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MAXIMAL CIRCUITS OF GRAPHS. I

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1. Introduction. All graphs considered are finite, undirected, and without loops or multiple edges. Circuits and paths are 'elementary', i.e., have no repeated vertices. $V(G)$ denotes the set of vertices of G. $[x]$ denotes the greatest integer less than or equal to x.

In [2, Theorem (2.7)], ERDŐS and GALLAI proved that, if $d \geqq 2$ and G is a graph on n vertices with more than $\frac{1}{2}d(n-1)$ edges, then G contains a circuit of length at least $d+1$. They pointed out that this result is best possible when n is of the form $t(d-1)+1$, in view of the graph consisting of t copies of K_d all having exactly one vertex in common. Here I obtain the slight improvement:

THEOREM 1. *If $d \geqq 2$, and $n=t(d-1)+p+1$ where $t \geqq 0$ and $0 \leqq p < d-1$, and G is a graph on n vertices with more than $t\binom{d}{2}+\binom{p+1}{2}$ edges, then G contains a circuit of length at least $d+1$.*

This result is best possible for every value of n, in view of the graph consisting of t copies of K_d and one copy of K_{p+1}, all having exactly one vertex in common.

An exactly analogous situation holds for paths, where ERDŐS and GALLAI [2, Theorem (2.6)] proved that, if $d \geqq 0$ and G is a graph on n vertices with more than $\frac{1}{2}dn$ edges, then G contains a path of length at least $d+1$. This is best possible when n is of the form $t(d+1)$, in view of the graph consisting of t disjoint copies of K_{d+1}. The analogous improvement, best possible for all values of n, is given in Corollary 1.1. (This result was first proved by FAUDREE and SCHELP [3, Theorem 5], who also characterized the extremal graphs.)

COROLLARY 1.1. *If $d \geqq 0$, and $n=t(d+1)+p$ where $t \geqq 0$ and $0 \leqq p < d+1$, and G is a graph on n vertices with more than $t\binom{d+1}{2}+\binom{p}{2}$ edges, then G contains a path of length at least $d+1$.*

PROOF. Add a new vertex to G, joined to all the vertices of G by edges, to form a new graph G^* with $n+1=t(d+1)+p+1$ vertices and more than

$$t\binom{d+1}{2}+\binom{p}{2}+n = t\binom{d+2}{2}+\binom{p+1}{2}$$

edges. By Theorem 1, G^* contains a circuit of length at least $d+3$, and so G contains a path of length at least $d+1$. This completes the proof.

If $a, b, c \geqq 0$, let $K(a, b, c)$ denote the graph consisting of K_{a+b} and $K_{b,c}$ with b of the vertices of K_{a+b} identified with the 'first' b vertices of $K_{b,c}$ (so that $K(a, b, c)$ has $a+b+c$ vertices). If $d \geqq 2$, $j \leqq \frac{1}{2}d$ and $n \geqq d+1-j$, let

$$f(n, j, d) := \binom{d-j+1}{2} + j(j+n-d-1),^*$$

the number of edges in the graph $K(d-2j+1, j, j+n-d-1)$, which has n vertices and in which the longest circuit has length d (if $j \leqq \frac{1}{2}d$). The proof of Theorem 1 uses:

THEOREM 2. *If $d \geqq 2$ and $n \geqq \frac{3}{2}d-1$, and G is a 2-connected graph on n vertices with more than $f\left(n, \frac{1}{2}d, d\right)$ edges, then G contains a circuit of length at least $d+1$.*

Note that this bound agrees (in effect) with that of Theorem 1 if $p = \frac{1}{2}d-1$, $\frac{1}{2}d-\frac{1}{2}$ or $\frac{1}{2}d$; otherwise it is less than that of Theorem 1. If d is even, Theorem 2 is best possible except for the restriction on the value of n. The following conjecture would be best possible for all values of n, in view of the graphs $K(d-2j+1, j, j+n-d-1)$ $\left(k \leqq j \leqq \left[\frac{1}{2}d\right]\right)$.

CONJECTURE. *If $d \geqq 2$, $2 \leqq k \leqq \frac{1}{2}d$ and $n \geqq d+1$, and G is a 2-connected graph on n vertices with more than*

$$\max\left(f(n, k, d), f\left(n, \left[\tfrac{1}{2}d\right], d\right)\right)$$

edges in which each vertex has valency at least k, then G contains a circuit of length at least $d+1$. (If nothing is known about the valencies, replace k by 2.)

Note that $f(n, k, d) \leqq f\left(n, \left[\frac{1}{2}d\right], d\right)$ whenever k is greater than about $\frac{1}{6}(5d-4n)$, so that the bound in the conjecture is always equal to $f\left(n, \left[\frac{1}{2}d\right], d\right)$ if $n > \frac{5}{4}d$. The conjecture is true for any values of n and $d (=: n-r-1)$ for which the conjecture on page 747 of [4] is true. (I have recently noticed that the latter conjecture can be false if $n \geqq \frac{3}{2}d+2$, in view of graphs consisting of three or more copies of $K\left(0, \left[\frac{1}{4}d\right], \left[\frac{1}{4}(d+10)\right]\right)$, disjoint except for two vertices which appear among the $\left[\frac{1}{4}(d+10)\right]$ vertices in each copy.)

Theorem 2 has the following corollary, which was proved by ERDŐS and GALLAI [2, Theorem (3.4)] subject to the stronger restriction that $n > k^2-k+6$.

