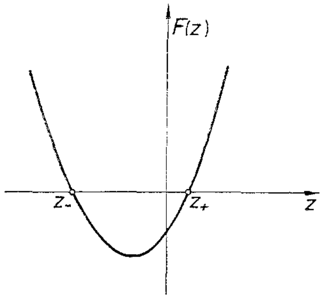

Fig. 13b

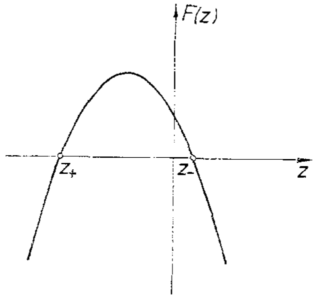

Fig. 13c

We shall show that the conditions of case d) are now fulfilled. To do this, we have yet to verify that $S'(x)>0$ or, owing to (11), that $y'\sin x-y\cos x>0$ for $x>\pi/2$. We have

$$(20)\quad y'\sin x-y\cos x=\frac{-A+p(\lambda+\mu-2\nu)\cos x}{1+p^2-2p\cos x}\sin x-(\lambda\alpha+\mu\beta+\nu x)\cos x>$$

$$>\left[\frac{\lambda+\mu-2\nu}{1+p^2-2p\cos x}p\sin x-\mu\beta\right]\cos x\geqq\left[\frac{\lambda+\mu-2\nu}{1+p^2-2p\cos x}p\sin x-\mu\sin\beta\right]\cos x,$$

the second factor being negative. On the other hand, we obtain, employing the sine theorem on the triangle ABC,

$$\sin\beta=\frac{p\sin x}{\sqrt{1+p^2-2p\cos x}},$$

hence the first factor on the right hand of (20) is

$$\frac{\lambda+\mu-2\nu}{1+p^2-2p\cos x}p\sin x-\mu\sin\beta=$$

$$=\frac{p\sin x}{1+p^2-2p\cos x}[(\lambda+\mu-2\nu)-\mu\sqrt{1+p^2-2p\cos x}].$$

It follows from (16) that $\mu>\lambda$ or $\mu-\nu>\lambda-\nu$. Since $0<B/p=(\mu-\nu)+(\lambda-\nu)$, we have $\mu-\nu>0$. On the other hand, $0>A=(\mu-\nu)p^2+(\lambda-\nu)$, whence $\lambda-\nu<0$ and $\lambda-2\nu<0$. Taking into account, further, that $1+p^2-2p\cos x>1$, we see that the expression in brackets is negative, and consequently the statement $y'\sin x-y\cos x>0$ for $x>\pi/2$ is true.

III.4. $A<0$, $B<0$ (see Fig. 13d). Then $0<z_-<z_+$.

We proceed to show that the assumptions of case d) are fulfilled again. For $x \geqq \pi/2$ is $y'>0$ by (9), hence $S'(x)>0$. Further, the vertex of the parabola $F(z)$ has the abszissa $z_0=\frac{A}{B}$. We claim that $z_0>1$. This inequality is equivalent to $-A+B>0$ or, by (19), to $-(\mu-\nu)p+(\lambda-\nu)<0$. It follows from (16) that $\lambda<\mu$ or $\lambda-\nu<\mu-\nu$. But $0>B/p=(\lambda-\nu)+(\mu-\nu)$, hence $\lambda-\nu<0$. Consequently, $-(\mu-\nu)p+(\lambda-\nu)<-(\mu-\nu)p+(\lambda-\nu)p=(\lambda-\mu)p<0$. Therefore, for the greater root of $F(z)$, $z_+>1$ holds, confirming our assertion.
Now, the proof of Lemma 5 is complete.

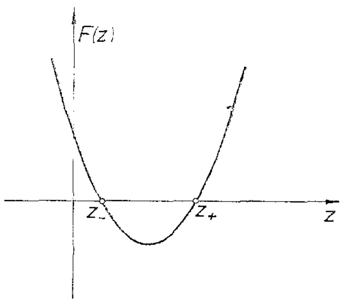

Fig. 13d

LEMMA 6. *Let C_a, C_1 and C_p be three circles of radii a $(0<a\leqq 1)$, 1 and p $(>0$, fixed), respectively, and mutually touching one another (Fig. 14). Then the density δ of C_a and C_1 with respect to the triangle Δ, determined by the centres of the three circles, attains its maximum only for $a=1$.*

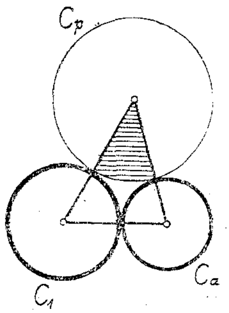

Fig. 14

PROOF. Obviously,

$$\delta=\frac{a^2\varphi_a+\varphi_1}{2\Delta}$$

where φ_a and φ_1 denote the central angles belonging to C_a and C_1. By elementary calculation we find

$$\sqrt{p}\delta = f(a, p) \equiv \frac{1}{\sqrt{a(a+p+1)}}\left[a^2 \operatorname{arctg}\sqrt{\frac{p}{a(a+p+1)}} + \operatorname{arctg}\sqrt{\frac{ap}{a+p+1}}\right].$$

To examine this function, we differentiate it partially and obtain

$$\frac{\partial f}{\partial a} = \frac{2a+p+1}{2[a(a+p+1)]^{3/2}} f_1(a, p)$$

with

$$f_1(a, p) = \frac{a^2(2a+3p+3)}{2a+p+1} \operatorname{arctg}\sqrt{\frac{p}{a(a+p+1)}} - \operatorname{arctg}\sqrt{\frac{ap}{a+p+1}} -$$

$$-\frac{a^2+(a+p)(a-1)}{(a+p)(2a+p+1)}\sqrt{ap(a+p+1)}.$$

Further differentiation yields

$$\frac{\partial f_1}{\partial a} = 2a\frac{4a^2+6ap+6a+3p^2+6p+3}{(2a+p+1)^2} f_2(a, p)$$

where

$$f_2(a, p) = \operatorname{arctg}\sqrt{\frac{p}{a(a+p+1)}} - \frac{1}{2}\,\frac{\sqrt{ap(a+p+1)}}{(a+1)^2(4a^2+6ap+6a+3p^2+6p+3)}\cdot$$

$$\cdot\left[\left(\frac{2a+p+1}{a+p}\right)^2(2a^2+3ap+3a+4p)+\frac{4a+p+3}{a}\right].$$

After some laborious calculations we obtain, putting $a^m p^n = (m, n)$,

$$4a(a+p)^3(a+1)^2(4a^2+6ap+6a+3p^2+6p+3)^2\sqrt{\frac{a(a+p+1)}{p}}\frac{\partial f_2}{\partial a} =$$

$$= f_3(a, p) \equiv -96\ (7,4) - 128\ (7,1) - 336\ (6,3) - 880\ (6,2) -$$
$$-320\ (6,1) - 456\ (5,4) - 1920\ (5,3) - 1688\ (5,2) - 288\ (5,1) -$$
$$-294\ (4,5) - 1918\ (4,4) - 2850\ (4,3) - 1298\ (4,2) - 104\ (4,1) -$$
$$-87\ (3,6) - 944\ (3,5) - 2086\ (3,4) - 1620\ (3,3) - 435\ (3,2) -$$
$$-12\ (3,1) - 9\ (2,7) - 216\ (2,6) - 658\ (2,5) - 700\ (2,4) -$$
$$-309\ (2,3) - 60\ (2,2) - 18\ (1,7) - 63\ (1,6) - 48\ (1,5) + 18\ (1,4) +$$
$$+18\ (1,3) - 3\ (1,2) + 3\ (0,7) + 18\ (0,6) + 36\ (0,5) + 30\ (0,4) + 9\ (0,3).$$

Since $\frac{\partial^2 f_3}{\partial a^2} < 0$, f_2 is a concave function of a. Note that $f_3(0, p) > 0$ and

$$f_3(1, p) = -24p^7 - 348p^6 - 1908p^5 - 5112p^4 - 7008p^3 - 4460p^2 - 852p < 0$$

for any positive p. Therefore, $f_3(a, p)$ and also $\frac{\partial f_2}{\partial a}$ passes from positive to negative values when a varies, increasing from 0 to 1.

We observe that

$$f_2(1,p) = \operatorname{arctg}\sqrt{\frac{p}{p+2}} - \frac{2p^3+14p^2+27p+13}{2(p+1)^2(3p^2+12p+13)}\sqrt{p(p+2)},$$

hence

$$\frac{d}{dp}f_2(1,p) = \frac{1}{2(p+1)^3(3p^2+12p+13)^2}\sqrt{\frac{p}{p+2}}\cdot$$

$$\cdot[21p^5+170p^4+527p^3+767p^2+496p+91] > 0.$$

Since $f_2(1, 0)=0$, we have $f_2(1, p)>0$. Combining this with $\lim_{a\to 0} f_2(a, p) = -\infty$, we deduce that $f_2(a, p)$ and also $\frac{\partial f_1}{\partial a}$ passes from negative to positive values when a increases. In view of $f_1(0, p)=0$ it follows that $f(a, p)$ as a function of a assumes its maximum only in a boundary point of the interval $0 \leqq a \leqq 1$. But it is easily proved, in a similar way as above, that

$$\frac{\sqrt{p+2}}{2}[f(1, p)-f(0, p)] = \operatorname{arctg}\sqrt{\frac{p}{p+2}} - \frac{\sqrt{p(p+2)}}{2(p+1)} > 0$$

for $p>0$. This completes the proof of Lemma 6.

Finally, it is very easy to prove the following two lemmas:[8]

LEMMA 7. *Let $\Delta_k = OTP_k$ $(k=1, 2)$ be right triangles ($\sphericalangle OTP_k = \pi/2$), where the sides TP_k do not have common interior points with the circle C of centre O (Fig. 15). If $\overline{OP_1} < \overline{OP_2}$ then*

$$\frac{C\cap\Delta_1}{\Delta_1} > \frac{C\cap\Delta_2}{\Delta_2}.$$

LEMMA 8. *Let $\Delta_k = OT_kP$ $(k=1, 2)$ be right triangles ($\sphericalangle OT_kP = \pi/2$), where the sides T_kP do not have common interior points with the circle C of centre O (Fig. 16). If $\sphericalangle T_1OP < \sphericalangle T_2OP$ then*

$$\frac{C\cap\Delta_1}{\Delta_1} < \frac{C\cap\Delta_2}{\Delta_2}.$$

Let us now return to the proof of our theorem.

For simplicity's sake, let us denote by $O_1V_1O_2V_2$ an arbitrary quadrangle of the tessellation T, where O_1, O_2 are the centres of the circles C_1, C_2 of $\{C_i\}$ and V_1, V_2 are the corresponding vertices of the tessellation S.

[8] See MOLNÁR [5] [6], [7], [8], [9].

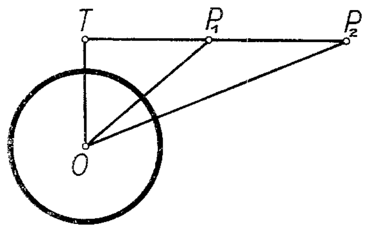

Fig. 15

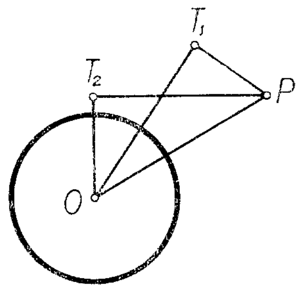

Fig. 16

We now proceed to show that in $O_1V_1O_2V_2$ the density of $\{C_i\}$ does not exceed $d(\varrho)$, i.e.

