On Extensions and Bimultiplication Algebras of Algebras

By

C. W. Kohls and L. J. Lardy[1])

Extension theory for rings was developed by Everett [1], and later redescribed in different ways by Hochschild, Rédei, and MacLane. The approach of MacLane [4] was generalized by Shukla [6] to algebras over a commutative ring with identity.

The main section of this paper treats the bimultiplication algebra of an algebra, which plays an important role in extension theory. We characterize the bimultiplication algebra of a commutative semiprime algebra, and describe the bimultiplication algebra of a matrix algebra over any algebra with zero two-sided annihilator as a matrix algebra over a bimultiplication algebra. First we review the necessary background, and give a direct elementary development of three essentially known results from the extension theory of algebras that are required for our paper [3].

1. Preliminaries. We review here the approach to extension theory used by MacLane [4] and Shukla [6]. First we introduce some conventions: All algebras will be K-algebras, where K is a commutative ring with identity. Homomorphisms between K-algebras will always be K-homomorphisms. Unless emphasis is desired, the prefix K will be omitted from both terms. The image of a set S under a mapping f is written $f[S]$.

The set of all bimultiplications of a K-algebra A is a K-algebra with identity, denoted by M_A. Clearly the mapping $\nu: A \to M_A$ onto the inner bimultiplications is a K-algebra homomorphism, and if A has an identity, we have $\nu(1) = 1$. Since also

(1) $$\sigma \nu_c = \nu_{\sigma c}, \quad \nu_c \sigma = \nu_{c\sigma}, \quad \text{and} \quad k \nu_c = \nu_{kc},$$

the set $\nu[A]$ is a two-sided ideal in the K-algebra M_A. The quotient algebra $M_A/\nu[A]$ of outer bimultiplications of A is denoted by P_A, and the kernel of ν by C_A. We thus have the exact sequence of K-algebras

$$0 \to C_A \to A \xrightarrow{\nu} M_A \to P_A \to 0\,.$$

When $C_A = 0$, so that ν is an injection of A into M_A, it follows from (1) and the associative law in M_A that any two bimultiplications are permutable. Any mapping into M_A or P_A whose range consists of mutually permutable elements will be called *regular*.

[1]) The research of the second author was supported (in part) by Air Force Contract AF 49 (638) — 1401.

For given K-algebras A and Λ, an extension of A by Λ is an exact sequence $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$ of K-algebras and K-homomorphisms. We depart from the convention in [4] and [6] for the case that Λ has an identity; we do *not* wish to require that E also have an identity. There are naturally induced regular homomorphisms $\mu \colon E \to M_A$ and $\theta \colon \Lambda \to P_A$; $\mu\alpha$ coincides with ν. Two extensions $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$ and $0 \to A \xrightarrow{\alpha'} E' \xrightarrow{\beta'} \Lambda \to 0$ are equivalent if there is a K-homomorphism $\varphi \colon E \to E'$ such that $\beta = \beta' \varphi$ and $\alpha' = \varphi\alpha$. These equations imply that φ is an somorphism of E to E'.

For given K-algebras A and Λ let the regular function σ assign to each $x \in \Lambda$ a bimultiplication σ_x of A, with $\sigma_0 = 0$. Let $h(x, y)$, $h(x \mid y)$, and $g(k, x)$ be functions from $\Lambda \times \Lambda$ and $K \times \Lambda$ to A related to σ by

$$\sigma_x + \sigma_y = \nu h(x, y) + \sigma_{x+y}, \quad \sigma_x \sigma_y = -\nu h(x \mid y) + \sigma_{xy}, \tag{2}$$
$$k\sigma_x = \nu g(k, x) + \sigma_{kx},$$

and satisfying the normalization conditions

$$h(0, y) = h(x, 0) = h(0 \mid y) = h(x \mid 0) = g(k, 0) = g(0, x) = 0, \tag{3}$$

for all $x, y \in \Lambda$ and $k \in K$. Then if h, g, and σ satisfy certain identities [6, pp.200—203], there exists an extension $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$ and elements $u_x \in E$ such that

$$\beta u_x = x, \quad \mu u_x = \sigma_x, \tag{4}$$

$$u_x + u_y = \alpha h(x, y) + u_{x+y}, \quad u_x u_y = -\alpha h(x \mid y) + u_{xy}, \tag{5}$$
$$k u_x = \alpha g(k, x) + u_{kx}.$$