COROLLARY 2.1. *If $n \geqq 3k+2$ and $k \geqq 0$, and G is a connected graph on n vertices with more than $\binom{k}{2}+k(n-k)$ edges, then G contains a path of length at least $2k+1$.*

PROOF. Add a new vertex to G, joined to all the vertices of G, to form a 2-connected graph G^* with $n+1$ vertices and more than

$$\binom{k}{2}+k(n-k)+n = \binom{k+2}{2}+(k+1)((n+1)-k-2)$$

* Throughout the paper the symbol := or =: indicates that the equation in which it occurs acts as the definition of (some part of) the expression on the same side of the equality sign as the colon.

edges. By Theorem 2 with $d=2k+2$, G^* contains a circuit of length at least $2k+3$, and so G contains a path of length at least $2k+1$.

2. Proofs of the theorems. LEMMA. *If $d \geqq 2$ and $d+1 \leqq n \leqq 2d-1$, and G is a graph on n vertices with more than $\binom{d}{2}+\binom{n-d+1}{2}$ edges, then G contains a circuit of length at least $d+1$.*

PROOF. Put $d+1=n-r$ in Corollary 11.1 of [4].

PROOF OF THEOREM 2 by induction on n. If $n=\frac{3}{2}d-1$, then $f\left(n, \frac{1}{2}d, d\right)= \binom{d}{2}+\binom{\frac{1}{2}d}{2}$ and the result follows by the Lemma. If $n=\frac{3}{2}d-\frac{1}{2}$, then $f\left(n, \frac{1}{2}d, d\right)= \binom{d}{2}+\binom{\frac{1}{2}(d+1)}{2}+\frac{1}{8}$, and the result follows similarly. So the induction starts.

If every vertex of G has valency at least $\frac{1}{2}(d+1)$, then the result follows by Theorem 4 of DIRAC [1]. If G contains a vertex v with valency $\leqq \frac{1}{2}d$, then $G \setminus \{v\}$ has more than $f\left(n-1, \frac{1}{2}d, d\right)$ edges, and the result follows by the induction hypothesis if $G \setminus \{v\}$ is 2-connected. So we may suppose that G contains at least one vertex with valency $\leqq \frac{1}{2}d$, and that, if v is any such vertex, then $G \setminus \{v\}$ is not 2-connected.

Let $\{a, b\}$ be a separating set of two vertices, and let L be a *lune* of G attached at a and b, i.e., a subgraph with $|V(L)| \geqq 3$ such that a and b are the only vertices of L joined to anything outside L, and $L \setminus \{a, b\}$ is connected; and choose a, b and L so that L is minimal (by inclusion). Suppose first that $L \setminus \{a, b\}$ contains a vertex v with valency $\leqq \frac{1}{2}d$, and consider the possibilities for a vertex w such that $\{v, w\}$ is a separating set. Certainly $w \notin L$, or there would be a smaller lune within L, attached at v and w. But if $w \notin L$, the only way in which we can avoid $\{v, a\}$ or $\{v, b\}$ being a separating set (giving a smaller lune) is to have $L=\{a, b, v, (a, v), (v, b)\}$, and now $\{(a, b)\} \cup G \setminus \{v, (a, v), (v, b)\}$ satisfies the hypotheses of the theorem and the result follows by induction. So we may suppose that every vertex of $L \setminus \{a, b\}$ has valency at least $\frac{1}{2}(d+1)$. By Lemma 12.4 of [5], a and b are connected by a path of length at least $\frac{1}{2}(d+1)$ in L.

Let L' be another minimal lune of G, attached at c and d. (Possibly $\{c, d\}= \{a, b\}$.) By the same argument, c and d are connected by a path of length at least $\frac{1}{2}(d+1)$ in L'. Since G is 2-connected, $\{a, b\}$ is connected to $\{c, d\}$ by two disjoint paths, which clearly do not contain any vertices of L or L' apart from a, b, c, d themselves. So G contains a circuit of length at least $d+1$. This completes the proof.

PROOF OF THEOREM 1 by induction on n. The result is (vacuously) true if $n \leqq d$, and it follows from the Lemma if $d+1 \leqq n \leqq 2d-1$; so suppose that $n \geqq 2d$. If G is 2-connected, the result follows from Theorem 2; so we may suppose that $G=G_1 \cup G_2$, where G_1 and G_2 either are disjoint or have exactly one vertex in common. Let G_i have n_i vertices, where

$$n_i = t_i(d-1)+p_i+1 \quad (0 \leqq p_i < d-1;\ i = 1, 2).$$

Then $n_1+n_2=n$ or $n+1$, and either

$$t_1+t_2 = t \quad \text{and} \quad p_1+p_2 = p-1 \quad \text{or} \quad p,$$

or

$$t_1+t_2 = t-1 \quad \text{and} \quad p_1+p_2 = d-1+p-1 \quad \text{or} \quad d-1+p.$$

Suppose that neither G_1 nor G_2 satisfies the hypotheses of the theorem. Then the number of edges in G is at most

$$t_1\binom{d}{2}+\binom{p_1+1}{2}+t_2\binom{d}{2}+\binom{p_2+1}{2} =: N.$$

If $t_1+t_2 = t$, then $(p_1+1)+(p_2+1)\leqq p+2$, and $p_1+1\leqq p+1$ and $p_2+1\leqq p+1$, and so

$$N\leqq t\binom{d}{2}+\binom{p+1}{2}+\binom{1}{2} = t\binom{d}{2}+\binom{p+1}{2},$$

contrary to hypothesis. If, on the other hand, $t_1+t_2=t-1$, then $(p_1+1)+(p_2+1)\leqq \leqq d+p+1$, and $p_1+1<d$ and $p_2+1<d$, and so

$$N<(t-1)\binom{d}{2}+\binom{d}{2}+\binom{p+1}{2} = t\binom{d}{2}+\binom{p+1}{2},$$

again contrary to hypothesis. Thus one of G_1 and G_2 must in fact satisfy the hypotheses of the theorem, and the result follows by induction.

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