Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 34 (1—2), (1979), 141—155.

AN A.S. INVARIANCE PRINCIPLE FOR LACUNARY SERIES $f(n_k x)$

By

I. BERKES (Budapest) and W. PHILIPP (Urbana)

1. Introduction

Let $f(x)$ $(-\infty < x < +\infty)$ be a measurable function such that

$$f(x+1) = f(x), \quad \int_0^1 f(x)\,dx = 0. \tag{1.1}$$

It is well known that if $f(x)$ is smooth enough and the sequence $\{n_k\}$ of integers grows rapidly then the sequence $f(n_k x)$ of functions $(0 \leqq x \leqq 1)$ behaves like a sequence of independent random variables. A typical result in this direction (see [5], [8]) is that if f satisfies (1.1) and the Lipschitz condition and

$$n_{k+1}/n_k \to \infty \tag{1.2}$$

then $f(n_k x)$ obeys the central limit theorem and the law of the iterated logarithm (CLT and LIL in the sequel). Here (1.2) is best possible in the sense that it cannot be weakened to

$$n_{k+1}/n_k \geqq q > 1 \tag{1.3}$$

even with a large q. This is shown by the example of Erdős and Fortet (see [5])

$$f(x) = \cos 2\pi x + \cos 2\pi m x, \quad n_k = m^k - 1$$

for which both the central limit theorem and the law of iterated logarithm fail to hold. On the other hand, there exist many sequences $\{n_k\}$ satisfying (1.3) but not (1.2) such that $f(n_k x)$ satisfies the CLT and LIL. E.g. $n_k = 2^k$ is such a sequence (see [4]). It was Gaposhkin who characterized all the sequences $\{n_k\}$ (among the sequences obeying (1.3)) such that $f(n_k x)$ obeys the CLT. Let us say that a sequence $\{n_k\}$ of integers belongs to class

B_2 if there is a constant C such that the number of solutions of the equation $n_k \pm n_l = \nu$ $(k > l)$ is at most C for any integer $\nu > 0$;

D_m if the (set-theoretic) union of the sequences $\{n_k\}$, $\{2n_k\}$, …, $\{mn_k\}$, considered as a single sequence, belongs to class B_2;

D_∞ if $\{n_k\}$ belongs to class D_m for all integers $m = 1, 2, \ldots$.

Gaposhkin showed (see [3]) that if $\{n_k\}$ belongs to D_∞ (and satisfies (1.3)) then $f(n_k x)$ obeys the CLT for all sufficiently smooth f; on the other hand, if $\{n_k\}$ does not belong to D_∞ (but satisfies (1.3)) then there exists a trigonometric polynomial f such that $f(n_k x)$ fails to obey the CLT.

The purpose of the present paper is to extend (the positive half of) Gaposhkin's theorem and to prove an a.s. invariance principle for the sequence $f(n_k x)$ under the assumption that $\{n_k\}$ satisfies (1.3) and belongs to D_∞. Our method (which differs from that of Gaposhkin) makes use of martingale tools; in fact, it is a combination of the methods of [1—2], [6]. In [2], § 3 an a.s. invariance principle was proved for $f(n_k x)$ assuming a condition for $\{n_k\}$ (the so called Λ^* condition) which is slightly more stringent than D_∞. The present improvement (which is now best possible) is obtained by utilizing ideas from [6].

Our main result is the following:

THEOREM 1. *Let $f(x)$ $(-\infty < x < +\infty)$ satisfy* (1.1) *and the Lipschitz condition. Assume that $\{n_k\}$ satisfies* (1.3) *and belongs to class D_∞. Assume finally that there exists a constant $C_1 > 0$ such that for any $M \geqq 0$, $N \geqq N_0$ we have*

$$\int_0^1 \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \geqq C_1 N. \tag{1.4}$$

Let $S_N = \sum_{k=1}^{N} f(n_k x)$. Then the sequence $\{S_N, N \geqq 1\}$ can be redefined on a new probability space (without changing its distribution) together with a Wiener-process $\zeta(t)$ such that

$$S_N = \zeta(\tau_N) + o(N^{1/2-\lambda}) \quad a.s. \tag{1.5}$$

where $\lambda > 0$ is an absolute constant and τ_N is an increasing sequence of random variables such that $\tau_N / b_N \to 1$ a.s. where

$$b_N = \int_0^1 \left(\sum_{k=1}^{N} f(n_k x) \right)^2 dx. \tag{1.6}$$

Condition (1.4) in Theorem 1 cannot be omitted as it is shown by the example $f(x) = \cos 2\pi x - \cos 4\pi x$, $n_k = 2^k$ (cf. [5]).

Actually, the proof of Theorem 1 will yield the following result which gives some information about what happens if we replace D_∞ by D_m in Theorem 1.

THEOREM 2. *Let $\varepsilon > 0$. Then there exists an $m = m(\varepsilon, f)$ with the property that if we replace the condition $\{n_k\} \in D_\infty$ by $\{n_k\} \in D_m$ in Theorem* 1 *then the statement remains valid with the modification that for the random variable τ_N in* (1.5) *we have*

$$1 - \varepsilon \leqq \varliminf_{N \to \infty} \frac{\tau_N}{b_N} \leqq \varlimsup_{N \to \infty} \frac{\tau_N}{b_N} \leqq 1 + \varepsilon \quad a.s. \tag{1.7}$$

instead of $\tau_N / b_N \to 1$ a.s.

In other words, if $\{n_k\}$ belongs to D_m with a large m ("large" here depends also on f) then the conclusion of Theorem 1 remains "almost" valid.

It is easy to see (cf. [2], Lemmas (2.1), (2.2) and their proofs) that Theorem 1 implies Donsker's invariance principle (functional CLT) and the functional LIL for $f(n_k x)$. These limit theorems need not be valid under the conclusion of Theorem 2 but even under Theorem 2 we can state at least that $f(n_k x)$ obeys Donsker's invariance principle and the functional LIL "approximately". Roughly speaking, the smaller the ε in (1.7) is, the more precisely $f(n_k x)$ satisfies the above mentioned limit theorems. (For precise details of this statement via "ε-limit theorems" see [2], § 4.)