One constructs E, an *Everett extension*, as the set $\Lambda \times A$, with operations defined by the equations

$$(x, a) + (y, b) = (x + y,\, h(x, y) + a + b), \tag{6}$$

$$(x, a)(y, b) = (xy,\, -h(x \mid y) + \sigma_x b + a\sigma_y + ab), \tag{7}$$

$$k(x, a) = (kx,\, g(k, x) + ka). \tag{8}$$

Given an extension $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$, one can always make $\Lambda \times A$ into an Everett extension of A by Λ that is equivalent to E, by choosing $\sigma_x \in \theta(x)$ and $u_x \in E$ such that (4) holds, with $\sigma_0 = 0$ and $u_0 = 0$, and then choosing $h(x, y)$, $h(x \mid y)$, and $g(k, x)$ in A such that (3) and (5) hold.

An Everett extension of A by Λ in which $h(x, y) = h(x \mid y) = g(k, x) = 0$ for all $x, y \in \Lambda$ and $k \in K$ will be called a *splitting* extension.

The *graph* Γ of a regular homomorphism $\theta \colon \Lambda \to P_A$ is the subalgebra of the direct sum $\Lambda \oplus M_A$ consisting of all pairs (x, σ) such that $\sigma \in \theta(x)$. When $C_A = 0$, the homomorphisms $a \to (0, \nu a)$ and $(x, \sigma) \to x$ yield an extension $0 \to A \to \Gamma \to \Lambda \to 0$.

Let $\mathscr{K}$ be any category. For $K, L \in \mathscr{K}$, if there exist $\varphi \colon K \to L$ and $\psi \colon L \to K$ such that $\varphi\psi$ is the identity for L, then L is called a *retract* of K, and φ is called a *retraction* of K onto L.

2. Some basic results on extensions.

Proposition 1. *Let A and Λ be K-algebras. Each regular homomorphism $\sigma: \Lambda \to M_A$ determines a splitting extension of A by Λ. Conversely, each splitting extension of A by Λ determines a regular homomorphism $\sigma: \Lambda \to M_A$.*

Proof. Let σ be a regular homomorphism. Let E be the set $\Lambda \times A$, with algebra operations defined by (6), (7), and (8), where $\sigma_x = \sigma(x)$ for all $x \in \Lambda$ and $h(x, y) = h(x|y) = g(k, x) = 0$ for all $x, y \in \Lambda$ and $k \in K$. It is easy to verify directly that E is a K-algebra, from which it follows that E is a splitting extension of A by Λ.

Conversely, if a splitting extension of A by Λ is given, then the associated set of bimultiplications is permutable. Let $\sigma: \Lambda \to M_A$ be defined by $\sigma(x) = \sigma_x$; then σ is a regular homomorphism.

We can immediately deduce from Proposition 1 that if A is a K-algebra with $A^2 = 0$, and $\theta: \Lambda \to P_A$ is a given regular homomorphism, then there exists a splitting extension of A by Λ inducing θ [6, Theorem 4]. For in this case, we have $P_A = M_A$.

We now show that the concepts of retraction and splitting extension are essentially the same. The result is well-known in the case of extensions of algebras A with $A^2 = 0$.

Theorem 1. *A homomorphism $\beta: E \to \Lambda$ is a retraction of E onto Λ if and only if E is equivalent to a splitting extension of the kernel of β by Λ.*

Proof. Assume that β is a retraction of E onto Λ, with $\gamma: \Lambda \to E$ a homomorphism such that $\beta\gamma$ is the identity on Λ. Let A be the kernel of β, and $\alpha: A \to E$ the identity. The homomorphism $\sigma = \mu\gamma$ is regular. Thus, by Proposition 1, it determines a splitting extension E'. Let $\varphi: E \to E'$ be defined by the equation $\varphi(e) = (\beta(e), e - \gamma\beta(e))$. Since $\beta\gamma$ is the identity on Λ, $\beta(e - \gamma\beta(e)) = 0$, whence $\varphi(e) \in E'$. A straightforward calculation shows that φ is a homomorphism. It then follows easily that E' is equivalent to E.

Now assume that $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$, where α is the identity, is equivalent under ψ to $0 \to A \xrightarrow{\alpha'} E' \xrightarrow{\beta'} \Lambda \to 0$, where E' is splitting. Then by (5), the mapping $\gamma': \Lambda \to E'$ defined by $\gamma'(x) = u'_x$ is a homomorphism, and $\beta'\gamma'$ is the identity on Λ. Define the homomorphism $\gamma: \Lambda \to E$ by $\gamma = \psi\gamma'$. Then $\beta\gamma = \beta\psi\gamma' = \beta'\gamma'$. It follows that β is a retraction of E onto Λ.

The following result is essentially [4, Corollary to Theorem 7], but we need the statement with no restrictions on the extensions or homomorphisms when Λ has an identity.

Proposition 2. *If $C_A = 0$, then there is a one-to-one correspondence between equivalence classes of extensions of A by Λ and regular homomorphisms θ from Λ to P_A. Each equivalence class contains the graph of the corresponding θ.*

Proof. With each extension, we associate the induced homomorphism from Λ to P_A. It is easy to verify that equivalent extensions induce the same homomorphism, and that if θ is any given regular homomorphism from Λ to P_A, then the extension $0 \to A \to \Gamma \to \Lambda \to 0$, where Γ is the graph of θ, induces θ.

Now suppose that the extension $0 \to A \to E \to \Lambda \to 0$ induces the homomorphism $\theta: \Lambda \to P_A$, and consider the extension $0 \to A \to \Gamma \to \Lambda \to 0$. Define the mapping $\varphi: E \to \Gamma$ by $\varphi(e) = (\beta e, \mu e)$. It follows readily from section 1 that φ is a homomorphism and that E is equivalent to Γ. Thus if two extensions induce the same homomorphism, then they are equivalent.

3. Characterizations of certain bimultiplication algebras. In this section we study the algebra of bimultiplications of K-algebras A with $C_A = 0$. Recall that a two-sided ideal I in a K-algebra A is said to be *dense in* A if the two-sided annihilator of I in A is zero. Observe that if A has a dense ideal, then $C_A = 0$. Also, it is easy to verify that the identity automorphism is the only endomorphism of A whose restriction to a dense ideal is the identity.

Proposition 3. *Let A be a commutative K-algebra, and assume that A is a dense ideal in the K-algebra B. Then B is commutative.*

Proof. We first show that $ab = ba$ for $a \in A$ and $b \in B$. For any $a' \in A$, the fact that A is commutative and an ideal in B implies that $a'(ab - ba) = (ab - ba)a' = a(ba') - (ba)a' = (ba')a - (ba)a' = b(a'a) - b(aa') = 0$. Since A is dense in B, we have $ab = ba$. Now for $b, b' \in B$ and any $a \in A$, it follows that $a(bb' - b'b) = (bb' - b'b)a = (ba)b' - b'(ba) = b'(ba) - b'(ba) = 0$. The density of A in B implies that $bb' = b'b$. Thus, B is commutative.

Definition. A *bimultiplication algebra* of a K-algebra A is a pair (B, φ), where B is a K-algebra and φ is an injection of A onto a dense ideal in B. Two bimultiplication algebras (B_1, φ_1) and (B_2, φ_2) are *equivalent* if there exists an isomorphism ψ of B_1 onto B_2 such that $\psi\varphi_1 = \varphi_2$. The equivalence class containing (B, φ) is written $(\bar{B}, \bar{\varphi})$.

Note that if A has an identity e, then $\varphi(e)$ is the identity of B, and $\varphi[A] = B$.

The collection of equivalence classes of all bimultiplication algebras of a given K-algebra A can be partially ordered by defining $(\bar{B}_1, \bar{\varphi}_1) \leqq (\bar{B}_2, \bar{\varphi}_2)$ if and only if there exists an injection ψ of B_1 into B_2 such that $\psi\varphi_1 = \varphi_2$.

Proposition 4. *For any K-algebra A with $C_A = 0$, the pair $(\bar{M}_A, \bar{\nu})$ is the largest element in the collection of equivalence classes of all bimultiplication algebras of A.*

Proof. It follows from (1) that (M_A, ν) is a bimultiplication algebra of A. Let (B, φ) be an arbitrary bimultiplication algebra of A. We define a mapping ψ from B into M_A by setting $\psi(b)a = \varphi^{-1}(b\varphi(a))$ and $a\psi(b) = \varphi^{-1}(\varphi(a)b)$ for each $b \in B$ and all $a \in A$. It is easy to check that ψ is an injection of B into M_A such that $\psi\varphi = \nu$.

The algebra of all A-module homomorphisms of a commutative K-algebra A into itself will be denoted by H_A. Let η denote the natural mapping from A into H_A: $\eta(a)a' = aa'$ for $a, a' \in A$. It is easy to verify that when A is a commutative algebra with $C_A = 0$, (H_A, η) is a bimultiplication algebra of A.

Proposition 5. *Let A be a commutative K-algebra with $C_A = 0$, and let (B, φ) be a bimultiplication algebra of A. Then $(\bar{B}, \bar{\varphi}) \leqq (\bar{H}_A, \bar{\eta})$, and (M_A, ν) is equivalent to (H_A, η).*

Proof. For a given bimultiplication algebra (B, φ) of A, let ψ denote the injection of B into H_A defined by $\psi(b)a = \varphi^{-1}(b\varphi(a))$. It is easy to check that $\psi\varphi = \eta$. Thus $(\bar{B}, \bar{\varphi}) \leqq (\bar{H}_A, \bar{\eta})$. By Proposition 4, $(\bar{H}_A, \bar{\eta}) = (\bar{M}_A, \bar{\nu})$.

We shall now characterize the algebra M_A of bimultiplications of a commutative semiprime algebra A, and give a more definitive result in the case of an algebra of continuous functions. The latter information is vital in [3]. The algebra of all continuous functions from a topological space X into a topological field F is designated by $C(X, F)$.

Theorem 2. *Let A be a commutative semiprime K-algebra. Assume that A is isomorphic to a subdirect sum of $\{D_x : x \in X\}$, where each D_x is an integral domain, and let F_x denote the field of quotients of D_x. Then (M_A, ν) is equivalent to (S, ι), where S is the largest subalgebra of the complete direct sum of $\{F_x : x \in X\}$ containing A as an ideal, and ι is the natural injection of A into S. If in addition each F_x coincides with the topological field F, X is a topological space, and A is a subalgebra of $C(X, F)$, then M_A is isomorphic to a subalgebra of $C(X, F)$.*

Proof. Clearly $C_A = 0$, so by Proposition 5 we may view (M_A, ν) as (H_A, η). Let $\sigma \in H_A$ and define σ^* as follows: For each $x \in X$, choose $f \in A$ such that $f(x) \neq 0$, and set $\sigma^*(x) = (\sigma f)(x)/f(x)$. To see that σ^* is well-defined, suppose that also $g(x) \neq 0$. Then, since $(\sigma f)g = \sigma(fg) = \sigma(gf) = (\sigma g)f$, we have $(\sigma f)(x)/f(x) = (\sigma g)(x)/g(x)$. If $f, g \in A$ with $f(x) \neq 0$, then

$$(\sigma^* g)(x) = \sigma^*(x)g(x) = ((\sigma f)(x)/f(x))g(x) = \\ = \sigma(fg)(x)/f(x) = ((\sigma g)(x)f(x))/f(x) = (\sigma g)(x).$$

Thus, $(\sigma^* g)(x) = (\sigma g)(x)$ for any $x \in X$.

Let S denote the subalgebra of all elements h in the complete direct sum of $\{F_x : x \in X\}$ such that $hA \subset A$ and let ι denote the injection of A into S. Clearly S is the largest subalgebra of the complete direct sum of $\{F_x : x \in X\}$ containing A as an ideal, and (S, ι) is a bimultiplication algebra of A. Define the mapping $\psi: H_A \to S$ by $\psi(\sigma) = \sigma^*$. It follows easily that ψ is an injection and that $\psi\eta = \iota$. Thus $(\bar{H}_A, \bar{\eta}) \leqq \leqq (\bar{S}, \bar{\iota})$, and by Proposition 5 (M_A, ν) is equivalent to (S, ι).

Now assume that each F_x coincides with the topological field F, X is a topological space, and A is a subalgebra of $C(X, F)$. Given any $\sigma^* \in S$ and any $x \in X$, choose $f \in A$ such that $f(x) \neq 0$, and let V be a neighborhood of $f(x)$ not containing 0. Select a neighborhood U of x such that $f[U] \subset V$. Then $\sigma^*(y) = (\sigma f)(y)/f(y)$ for all $y \in U$, so σ^* is continuous at x. It follows that M_A is isomorphic to a subalgebra of $C(X, F)$.

Corollary. *If A is a p-ring (Boolean ring), then M_A and P_A are also p-rings (Boolean rings).*

Proof. Any p-ring A is isomorphic to a subdirect sum of copies of the ring of integers modulo p [5, Theorem 45]. By Theorem 2, M_A is isomorphic to a subring of the corresponding complete direct sum. Hence, by [5, