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A NINE-FOLD PACKING

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Let d_k represent the density of closest k-fold packing of equal circles in the plane when the centres of the circles are at the points of a lattice Λ. It is well known that $d_1=\pi/2\sqrt{3}$, this result being essentially due to Lagrange. HEPPES [4] proved that $d_k/kd_1=1$ for $k=2, 3, 4$ and BLUNDON [1] found the values of d_5 and d_6.

BLUNDON [2] pointed out an inequality in the Heppes paper which leads readily to an estimate for d_k and later [3] he gave a sharper estimate,

$$d_k/kd_1 \geqq f(x/k), \tag{1}$$

where $c=[k\theta]$ and $\theta=\frac{1}{13}(6-\sqrt{10})=0.21828\ldots$, and where

$$f(x)=(1-x^2)/(1-4x^2)^{1/2}.$$

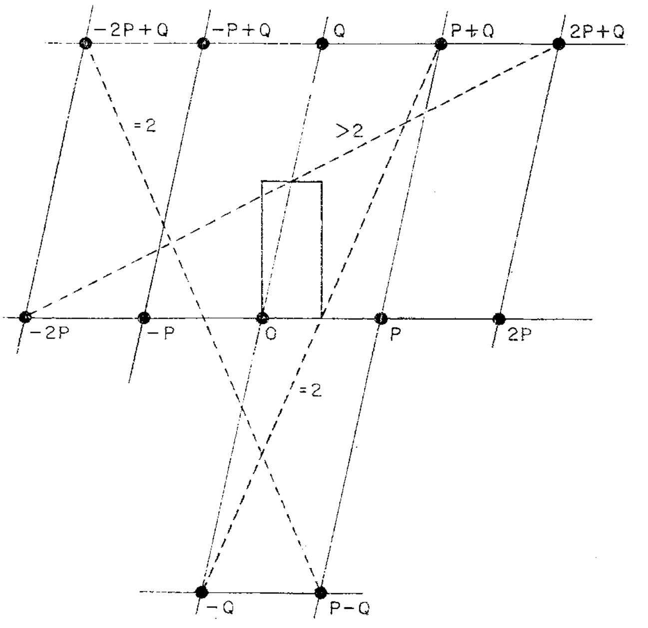

Fig. 1

The values of d_k for $k=1, 2, \ldots, 6$ are all given by equality in (1). Further, (1) gives the best known estimates for d_k for small $k \geqq 7$.

In particular, (1) gives $d_9/9k_1 \geqq 1.01298\ldots$. The purpose of this paper is to give an improved estimate for d_9. We prove the following

THEOREM. $d_9/9d_1 \geqq \frac{25}{63}\sqrt{7} = 1.04990\ldots$, *and the corresponding lattice is that generated by the points* $P(2/5, 0)$ *and* $Q\left(1/5, \frac{1}{5}\sqrt{21}\right)$.

PROOF. Let circles of unit radius be centred at the points of Λ. By symmetry, it is sufficient to prove that no point in the open rectangle with vertices at $O, \frac{1}{2}P, \frac{1}{4}P+\frac{1}{2}Q, -\frac{1}{4}P+\frac{1}{4}Q$ is covered by more than nine circles of Λ. The only circles having points in common with this rectangle are those centred at the twelve lattice points $-2P+Q, -P+Q, Q, P+Q, 2P+Q, -2P, -P, O, P, 2P, -Q, P-Q$.

Since $|P+2Q|=2$, no point of the rectangle can be common to the circles centred at $P+Q$ and $-Q$. Similar considerations apply to the pair $2P+Q$, $-2P$ and also to the pair $P-Q$ and $-2P+Q$. It follows that no point of the rectangle and hence no point of the plane can be covered by more than nine circles.

The estimate for d_9 follows at once from the fact that the determinant of Λ is $\frac{2}{25}\sqrt{21}$ and the determinant of the lattice giving best single packing for circles of unit radius is $2\sqrt{3}$.

References

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