Let us say that a rational number $r>0$ is of order k if in the reduced form $r=p/q$ the greater of p and q is k. The following lemma is easy to prove (cf. the proof of Lemma (3.1) in [2]):

LEMMA (1.1). *The (set-theoretic) union of the sequences* $\{n_k\}, \{2n_k\}, \dots, \{mn_k\}$ *satisfies the Hadamard gap condition if and only if for any subsequences* n_{k_i}, n_{l_i} *and any rational number* $r>0$ *of order* $\leqq m$ *the relations*

$$\lim_{i\to\infty} \frac{n_{k_i}}{n_{l_i}} = r, \quad \frac{n_{k_i}}{n_{l_i}} \neq r \quad (i=1, 2, \dots)$$

are impossible.

The condition of Lemma (1.1) is satisfied e.g. if

a) $n_{k+1}/n_k > m \quad (k=1, 2, \dots)$,

b) $\lim\limits_{k\to\infty} n_{k+1}/n_k = \alpha$ where α is a rational number of order $> m$.

Since the Hadamard gap condition implies condition B_2, in examples a), b) the sequence $\{n_k\}$ belongs to D_m. For examples for sequences D_∞ see [3] or [2], § 3.

It follows from example a), Theorem 2 and our remarks above that if $\{n_k\}$ satisfies (1.3) with a large q then $f(n_k x)$ almost satisfies the CLT, the LIL and their functional versions (this was also proved in [2], § 4). Example b) shows that the same conclusion holds if n_{k+1}/n_k tends to a rational number of great order (e.g. to a rational number very close to an integer).

2. Two preparatory lemmas

In what follows, $\|f\|$ and $\|f\|_\infty$ will denote the L_2 and L_∞ norm of f, resp. For two numerical sequences a_n, b_n the relation $a_n \sim b_n$ means $\lim\limits_{n\to\infty} a_n/b_n = 1$.

LEMMA (2.1). *Let* $g(x)$ $(-\infty<x<+\infty)$ *be a measurable function such that*

$$g(x+1)=g(x), \quad \int_0^1 g(x)\,dx = 0.$$

Then for any real $a<b$ *and* $\lambda>0$ *we have*

$$\left|\int_a^b g(\lambda x)\,dx\right| \leqq \frac{2}{\lambda}\int_0^1 |g(x)|\,dx.$$

This is Lemma (3.2) of [1].

For the formulation of the next lemma we notice that if f satisfies (1.1) and the Lipschitz α condition then

$$\|f-s_n(f)\|_\infty \leqq An^{-\alpha/2} \tag{2.1}$$

where A is a positive constant and $s_n(f)$ denotes the n-th partial sum of the Fourier-series of f. (See [9], p. 64.)

LEMMA (2.2). *Let $f(x)$ satisfy* (1.1) *and the Lipschitz α condition and let $1 \leqq n_1 < n_2 < \ldots < n_N$ be a sequence of positive numbers satisfying* (1.3). *Then, if $N \geqq N_0$ where N_0 depends on $f(x)$ and q, we have for any real a*

$$\int_a^{a+1} \left(\sum_{k=1}^N f(n_k x) \right)^2 dx \leqq C_2(\|f\|^3 + \|f\|^2 + \|f\|)N \tag{2.2}$$

and

$$\int_0^1 \left(\sum_{k=1}^N f(n_k x) \right)^4 dx \leqq C_3 N^2 \tag{2.3}$$

where C_2 depends on q and on the numbers A, α in (2.1) *and C_3 depends on $f(x)$ and q.*

In view of the remark preceding Lemma (2.2), relation (2.2) follows from Lemma (3.3) of [1]; on the other hand, by Lemma (3.4) of [1] we have for $N \geqq N_0$

$$\sum_{k=1}^N f(n_k x) = \xi_1 + \xi_2 \tag{2.4}$$

where ξ_1 and ξ_2 are random variables (functions) on [0, 1] such that, if P denotes the Lebesgue measure, then we have

$$P(|\xi_1| \geqq y\sqrt{N}) \leqq C_4 e^{-C_5 y} \quad (y \geqq 0) \quad \text{and} \quad \|\xi_2\|_\infty \leqq 1 \tag{2.5}$$

where C_4 and C_5 depend on $f(x)$ and q. (As a matter of fact, Lemma (3.4) of [1] assumes $\|f - s_n(f)\| \leqq An^{-\alpha}$ instead of (2.1) and states correspondingly $\|\xi_2\| \leqq 1$ instead of $\|\xi_2\|_\infty \leqq 1$ but the proof there applies with trivial changes in the present case too.) Evidently (2.4) and (2.5) imply (2.3).

3. Main lemma

We first approximate the functions $f(n_k x)$ by step-functions $\varphi_k(x)$ as follows. By assumption, $f(x)$ satisfies in [0, 1] the Lipschitz α condition for some $0 < \alpha \leqq 1$. Let now $2^l \leqq n_k < 2^{l+1}$, put $p = \left[l + \frac{20}{\alpha} \log k \right]$ and let $\varphi_k(x)$ denote the function in [0, 1] which is constant in the intervals $[i2^{-p}, (i+1)2^{-p})$ $(0 \leqq i \leqq 2^p - 1)$ and these constant values coincide with the respective values of $f(n_k x)$ at the points $i2^{-p}$ $(0 \leqq i \leqq 2^p - 1)$. By the Lipschitz α condition we have

$$|f(n_k x) - \varphi_k(x)| \leqq C \left(\frac{n_k}{2^p} \right)^\alpha \leqq C \left(\frac{2^{l+1}}{2^{l+(20/\alpha)\log k - 1}} \right)^\alpha \leqq C \cdot 2^{-20 \log k} \leqq Ck^{-10}. \tag{3.1}$$

(Here and in the sequel, C will denote positive constants, not always the same, depending (at most) on $f(x)$ and q.) Let us now divide the set of positive integers into disjoint blocks $I_1, J_1, I_2, J_2, \ldots$ in such a way that I_k contains $[k^{1/2}]$ integers, J_k contains $[k^{1/4}]$ integers $(k = 1, 2, \ldots)$. Let

$$T_k = \sum_{\nu \in I_k} f(n_\nu x), \quad D_k = \sum_{\nu \in I_k} \varphi_\nu(x). \tag{3.2}$$

Then by (3.1)

$$|D_k-T_k| \leqq C \sum_{\nu\in I_k} \nu^{-10} \leqq C \sum_{\nu=[(k-1)^{1/2}]}^{\infty} \nu^{-10} \leqq Ck^{-4} \tag{3.3}$$

and thus using $|D_k|\leqq Ck^{1/2}$, $|T_k|\leqq Ck^{1/2}$ and the mean value theorem we get

$$|D_k^2-T_k^2| \leqq C, \quad |D_k^4-T_k^4| \leqq C. \tag{3.4}$$

Now we formulate our

MAIN LEMMA. *We have*

$$\left|E(D_k|D_1, \dots, D_{k-1})\right| \leqq Ck^{-2} \quad (k \geqq k_0) \tag{3.5}$$

$$E(D_k^2|D_1, \dots, D_{k-1}) \leqq Ck^{1/2} \quad (k \geqq k_0) \tag{3.6}$$

$$\sum_{k=1}^{N} E(D_k^2|D_1, \dots, D_{k-1}) \sim d_N \quad \textit{a.s.} \quad \textit{as} \quad N\to\infty \tag{3.7}$$

$$ED_k^4 \leqq Ck \quad (k \geqq k_0) \tag{3.8}$$

where $d_N=\sum_{k=1}^{N} ED_k^2$. *Also,* $CN^{3/2}\leqq d_N\leqq CN^{3/2}$ *for* $N\geqq N_0$.

PROOF. We begin with the proof of (3.5). Let $\mathscr{F}_{k-1}$ denote the σ-field generated by $D_1, \dots, D_{k-1}$. In view of (3.3) it suffices to show that

$$\left|E(T_k|\mathscr{F}_{k-1})\right| \leqq Ck^{-2}. \tag{3.9}$$

Let $b=b(k)$ denote the largest integer of the block I_{k-1}, let l be an integer such that $2^l\leqq n_b<2^{l+1}$ and put $w=\left[l+\frac{20}{\alpha}\log b\right]$. From the definition of φ_k it follows that every φ_ν, $1\leqq\nu\leqq b$ takes a constant value on each interval of the form

$$[i2^{-w}, (i+1)2^{-w}) \quad (0 \leqq i \leqq 2^w-1) \tag{3.10}$$

and thus every set $\{D_1=a_1, \dots, D_{k-1}=a_{k-1}\}$ where $a_1, \dots, a_{k-1}$ are constants, can be obtained as an union of intervals of the form (3.10). In other terms, $\mathscr{F}_{k-1}$ is purely atomic and each of its atoms is a union of intervals of the form (3.10). Hence to prove (3.9) it suffices to show that

$$\left|2^w \int_{i2^{-w}}^{(i+1)2^{-w}} T_k\,dx\right| \leqq Ck^{-2} \quad (0 \leqq i \leqq 2^w-1). \tag{3.11}$$

Let $c=c(k)$ denote the smallest integer of the block I_k. By (1.3) we have

$$\sum_{\nu\in I_k} \frac{1}{n_\nu} \leqq \sum_{j=c}^{\infty} \frac{1}{n_\nu} \leqq \frac{1}{n_c}(1+q^{-1}+q^{-2}+\dots) = \frac{q}{q-1}\frac{1}{n_c}$$

and

$$\frac{n_b}{n_c} \leqq q^{-(c-b)} = q^{-[(k-1)^{1/4}]-1} \leqq q^{-(k-1)^{1/4}}.$$

Hence applying Lemma (2.1) and using the trivial relation $b \leqq 2k^{3/2}$ we get

$$(3.12) \qquad \left|2^w \int_{i2^{-w}}^{(i+1)2^{-w}} T_k\,dx\right| = \left|2^w \int_{i2^{-w}}^{(i+1)2^{-w}} \sum_{\nu \in I_k} f(n_\nu x)\right| dx \leqq$$

$$\leqq 2^w C \sum_{\nu \in I_k} \frac{2}{n_\nu} \leqq C\frac{2^w}{n_c} \leqq C\frac{2^{l+(20/\alpha)\log b}}{n_c} \leqq C\frac{n_b}{n_c} b^{20/\alpha} \leqq Cq^{-(k-1)^{1/4}} k^{30/\alpha} \leqq Ck^{-2}$$

and thus (3.11) is proved.

To prove (3.6) it suffices to show (in view of (3.4)) that $E(T_k^2 | \mathscr{F}_{k-1}) \leqq Ck^{1/2}$ and since $\mathscr{F}_{k-1}$ is atomic and each of its atoms is a union of intervals of the form (3.10), the last relation will follow if we show that

$$2^w \int_{i2^{-w}}^{(i+1)2^{-w}} T_k^2\,dx \leqq Ck^{1/2} \quad (0 \leqq i \leqq 2^w - 1,\ k \geqq k_0).$$

The integral on the left hand side is equal to

$$(3.13) \qquad 2^w \int_{i2^{-w}}^{(i+1)2^{-w}} \Big(\sum_{\nu \in I_k} f(n_\nu x)\Big)^2 dx = \int_i^{i+1} \Big(\sum_{\nu \in I_k} f(m_\nu t)\Big)^2 dt$$

where $m_\nu = 2^{-w} n_\nu$. Evidently $m_{\nu+1}/m_\nu \geqq q > 1$ for all the ν's appearing here. If $c = c(k)$ denotes the smallest integer of I_k as above, then the smallest of the m_ν's is $m_c = n_c/2^w$ which is at least 1 (in fact it is $\geqq Ck^2$ by a part of the estimate (3.12)). Hence by Lemma (2.2) the integral on the right side of (3.13) is $\leqq Ck^{1/2}$ for $k \geqq k_0$ which was to be proved.

To prove (3.8) it suffices to remark that, by Lemma (2.2), we have $ET_k^4 \leqq Ck$ which, together with (3.4), implies (3.8).

We now turn to the proof of (3.7). We proceed in three steps.

a) Let $\mathscr{L}_1 \subset \mathscr{L}_2 \subset \dots$ be any increasing sequence of σ-fields such that D_k is $\mathscr{L}_k$ measurable. Then the relations

$$(3.14) \qquad \sum_{k=1}^{N} D_k^2 \sim d_N \quad \text{a.s.}$$

and

$$(3.15) \qquad \sum_{k=1}^{N} E(D_k^2 | \mathscr{L}_{k-1}) \sim d_N \quad \text{a.s.}$$

are equivalent. Indeed, the sequence $H_k = D_k^2 - E(D_k^2 | \mathscr{L}_{k-1})$ is a square integrable martingale difference sequence (and consequently orthogonal) with $EH_k^2 \leqq \leqq 4ED_k^4 \leqq Ck$ by Minkowski's inequality and (3.8). Hence the Rademacher—Mensov convergence theorem implies the a.s. convergence of $\sum_{k=1}^{\infty} k^{-3/2} H_k$ and thus by the Kronecker lemma we have

$$(3.16) \qquad \frac{1}{N^{3/2}} \sum_{k=1}^{N} H_k \to 0 \quad \text{a.s.}$$

By condition (1.4) of the theorem $ET_k^2 \geqq Ck^{1/2}$ for $k \geqq k_0$ hence by (3.4) $ED_k^2 \geqq$

$\geqq Ck^{1/2}$ and thus $d_N \geqq CN^{3/2}$ for $N \geqq N_0$. Also, by Lemma (2.2), $ET_k^2 \leqq Ck^{1/2}$ for $k \geqq k_0$, hence by (3.4) $ED_k^2 \leqq Ck^{1/2}$ and $d_N \leqq CN^{3/2}$. (We thus proved the last statement of the main lemma.) Therefore (3.16) implies $\sum_{k=1}^{N} H_k = o(d_N)$ which really shows that (3.14) and (3.15) are equivalent.

b) We now prove (3.7) in the special case when f is a trigonometric polynomial:

$$f = \sum_{k=1}^{m} (a_k \cos 2\pi kx + b_k \sin 2\pi kx). \tag{3.17}$$

By a) it suffices to show (3.14) or, what is the same,

$$\sum_{k=1}^{N} (D_k^2 - ED_k^2) = o(d_N) \quad \text{a.s.} \tag{3.18}$$

By (3.4) and $d_N \geqq CN^{3/2}$, (3.18) will follow from

$$\sum_{k=1}^{N} (T_k^2 - ET_k^2) = o(N^{3/2}) \quad \text{a.s.} \tag{3.19}$$

Let us express the left side of (3.19) as a trigonometric polynomial, using (3.17). By (3.17),

$$f(n_\nu x) = \sum_{j=1}^{m} (a_k \cos 2\pi j n_\nu x + b_k \sin 2\pi j n_\nu x)$$

and thus

$$T_k = \sum_{\nu \in I_k} f(n_\nu x) = \sum (c_l \cos 2\pi\lambda_l x + d_l \sin 2\pi\lambda_l x) \tag{3.20}$$

where all the λ_l's are of the form tn_ν, $1 \leqq t \leqq m$, $\nu \in I_k$. Denoting by N_m the (set-theoretic) union of the sequences $\{n_k\}, \{2n_k\}, \dots, \{mn_k\}$, this means that all the λ_l's belong to N_m. Also, for the coefficients c_l, d_l in (3.20) we have

$$|c_l| \leqq \overline{M}, \quad |d_l| \leqq \overline{M} \tag{3.21}$$

where $\overline{M}$ depends only on $f(x)$ and $\{n_k\}$. Indeed, the trigonometric sums $f(n_\nu x)$ and $f(n_\mu x)$, $\nu < \mu$ can overlap (i.e. contain a term with the same frequency) only if $n_\mu \leqq mn_\nu$ i.e. overlapping is impossible if $\mu - \nu \geqq p$ where p is the smallest integer such that $q^p > m$. This remark shows that $|c_l| \leqq pM_1$, $|d_l| \leqq pM_1$ where $M_1 = \max(|a_1|, |b_1|, \dots, |a_m|, |b_m|)$ and hence (3.21) is valid.

We notice also that the trigonometric sums in (3.20) are pairwise non-overlapping for $k \geqq k_0$. This follows from the fact that the largest λ_l in T_{k-1} is mn_b and the smallest λ_l in T_k is n_c where b and c are the largest integer of the block I_{k-1} and the smallest integer of the block I_k, resp. By the separation of I_{k-1} and I_k by the block J_{k-1} of length $[(k-1)^{1/4}]$ and because of (1.3) we have $mn_b < n_c$ for $k \geqq k_0$.

Squaring (3.20) and using well known trigonometric identities we get

$$T_k^2 = \frac{1}{2} \sum (c_l^2 + d_l^2) + \sum (e_i \cos 2\pi\varrho_i x + f_i \sin 2\pi\varrho_i x) \tag{3.22}$$

where $|e_i|\leqq\overline{M}^2$, $|f_i|\leqq\overline{M}^2$ and the ϱ_i's are the numbers of the form $\lambda_s\pm\lambda_r$ with the λ's appearing in (3.20). Hence summing (3.22) for $M+1\leqq k\leqq M+N$ (but not collecting the terms with equal frequencies) we get that

$$\sum_{k=M+1}^{M+N} T_k^2 = B+\sum (r_i\cos 2\pi\theta_i x+s_i\sin 2\pi\theta_i x) \tag{3.23}$$

where B is a constant, $|r_i|\leqq\overline{M}^2$, $|s_i|\leqq\overline{M}^2$ and the θ_i's are the numbers of the form $\lambda_s\pm\lambda_r$ where λ_s and λ_r are from the same T_k, $M+1\leqq k\leqq M+N$. Since the T_k's are non-overlapping for $k\geqq k_0$ and N_m satisfies condition B_2, there is a constant C_1 such that at most C_1 of the θ_i's can be equal. Hence collecting the terms with equal frequency on the right hand side of (3.23) we get

$$\sum_{k=M+1}^{M+N} T_k^2 = B+\sum (u_j\cos 2\pi jx+v_j\sin 2\pi jx) \tag{3.24}$$

where the sum on the right hand side is finite and $|u_j|\leqq C_1\overline{M}^2$, $|v_j|\leqq C_1\overline{M}^2$. Also, the number of terms on the right hand side of (3.20) is $\leqq mk^{1/2}$, hence in the second sum of (3.22) is $\leqq m^2k$ and on the right side of (3.23), (3.24) is $\leqq \sum_{k=M+1}^{M+N} m^2k\leqq$ $\leqq m^2N(M+N)\leqq m^2[(M+N)^2-M^2]$. The number B in (3.24) is evidently equal to the expectation of $\sum_{k=M+1}^{M+N} T_k^2$ (since the integral of the trigonometric sum on the right hand side is 0). Hence (3.24) implies

$$E\left(\sum_{k=M+1}^{M+N}(T_k^2-ET_k^2)\right)^2 = \frac{1}{2}\sum (u_j^2+v_j^2)\leqq$$

$$\leqq \frac{1}{2}C_1^2\overline{M}^4m^2\big((M+N)^2-M^2\big) = C_2\big((M+N)^2-M^2\big).$$

Applying the Gal—Koksma law of large numbers (see [6], p. 134) for the variables $T_k^2-ET_k^2$, we get

$$\sum_{k=1}^{N}(T_k^2-ET_k^2) = O(N\log^3 N) \quad \text{a.s.}$$

and thus (3.19) is proved.

c) Let now f be any function satisfying (1.1) and the Lipschitz α condition and fix an $\varepsilon>0$. Since the Fourier series of f converges uniformly to f (even (2.1) is valid) we can write $f=f_1+f_2$ where f_1 is a suitable partial sum of the Fourier series of f (hence it is a trigonometric polynomial) and $\|f_2\|_\infty\leqq\varepsilon$. (Evidently f_1 and f_2 also satisfy (1.1) and they are also Lipschitz α functions.) In the same way as we constructed the step-function $\varphi_k(x)$ from $f(n_kx)$, we can construct $\varphi_k^{(1)}(x)$ and $\varphi_k^{(2)}(x)$ from $f_1(n_kx)$ and $f_2(n_kx)$, resp. Then we have

$$T_k = T_k^{(1)}+T_k^{(2)} \quad\text{and}\quad D_k = D_k^{(1)}+D_k^{(2)}$$

where

$$T_k^{(1)} = \sum_{\nu\in I_k} f_1(n_\nu x), \quad T_k^{(2)} = \sum_{\nu\in I_k} f_2(n_\nu x)$$

$$D_k^{(1)} = \sum_{\nu\in I_k} \varphi_\nu^{(1)}(x), \quad D_k^{(2)} = \sum_{\nu\in I_k} \varphi_\nu^{(2)}(x).$$

Evidently (3.3), (3.4) hold for the D_k, T_k's with superscripts, too:

$$|D_k^{(i)}-T_k^{(i)}| \leqq Ck^{-4}, \quad |(D_k^{(i)})^2-(T_k^{(i)})^2| \leqq C, \quad |(D_k^{(i)})^4-(T_k^{(i)})^4| \leqq C \quad i=1,2. \tag{3.25}$$

If $b=b(k)$ is the largest integer of I_{k-1}, $2^l \leqq n_b < 2^{l+1}$, $w=\left[l+\frac{20}{\alpha}\log b\right]$ then, as we showed, every φ_ν, $1\leqq\nu\leqq b$ and therefore also D_{k-1}, takes a constant value on each interval of the form (3.10). In other words, if $\mathscr{G}_{k-1}$ denotes the σ-field generated by the intervals (3.10), then D_{k-1} is $\mathscr{G}_{k-1}$ measurable. Since $\varphi_\nu^{(1)}$ and $\varphi_\nu^{(2)}$ are step-functions with the same intervals of constancy as φ_ν, not only D_{k-1} but also $D_{k-1}^{(1)}$ and $D_{k-1}^{(2)}$ are $\mathscr{G}_{k-1}$ measurable.

Let $d_N^{(1)} = \sum_{k=1}^{N} E(D_k^{(1)})^2$. Since f_1 is a trigonometric polynomial, the relation

$$\sum_{k=1}^{N} (D_k^{(1)})^2 \sim d_N^{(1)} \quad \text{a.s.} \tag{3.26}$$

is exactly what we proved in b). As we remarked above, $D_k^{(1)}$ is $\mathscr{G}_k$ measurable and hence by the equivalence statement of a) (3.26) implies

$$\sum_{k=1}^{N} E\big((D_k^{(1)})^2|\mathscr{G}_{k-1}\big) \sim d_N^{(1)} \quad \text{a.s.} \tag{3.27}$$

We now prove two simple estimates

$$E\big((D_k^{(1)})^2|\mathscr{G}_{k-1}\big) \leqq Ck^{1/2} \quad (k \geqq k_0) \tag{3.28}$$

$$E\big((D_k^{(2)})^2|\mathscr{G}_{k-1}\big) \leqq C\varepsilon k^{1/2} \quad (k \geqq k_0) \tag{3.29}$$

which, together with (3.27), will easily lead to our aim (3.7).

The proofs of (3.28) and (3.29) are the same, we prove e.g. (3.29). In view of (3.25) it suffices to show

$$E\big((T_k^{(2)})^2|\mathscr{G}_{k-1}\big) \leqq C\varepsilon k^{1/2} \quad (k \geqq k_0)$$

and since $\mathscr{G}_{k-1}$ is atomic with atoms of the form (3.10), the last inequality is equivalent to

$$2^w \int_{i2^{-w}}^{(i+1)2^{-w}} (T_k^{(2)})^2\,dx \leqq C\varepsilon k^{1/2} \quad (0 \leqq i \leqq 2^w-1,\ k \geqq k_0). \tag{3.30}$$

Here the left-hand side can be written as

$$2^w \int_{i2^{-w}}^{(i+1)2^{-w}} (T_k^{(2)})^2\,dx = 2^w \int_{i2^{-w}}^{(i+1)2^{-w}} \Big(\sum_{\nu\in I_k} f_2(n_\nu x)\Big)^2 dx = \int_i^{i+1} \Big(\sum_{\nu\in I_k} f_2(m_\nu t)\Big)^2 dt \tag{3.31}$$

where $m_\nu = n_\nu/2^w$. Exactly in the same way as in the case of the second integral in (3.13), the numbers m_ν are all greater than 1 and they satisfy $m_{\nu+1}/m_\nu \geqq q > 1$. Let us also observe that since f_1 is a partial sum of the Fourier series of f i.e. $f_2 = f - s_k(f)$ with a certain k, (2.1) is inherited for f_2 with the same A, α. Since $\|f_2\| \leqq \|f_2\|_\infty \leqq \varepsilon$, an application of Lemma (2.2) gives that the last integral in (3.31) is at most $C\varepsilon k^{1/2}$ for $k \geqq k_0$ and thus (3.30) is valid.

To deduce (3.7) from (3.27), (3.28), (3.29) let us first integrate (3.29) to get $E(D_k^{(2)})^2 \leqq C\varepsilon k^{1/2}$ for $k \geqq k_0$. On the other hand, by assumption (1.4) of the theorem we have $ET_k^2 \geqq Ck^{1/2}$ for $k \geqq k_0$ and thus (3.4) implies $ED_k^2 \geqq Ck^{1/2}$. We thus get $E(D_k^{(2)})^2/ED_k^2 \leqq C\varepsilon$ $(k \geqq k_0)$ whence we obtain, using $D_k = D_k^{(1)} + D_k^{(2)}$ and Minkowski's inequality,

$$1 - C\sqrt{\varepsilon} \leqq E(D_k^{(1)})^2/ED_k^2 \leqq 1 + C\sqrt{\varepsilon} \quad (k \geqq k_0)$$

and consequently

$$(3.32) \qquad (1 - C\sqrt{\varepsilon})d_N < d_N^{(1)} < (1 + C\sqrt{\varepsilon})d_N \quad (N \geqq N_0).$$

Summing up (3.29) for $k = 1, 2, \dots, N$ and using $d_N \geqq CN^{3/2}$ we obtain

$$(3.33) \qquad \sum_{k=1}^{N} E((D_k^{(2)})^2 | \mathcal{G}_{k-1}) \leqq C\varepsilon d_N \quad (N \geqq N_0).$$

Also, (3.28), (3.29) and Schwarz's inequality imply $|E(D_k^{(1)} D_k^{(2)} | \mathcal{G}_{k-1})| \leqq C\sqrt{\varepsilon}\, k^{1/2}$ whence

$$(3.34) \qquad \left| \sum_{k=1}^{N} E(2D_k^{(1)} D_k^{(2)} | \mathcal{G}_{k-1}) \right| \leqq C\sqrt{\varepsilon}\, d_N \quad (N \geqq N_0).$$

Adding (3.27), (3.33), (3.34) and using $D_k = D_k^{(1)} + D_k^{(2)}$ and (3.32) we see that

$$(1 - C\sqrt{\varepsilon})\, d_N \leqq \sum_{k=1}^{N} E(D_k^2 | \mathcal{G}_{k-1}) \leqq (1 + C\sqrt{\varepsilon})\, d_N$$

a.s. for sufficiently large N which implies, since $\varepsilon > 0$ was arbitrary,

$$\sum_{k=1}^{N} E(D_k^2 | \mathcal{G}_{k-1}) \sim d_N \quad \text{a.s.}$$

Since D_k is $\mathcal{G}_k$ measurable, the last relation implies (3.14) and (3.7) by the equivalence statement of a). Hence the proof of the main lemma is complete.

REMARK 1. In the proof of the lemma above, the assumption $\{n_k\} \in D_\infty$ was used only in the proof of (3.7); relations (3.5), (3.6), (3.8) are valid under the mere assumption (1.3). We also see that if f is a trigonometric polynomial of order m:

$$f = \sum_{k=1}^{m} (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$$

then for the validity of (3.7) it suffices to assume $\{n_k\} \in D_m$ instead of $\{n_k\} \in D_\infty$ (see step b) of the proof of (3.7)). Together with step c), this shows that if $\{n_k\} \in D_m$

and $\|f-s_m(f)\|_\infty=\varepsilon_0$ then instead of (3.7) we have

$$(1-C\sqrt{\varepsilon_0})\, d_N \leqq \sum_{k=1}^N E(D_k^2|\mathcal{G}_{k-1}) \leqq (1+C\sqrt{\varepsilon_0})\, d_N \tag{3.35}$$

a.s. for sufficiently large N where $\mathcal{G}_k$ is the σ-field defined above. Since D_k is $\mathcal{G}_k$ measurable and by step a) we have (under (1.3))

$$\sum_{k=1}^N D_k^2 - \sum_{k=1}^N E(D_k^2|\mathcal{L}_{k-1}) = o(d_N) \quad \text{a.s.}$$

for any increasing sequence $\mathcal{L}_k$ of σ-fields such that D_k is $\mathcal{L}_k$ measurable, (3.35) implies

$$(1-C\sqrt{\varepsilon_0})\, d_N \leqq \sum_{k=1}^N E(D_k^2|D_1, \dots, D_{k-1}) \leqq (1+C\sqrt{\varepsilon_0})\, d_N$$

a.s. for sufficiently large N. In other words, if $\{n_k\}\in D_m$ for a large m (here "large" depends also on f) then (3.7) is satisfied "approximately".

Let

$$\bar{D}_k = D_k - E(D_k|D_1, \dots, D_{k-1}).$$

Then for the $\bar{D}_k$'s the main lemma implies the following

LEMMA (3.2). *We have*

$$E(\bar{D}_k|\bar{\mathcal{F}}_{k-1}) = 0 \tag{3.36}$$

$$E(\bar{D}_k^2|\bar{\mathcal{F}}_{k-1}) \leqq Ck^{1/2} \quad (k \geqq k_0) \tag{3.37}$$

$$\sum_{k=1}^N E(\bar{D}_k^2|\bar{\mathcal{F}}_{k-1}) \sim d_N \quad a.s. \tag{3.38}$$

$$E\bar{D}_k^4 \leqq Ck \quad (k \geqq k_0) \tag{3.39}$$

where $\bar{\mathcal{F}}_{k-1}$ denotes the σ-field generated by $\bar{D}_1, \dots, \bar{D}_{k-1}$.

PROOF. We have $|\bar{D}_k - D_k| \leqq Ck^{-2}$ (see (3.5)), $|D_k| \leqq Ck^{1/2}$, $|\bar{D}_k| \leqq Ck^{1/2}$ and hence by the mean value theorem

$$|\bar{D}_k^2 - D_k^2| \leqq C, \quad |\bar{D}_k^4 - D_k^4| \leqq C. \tag{3.40}$$

The second relations of (3.40) and (3.8) evidently imply (3.39), furthermore (3.6) and the first relation of (3.40) imply $E(\bar{D}_k^2|D_1, \dots, D_{k-1}) \leqq Ck^{1/2}$ from which (3.37) follows by taking conditional expectations of both sides with respect to $\bar{\mathcal{F}}_{k-1}$ (which is contained in the σ-field generated by $D_1, \dots, D_{k-1}$). In step a) of the proof of (3.7) we saw that (3.7) is equivalent to $\sum_{k=1}^N D_k^2 \sim d_N$ a.s.; now the first relation of (3.40) and $d_N \geqq CN^{3/2}$ show that also $\sum_{k=1}^N \bar{D}_k^2 \sim d_N$ a.s. is an equivalent statement. Finally, the martingale argument of step a) and (3.39) show that $\sum_{k=1}^N \bar{D}_k^2 \sim d_N$ implies (3.38).

REMARK 2. The main lemma concerns the "long" block sums D_k and T_k. Defining the "short" block sums

$$T'_k = \sum_{\nu \in J_k} f(n_\nu x), \quad D'_k = \sum_{\nu \in J_k} \varphi_\nu(x) \tag{3.41}$$

an analogous statement holds for these sums:

$$\left|E(D'_k|\mathcal{F}'_{k-1})\right| \leqq Ck^{-2}, \quad E\big((D'_k)^2|\mathcal{F}'_{k-1}\big) \leqq Ck^{1/4},$$

$$\sum_{k=1}^{N} E\big((D'_k)^2|\mathcal{F}'_{k-1}\big) \sim d'_N \quad \text{a.s.,} \quad E(D'_k)^4 \leqq Ck^{1/2}$$

where $\mathcal{F}'_{k-1}$ denotes the σ-field generated by $D'_1, \ldots, D'_{k-1}$ and $d'_N = \sum_{k=1}^{N} E(D'_k)^2$. Also, $CN^{5/4} \leqq d'_N \leqq CN^{5/4}$ for $N \geqq N_0$. The analogue of Lemma (3.2) also holds for the centered sums $\bar{D}'_k = D'_k - E(D'_k|\mathcal{F}'_{k-1})$.

4. Conclusion of the proof

Using the main lemma and Lemma (3.2) we can complete the proofs of Theorem 1, 2 in a standard way, following [1] or [6]. We prove here Theorem 1; the proof of Theorem 2 is the same (see Remark 1 after the proof of the main lemma). Let

$$V_N = \sum_{k=1}^{N} E(\bar{D}_k^2|\bar{D}_1, \ldots, \bar{D}_{k-1}),$$

then $V_N \sim d_N$ a.s. by (3.38). Also, using (3.39) and $d_N \geqq CN^{3/2}$ (see the main lemma) we see that the sum $\sum_{k=1}^{\infty} d_k^{-3/2} E\bar{D}_k^4$ is convergent. By Beppo Levi's theorem this implies the a.s. convergence of the series

$$\sum_{k=1}^{\infty} d_k^{-3/2} E(\bar{D}_k^4|\bar{D}_1, \ldots, \bar{D}_{k-1})$$

and since the general term of the series

$$\sum_{k=1}^{\infty} \frac{1}{V_k^{3/4}} \int_{x^2 > V_k^{3/4}} x^2\, dP(\bar{D}_k < x|\bar{D}_1, \ldots, \bar{D}_{k-1}) \tag{4.1}$$

can be majorized by

$$\frac{1}{V_k^{3/2}} \int_{-\infty}^{+\infty} x^4\, dP(\bar{D}_k < x|\bar{D}_1, \ldots, \bar{D}_{k-1}) = \frac{1}{V_k^{3/2}} E(\bar{D}_k^4|\bar{D}_1, \ldots, \bar{D}_{k-1}) \sim$$

$$\sim \frac{1}{d_k^{3/2}} E(\bar{D}_k^4|\bar{D}_1, \ldots, \bar{D}_{k-1}),$$

it follows that the series (4.1) is also a.s. convergent. Thus we can apply Theorem (4.4) of [7] to the martingale difference sequence $\bar{D}_k$ with $f(x) = x^{3/4}$ and we get that there exists a Wiener-process $\zeta(t)$ such that

$$\bar{D}_1+\ldots+\bar{D}_k = \zeta(V_k)+o(V_k^{1/2-\eta}) \quad \text{a.s.} \tag{4.2}$$

with an absolute constant $\eta>0$. (Strictly speaking, we first have to redefine the sequence $\bar{D}_k$ on a new, larger probability space and $\zeta(t)$ will be defined over this new space; in the sequel, however, we will speak as if (4.2) were valid for the original sequence. This little inaccuracy essentially simplifies the formulas (we do not have to use stars or superscripts for the "redefined" variables) and does not cause any trouble.) Replacing $\bar{D}_1+\ldots+\bar{D}_k$ with $T_1+\ldots+T_k$ on the left hand side of (4.2), we commit an error $O(1)$ (since $|\bar{D}_k-T_k|\leqq|\bar{D}_k-D_k|+|D_k-T_k|=O(k^{-2})$ by (3.3) and (3.5)); hence (4.2) and $V_k\sim d_k$ a.s. imply

$$T_1+\ldots+T_k = \zeta(V_k)+o(d_k^{1/2-\eta}) \quad \text{a.s.} \tag{4.3}$$

(In what follows η will denote positive absolute constants, possibly different at different places.) We also remark that by replacing the left hand side of (4.3) by $T_1+T_1'+\ldots+T_k+T_k'$ (T_i' are the short block sums defined in (3.41)) we only add a term which is $o(d_k^{1/2-\eta})$, so it does not bother the right side of (4.3). (Indeed, (4.3) has the exact analogue

$$T_1'+\ldots+T_k' = \zeta(V_k')+o(d_k'^{1/2-\eta}) \quad \text{a.s.} \tag{4.4}$$

for the short block sums where $V_k'=\sum_{j=1}^{k} E(\bar{D}_j'^2|\bar{D}_1', \ldots, \bar{D}_{j-1}')$, $d_k'=\sum_{j=1}^{k} E(D_j')^2$, cf. Remark 2 at the end of § 3. Now it is sufficient to observe that the analogue of (3.37) to the 'primed' variables $\bar{D}_k'$ i.e. $E(\bar{D}_k'^2|\mathcal{F}_{k-1}')\leqq Ck^{1/4}$ implies $V_k'=O(k^{5/4})$ and thus $d_k'\leqq Ck^{5/4}$ and the standard estimate $\zeta(t)=o(t^{1/2}\log t)$ show that the right hand side of (4.4) is $o(k^{5/8}\log k)+o(k^{5/8-\eta})$ which is dominated by the remainder term $o(d_k^{1/2-\eta})=o(k^{3/4-3\eta/2})$ in (4.3) if η is small enough.) Hence (4.3) implies

$$T_1+T_1'+\ldots+T_k+T_k' = \zeta(V_k)+o(d_k^{1/2-\eta}) \quad \text{a.s.}$$

which can be rewritten as

$$S_{N_k} = \zeta(V_k)+o(d_k^{1/2-\eta}) \quad \text{a.s.} \tag{4.5}$$

where $S_N=\sum_{\nu=1}^{N} f(n_\nu x)$ and $N_k=\sum_{i=1}^{k}([i^{1/2}]+[i^{1/4}])\sim\frac{2}{3}k^{3/2}$. Since $CN^{3/2}\leqq d_N\leqq CN^{3/2}$ the remainder term in (4.5) can be also written as $o(N_k^{1/2-\eta})$. Hence if we define a sequence τ_n of random variables by

$$\begin{cases}\tau_0 = 0 \\ \tau_{N_k} = V_k \quad \text{for} \quad k=1,2,\ldots \\ \tau_n \quad \text{is linear in the intervals } N_k\leqq n\leqq N_{k+1} \text{ for } k=0,1,\ldots\ (N_0=0)\end{cases} \tag{4.6}$$

then (4.5) simply says that the relation

$$S_N = \zeta(\tau_N)+o(N^{1/2-\eta}) \quad \text{a.s.} \tag{4.7}$$

is valid for the indices $N=N_k$. To get (4.7) for general N it suffices to show

(4.8) $$\max_{N_k \leqq N \leqq N_{k+1}} |S_N - S_{N_k}| = o(N_k^{1/2-\eta}) \quad \text{a.s.}$$

and

(4.9) $$\max_{N_k \leqq N \leqq N_{k+1}} |\zeta(\tau_N) - \zeta(\tau_{N_k})| = o(N_k^{1/2-\eta}) \quad \text{a.s.}$$

The first relation is trivial since

$$|S_N - S_{N_k}| = \left| \sum_{\nu=N_k+1}^{N} f(n_\nu x) \right| \leqq C(N-N_k) \leqq C(N_{k+1}-N_k) =$$
$$= C\big([(k+1)^{1/2}] + [(k+1)^{1/4}]\big) \sim Ck^{1/2} \leqq CN_k^{1/2-\eta}.$$

To see (4.9) let us note that

$$\tau_{N_k} = V_k = O(k^{3/2}) \quad \text{a.s.}$$

and

$$\max_{N_k \leqq N \leqq N_{k+1}} |\tau_N - \tau_{N_k}| = \tau_{N_{k+1}} - \tau_{N_k} = V_{k+1} - V_k = O(k^{1/2}) \quad \text{a.s.}$$

by (3.37) and thus Lemma (3.6) of [1] (with $r=3/2$, $s=1/2$) shows the left side of (4.9) is $O(k^{1/4}\log k) = O(N_k^{1/2-\eta})$ a.s. Hence (4.7) is proved and it remains to show that $\tau_N/b_N \to 1$ a.s. where b_N is defined in (1.6). To this end we notice that $\tau_N/e_N \to 1$ a.s. where e_N is the numerical sequence defined (in analogy with (4.6)) by

$$\begin{cases} e_0 = 0 \\ e_{N_k} = d_k \quad \text{for} \quad k = 1, 2, \dots \\ e_n \quad \text{is linear in the intervals } N_k \leqq n \leqq N_{k+1} \text{ for } k = 0, 1, \dots . \end{cases}$$

(This follows trivially from the piecewise linearity of τ_n and e_n and the relation $V_k/d_k \to 1$ a.s. which is identical with (3.38).) Since $N_k \sim \frac{2}{3} k^{3/2}$ and $e_{N_k} = d_k \geqq Ck^{3/2}$ we have $e_n \geqq Cn$ and thus the remainder term in (4.7) can also be written as $o(e_N^{1/2-\eta})$. Hence (4.7) and $\tau_n \sim e_n$ a.s. imply that the distribution of $S_N/\sqrt{e_N}$ tends to the standard normal distribution. Since the L_4 norm of $S_N/\sqrt{e_N}$ remains bounded (this follows from the second relation of Lemma (2.2) and $e_n \geqq Cn$), the second moment of $S_N/\sqrt{e_N}$ converges to the second moment of the standard normal distribution, i.e. to 1. In other words, $ES_N^2/e_N \to 1$ and since here $ES_N^2 = b_N$ (see (1.6)), we see that the sequences e_N and b_N are asymptotically equal. Thus $\tau_N/e_N \to 1$ a.s. implies $\tau_N/b_N \to 1$ a.s. and this completes the proof of Theorem 1.

References

[1] I. Berkes, On the asymptotic behaviour of $\sum f(n_k x)$ I. Main theorems, *Zeitschrift für Wahrscheinlichkeitstheorie verw. Gebiete*, **34** (1976), 319—345.

[2] I. Berkes, On the asymptotic behaviour of $\sum f(n_k x)$ II. Applications, *Zeitschrift für Wahrscheinlichkeitstheorie verw. Gebiete*, **34** (1976), 347—365.

[3] V. F. Gaposhkin, On the central limit theorem for some weakly dependent sequences (in Russian), *Teor. Verojatn. i Primenen*, **15** (1970), 666—684. English translation: *Theory Prob. and Appl.*, **15** (1970), 649—666.

[4] M. Kac, On the distribution of values of sums of the type $\Sigma f(2^k t)$, *Ann. Math.*, **47** (1946), 33—49.
[5] M. Kac, Probability methods in some problems of analysis and number theory, *Bull. Amer. Math. Soc.*, **55** (1949), 641—665.
[6] W. Philipp and W. F. Stout, Almost sure invariance principles for partial sums of weakly dependent random variables. *Memoirs of the AMS*, no. 161.
[7] V. Strassen, Almost sure behaviour of sums of independent random variables and martingales, *Proc. 5th Berkeley Sympos. Math. Statist. and Probab.* Vol. II. (Part I), 315—343, Univ. of California Press (Berkeley, 1967).
[8] S. Takahashi, The law of the iterated logarithm for a gap sequence with infinite gaps, *Tohoku Math. Journ.*, **15** (1963), 281—288.
[9] A. Zygmund, *Trigonometric series*, Vol. I., Cambridge University Press (1959).

(Received September 20, 1977)

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15
HUNGARY

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801
USA