$$d_{12}(V_1V_2) \leqq d(\varrho) = \frac{\arccos\frac{1}{1+\varrho}}{\sqrt{2\varrho+\varrho^2}},$$

and distinguish the following two cases:

a) *$O_1V_1O_2V_2$ is convex.* In this case we decompose $O_1V_1O_2V_2$ into two triangles $\Delta_1 = O_1O_2V_1$ and $\Delta_2 = O_1O_2V_2$. Obviously, the inequality $d_{12}(V_1V_2) \leqq d(\varrho)$ is valid if we can show that

$$d_{12}(V_i) = \frac{C_1\cap\Delta_i + C_2\cap\Delta_i}{\Delta_i} \leqq d(\varrho) \quad (i = 1, 2),$$

and it suffices for $i=1$.

Consider three circles of radii r, r, ϱ^* and of centres A, B, C, respectively, mutually touching one another. We denote with $d(r, r, \varrho^*)$ the density of the circles of centres A, B with respect to the triangle ABC; obviously $d(r, r, \varrho^*) = d\left(1, 1, \frac{\varrho^*}{r}\right) = d\left(\frac{\varrho^*}{r}\right)$.

Let $\varrho^* \geqq \varrho$ be the radius of the supporting circle C centred at V_1 which touches C_1, C_2.

If the segment O_1O_2 has no common interior points with C (Fig. 17) then, in view of Lemma 5, $d_{12}(V_1)$ attains its maximum for one of the following configurations:

(i) C_1 and C_2 (radii r_1 and r_2, $r_1 \leqq r_2$) touch one another (Fig. 18). Making use of Lemma 6 and, if necessary, of Lemma 7, we obtain $d_{12}(V_1) \leqq d(r_2, r_2, \varrho^*) = d\left(1, 1, \frac{\varrho^*}{r_2}\right) \leqq d(\varrho)$.

(ii) The segment O_1O_2 touches C (Fig. 19). We draw the tangents from V_1 to C_1 and C_2 and denote the points of tangency with T_1 and T_2. In view of Lemma 8 we get

$$\frac{C_i\cap O_iPV_1}{O_iPV_1} < \frac{C_i\cap O_iT_iV_1}{O_iT_iV_1} = d(r_i, r_i, \varrho^*) \quad (i = 1, 2),$$

where P is the foot of the perpendicular from V_1 to O_1O_2. Thus $d_{12}(V_1) \leqq \leqq d(r_2, r_2, \varrho^*) \leqq d(\varrho)$.

If the segment O_1O_2 has common interior points with C, the inequality $d_{12}(V_1) \leqq \leqq d(\varrho)$ can be proved in the same way as in case (ii) of a).

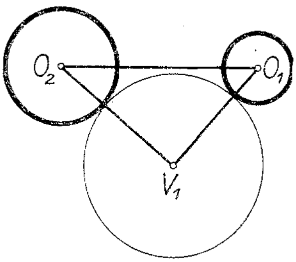

Fig. 17

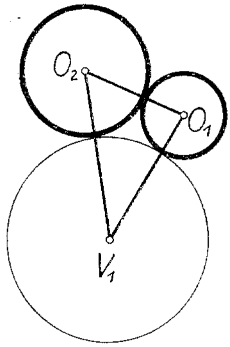

Fig. 18

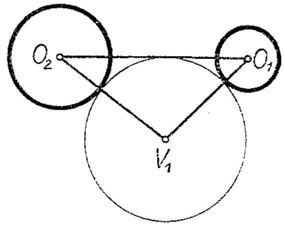

Fig. 19

b) $O_1V_1O_2V_2$ *is concave.* Let $\overline{O_1V_1} < \overline{O_1V_2}$ (Fig. 20), then by Lemma 4 we have $d_{12}(V_1V_2) \leqq d_{12}(V_1)$. But we have already seen that $d_{12}(V_1) \leqq d(\varrho)$.

This completes the proof of our statement that in each quadrangle of the tessellation T the density of $\{C_i\}$ is not greater than $d(\varrho)$. In order to deduce, finally, the inequality $d \leqq d(\varrho)$, we remark that, in view of $\sup r_i \leqq 1$, the circumradii of

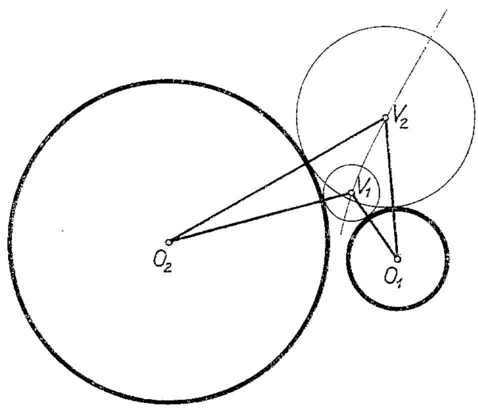

Fig. 20

the quadrangles of the tessellation T have also a finite upper bound b. Denoting by $Q_{ij} = O_iAO_jB$ the quadrangle of the tessellation T corresponding to the circles C_i, C_j and taking into account that

$$C_i(AO_iB) + C_j(AO_jB) \leqq d(\varrho)Q_{ij},$$

we obtain

$$\frac{1}{\pi R^2}\sum_i (C_i\cap C(R)) \leqq \frac{1}{\pi R^2}\sum_{C\cap C(R)\neq\emptyset} C_i \leqq \frac{d(\varrho)}{\pi R^2}\sum Q_{ij} \leqq$$

$$\leqq \frac{\pi(R+2+2b)^2}{\pi R^2} d(\varrho) = \left(1+\frac{2+2b}{R}\right)^2 d(\varrho).$$

From this the desired inequality $d \leqq d(\varrho)$ follows immediately.

REMARK. The Lemmas 2, 4, 5, 7, 8 continue to be valid whenever the "measure" of C_i is an arbitrary positive value $\varphi(r_i)$. The system of values $\{\varphi(r_i)\}$ associated to $\{C_i\}$ is called a functional system of $\{C_i\}$ and the corresponding density a functional density.

Lemma 6, however, is no longer valid for an arbitrary functional system. The case of a decreasing function $\varphi(r)$ yields a trivial counterexample. It is easy to give counterexamples also for certain increasing functions $\varphi(r)$. But it seems likely that Lemma 6 continues to hold for some particular functional systems of $\{C_i\}$.

References

[1] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum.* Zweite Auflage (Berlin—Heidelberg—New York, 1972).

[2] L. FEJES TÓTH and J. MOLNÁR, Unterdeckung und Überdeckung der Ebene durch Kreise, *Math. Nachrichten,* **18** (1958), 235—243.

[3] A. FLORIAN, Ausfüllung der Ebene durch Kreise, *Rend. Circ. Mat. Palermo,* **(2) 9** (1960), 300—312.

[4] L. HÁRS, Weighted circle systems (Thesis, in Hungarian), L. Eötvös University (Budapest, 1977).

[5] J. MOLNÁR, Collocazioni di cerchi sulla superficie di curvatura constante, *Celebrazioni archimedee del secolo XX, Siracusa, 11—16 aprile 1961,* pp. 61—72.

[6] J. MOLNÁR, Körelhelyezések állandó görbületű felületeken, *Magyar Tud. Akad. III. Oszt. Közl.,* **12** (1962), 223—263 (in Hungarian).

[7] J. MOLNÁR, Kreislagerungen auf Flächen konstanter Krümmung, *Math. Annalen,* **158** (1965), 365—376.

[8] J. MOLNÁR, Collocazioni di cerchi di Minkowski, *Rend. Sem. Mat. di Messina,* **10** (1966), 100—104.

[9] J. MOLNÁR, Aggregati di cerchi di Minkowski, *Annali di Matematica pura ed applicata,* **71** (1966), 101—108.

[10] J. MOLNÁR, On the ϱ-system of unit circles, *Annales Univ. Sci. Budapest, Sectio Math.,* **20** (1977), 195—203.

[11] J. MOLNÁR, Sur les empilements de ϱ-système de cercles dans un domaine convexe, *Coll. on Intuitive Geometry* (Tihany, 1975).

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