A Degenerate Staudt-Clausen Theorem

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1. Put

$$\frac{x}{(1+\lambda x)^{\mu}-1} = \sum_{m=0}^{\infty} \beta_m(\lambda) \frac{x^m}{m!} \qquad (\lambda \mu = 1), \tag{1.1}$$

so that $\beta_m(\lambda)$ is a polynomial in λ with rational coefficients. Comparison of (1.1) with

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}, \tag{1.2}$$

which defines the Bernoulli numbers, suggests that $\beta_m(\lambda)$ may have some arithmetic properties analogous to those of B_m. We shall see that this is indeed the case for rational λ.

To begin with, if we put $y = (1+\lambda x)^{\mu} - 1$, then $\lambda x = (1+y)^{\lambda} - 1$, so that

$$\frac{x}{y} = \frac{(1+y)^{\lambda}-1}{\lambda y} = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{\lambda-1}{k} y^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{\lambda-1}{k} \sum_{s=0}^{\infty} (-1)^{k-s} \binom{k}{s} \sum_{m=k}^{\infty} \binom{s\mu}{m} \lambda^m x^m .$$

Comparison with (1.1) yields

$$\begin{aligned} \beta_m(\lambda) &= \sum_{k=0}^{m} \frac{1}{k+1} \binom{\lambda-1}{k} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} (s\mu)_m \lambda^m \\ &= \sum_{k=0}^{m} \frac{1}{k+1} \binom{\lambda-1}{k} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} s(s-\lambda) \ldots (s-(m-1)\lambda), \end{aligned} \tag{1.3}$$

where $(x)_m = x(x-1)\ldots(x-m+1)$. This explicit formula for $\beta_m(\lambda)$ may be compared with the well-known formula for the Bernoulli numbers

$$B_m = \sum_{k=0}^{m} \frac{1}{k+1} \sum_{s=0}^{k} (-1)^s \binom{k}{s} s^m ; \tag{1.4}$$

incidentally we see that

$$\beta_m(0) = B_m. \tag{1.5}$$

It also follows immediately from (1.1) that

$$\beta_m(1) = 0 \qquad (m \geq 1) \tag{1.6}$$

and

$$\beta_m(-\lambda) = (-1)^m \beta_m(\lambda) \qquad (m \geq 2); \tag{1.7}$$

note that $\beta_1(\lambda) = (\lambda - 1)/2$.

2. Suppose now that λ is a rational number a/b, where $(a,b) = 1$. Since

$$\sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} s(s-\lambda) \ldots (s-(m-1)\lambda)$$

is the k-th difference of a polynomial with integral coefficients (primes dividing b are ignored), it is divisible by $k!$ By a familiar argument $k!/(k+1)$ is integral except when $k+1 = 4$ or a prime p. In the latter case, if $p \nmid b$, the corresponding term in the right member of (1.3) becomes

$$\frac{1}{p} \binom{\lambda-1}{p-1} \sum_{s=0}^{p-1} (-1)^{p-1-s} \binom{p-1}{s} s(s-\lambda) \ldots (s-(m-1)\lambda), \tag{2.1}$$

which is certainly integral (mod p) unless $p \mid a$, in which case (2.1) reduces to

$$\frac{1}{p} \sum_{s=0}^{p-1} s^m + A,$$

where A is integral (mod p). Finally it is clear that we get $-1/p$ if $p-1 \mid m$ while otherwise (2.1) is integral (mod p). Taking next the case $k+1 = 4$ we may evidently assume that $2 \mid a$. In place of (2.1) we now have

$$\tfrac{1}{4} \binom{\lambda-1}{3} \sum_{s=0}^{3} (-1)^{3-s} \binom{3}{s} s(s-\lambda) \ldots (s-(m-1)\lambda). \tag{2.2}$$

If $4 \nmid a$ we find that (2.2) is integral (mod. 2), while if $4 \mid a$ we get

$$\tfrac{1}{4} \sum_{s=0}^{3} (-1)^{3-s} \binom{3}{s} s^m = \tfrac{1}{4} (3^m - 3 \cdot 2^m + 3 \cdot 1^m);$$

for m even this is integral, while for m odd > 1 we get a contribution of 1/2. Summing up we may state the following

Theorem 1. *Let $\lambda = a/b$, $(a,b) = 1$. Then for m even*

$$\beta_m(\lambda) = A_m - \sum_{\substack{p-1 \mid m \\ p \mid a}} \frac{1}{p}, \tag{2.3}$$

where A_m is a rational number whose denominator contains only primes occuring in b. For m odd $\beta_1(\lambda) = (\lambda-1)/2$ and

$$\beta_m(\lambda) = A_m - \tfrac{1}{2} \qquad (m > 1) \tag{2.4}$$

provided $2 \mid a$, $4 \nmid a$, while if $2 \nmid a$ or $4 \mid a$ then $\beta_m(\lambda) = A_m$. In particular when λ is a rational integer then A_m is also an integer.

The theorem, particularly (2.3), may be compared with the STAUDT-CLAUSEN theorem

$$B_{2m} = A_{2m} - \sum_{p-1 \mid 2m} \frac{1}{p},$$

where A_{2m} denotes an integer; for the proof compare [3; p. 32]. It may be of interest to note that a result like Theorem 1 holds also for the coefficients $b_m(\lambda)$ defined by

$$\frac{e^{\lambda x} - 1}{\lambda (e^x - 1)} = \sum_{m=0}^{\infty} b_m(\lambda) \frac{x^m}{m!}.$$

3. Consider next

$$\frac{1}{(1+\lambda x)^{\mu/a} - 1} - \frac{a}{(1+\lambda x)^{\mu} - 1}$$
$$= \frac{1}{(1+\lambda x)^{\mu} - 1} \left\{ \sum_{k=0}^{a-1} (1+\lambda x)^{k\mu/a} - a \right\}$$
$$= \frac{(1+\lambda x)^{\mu/a} - 1}{(1+\lambda x)^{\mu} - 1} \sum_{k=1}^{a-1} \frac{(1+\lambda x)^{k\mu/a} - 1}{(1+\lambda x)^{\mu/a} - 1}.$$

It follows that if the integer a is not divisible by the prime p and λ is integral (mod p), then the extreme right member is of the form

$$\sum_{m=0}^{\infty} A_m x^m/m!,$$

where the A_m are integral (mod p). Hence applying (1.1) we infer that

$$\frac{a^{-m} \beta_m(a\lambda) - \beta_m(\lambda)}{m} \tag{3.1}$$

is integral (mod p). This implies that if $p^r \mid m$ then

$$\beta_m(a\lambda) \equiv a^m \beta_m(\lambda) \qquad (\mathrm{mod}\ p^r). \tag{3.2}$$

When $p - 1 \mid m$ and $p \mid \lambda$, each member of (3.2) is fractional (mod p); however the difference is divisible by p^r, as is clear from (3.1). In this case (3.2) may be written in the simpler form

$$\beta_m(a\lambda) \equiv \beta_m(\lambda) \qquad (\mathrm{mod}\ p^r) \quad (p-1 \mid m). \tag{3.3}$$

Since when $p \mid b$, (3.2) implies

$$a^m \beta_m(\lambda) \equiv \beta_m(a\lambda) \equiv b^m \beta_m(a\lambda/b) \qquad (\mathrm{mod}\ p^r)$$

we may state the following

Theorem 2. *Let λ be integral* (mod p) *and $p \nmid ab$; also let $p^r \mid m$. Then we have*

$$\beta_m(a\lambda/b) \equiv (a/b)^m \beta_m(\lambda) \qquad (\mathrm{mod}\ p^r). \tag{3.4}$$

In particular if $\lambda \not\equiv 0 \pmod p$, (3.4) implies

(3.5) $$\beta_m(\lambda) \equiv \lambda^m \beta_m(1) \equiv 0 \qquad (\operatorname{mod} p^r) \quad (m \geq 1)$$

a result that will be improved below.

4. In place of the explicit expression (1.3), a formula of a different type can be obtained as follows: Let λ be a positive integer; then we have

$$\frac{x}{(1+\lambda x)^\mu - 1} = \frac{1}{\lambda} \sum_{k=0}^{\lambda-1} (1+\lambda x)^{k\mu}$$
$$= \frac{1}{\lambda} \sum_{k=0}^{\lambda-1} \sum_{m=0}^{\infty} \binom{k\mu}{m} \lambda^m x^m .$$

It follows that

(4.1) $$\begin{aligned} \beta_m(\lambda) &= \frac{1}{\lambda} \sum_{k=0}^{\lambda-1} \lambda^m (k\mu)_m \\ &= \frac{1}{\lambda} \sum_{k=0}^{\lambda-1} k(k-\lambda)(k-2\lambda) \ldots (k-(m-1)\lambda). \end{aligned}$$

This formula is a good deal simpler than (1.3) but is of course only meaningful for integral λ. For example by means of (4.1) it is evident that the only primes occuring in the denominator of $\beta_m(\lambda)$ must divide λ; indeed it is easy by means of (4.1) to give another proof of Theorem 1 for the case of integral λ.

If we put

$$x(x-1)\ldots(x-m+1) = \sum_{r=0}^{m-1} (-1)^r S(m,r)\, x^{m-r} ,$$

and recall that

$$\sum_{k=0}^{\lambda-1} k^m = \frac{B_{m+1}(\lambda) - B_{m+1}}{m+1} ,$$

where $B_{m+1}(\lambda)$ is the Bernoulli polynomial of degree $m+1$, then (4.1) becomes

(4.2) $$\beta_m(\lambda) = \frac{1}{\lambda} \sum_{r=1}^{m} (-1)^{m-r} S(m, m-r)\, \lambda^{m-r} \frac{B_{r+1}(\lambda) - B_{r+1}}{r+1} .$$

Making use of

$$B_r(\lambda) = \sum_{s=0}^{r} \binom{r}{s} B_s \lambda^{r-s} ,$$

we get

(4.3) $$\beta_m(\lambda) = \sum_{s=0}^{m} B_s \lambda^{m-s} \sum_{r=s}^{m} \frac{(-1)^{m-r}}{r+1} \binom{r+1}{s} S(m, m-r) .$$

Since both members of (4.3) are polynomials and the equality holds for infinitely many values of λ, it follows that (4.3) holds for all λ. The same is true of (4.2) also. If we prefer, the coefficients in (4.3) can be expressed in terms of BERNOULLI numbers of higher order. Incidentally (4.3) shows that $\beta_m(\lambda)$ is a polynomial of degree $\leq m$.

Returning to (4.1) we shall prove a divisibility property of $\beta_m(\lambda)$. In the first place if $p \nmid \lambda$ then the quotient

$$k(k-\lambda)\ \ldots\ (k-(m-1)\lambda)/m!$$

is integral (mod p). Consequently if p^t is the highest power of p dividing $m!$ it follows that

$$\beta_m(\lambda) \equiv 0 \qquad (\bmod\ p^t) \qquad (p \nmid \lambda)\,, \tag{4.4}$$

which may be compared with (3.5). It should however by observed that (4.4) has been proved only for integral λ.

Let us now examine the case $p \mid \lambda$. It is convenient to assume first that $\lambda = p^t$, $t \geq 1$. We shall require the following

Lemma. *Let* $p^r \mid m$, $p \geq 3$, $t \geq 1$; *then*

$$\prod_{s=0}^{m-1} (x-sp^t) \equiv x^m \qquad (\bmod\ p^{r+t})\,. \tag{4.5}$$

Indeed LUBELSKI [2] has proved the identical congruence ($p \geq 3$)

$$\prod_{s=0}^{p^r-1} (x-sp) \equiv x^{p^r} \qquad (\bmod\ p^{r+1})\,. \tag{4.6}$$

Putting (4.6) in homogeneous form, we get

$$\prod_{s=0}^{p^r-1} (x-spy) \equiv x^{p^r} \qquad (\bmod\ p^{r+1}\,y)\,.$$

If now we take $y = p^{t-1}$, we get

$$\prod_{s=0}^{p^r-1} (x-sp^t) \equiv x^{p^r} \qquad (\bmod\ p^{r+t})\,. \tag{4.7}$$

Finally we have

$$\begin{aligned}\prod_{s=0}^{np^r-1} (x-sp^t) &= \prod_{k=0}^{n-1}\ \prod_{s=0}^{p^r-1} (x-(s+kp^r)\,p^t)\\ &\equiv \prod_{k=0}^{n-1}\ \prod_{s=0}^{p^r-1} (x-sp^t)\\ &\equiv \prod_{k=0}^{n-1} x^{p^r} \equiv x^{np^r} \qquad (\bmod\ p^{r+t})\,,\end{aligned}$$

which proves the Lemma.

By the Lemma and (4.1) it is clear that

(4.8) $$p^t \beta_m(p^t) \equiv \sum_{k=0}^{p^t-1} k^m \pmod{p^{r+t}},$$

provided $p^r \mid m$, $p \geq 3$, $t \geq 1$. But

$$\begin{aligned}\sum_{k=0}^{p^t-1} k^m &= \frac{B_{m+1}(p^t) - B_{m+1}}{m+1} \\ &= p^t B_m + \tfrac{1}{2} m p^{2t} B_{m-1} + \cdots \\ &\equiv p^t B_m \pmod{p^{r+2t}},\end{aligned}$$

so that (4.8) becomes

(4.9) $$\beta_m(p^t) \equiv B_m \pmod{p^r}.$$

Now if $p-1 \nmid m$, $p^r \mid m$, it is known that $B_m \equiv 0 \pmod{p^r}$. Consequently (4.9) reduces to

(4.10) $$\beta_m(p^t) \equiv 0 \pmod{p^r} \quad (p-1 \nmid m).$$

On the other hand, if $(p-1)\, p^r \mid m$ then [1; Theorem 3]

$$B_m + \frac{1}{p} - 1 \equiv 0 \pmod{p^r}.$$

Thus in this case we get

(4.11) $$\beta_m(p^t) + \frac{1}{p} - 1 \equiv 0 \pmod{p^r} \quad (p-1 \mid m).$$

Finally, using (3.4), (4.10) and (4.11) can be stated in the following more general form.

Theorem 3. *Let* $p \geq 3$, $p^r \mid m$, $p^t \mid \lambda$, $t \geq 1$. *Then*

(4.12) $$\beta_m(\lambda) \equiv 0 \pmod{p^r} \quad (p-1 \nmid m)$$

while

(4.13) $$\beta_m(\lambda) + \frac{1}{p} - 1 \equiv 0 \pmod{p^r} \quad (p-1 \mid m).$$

References

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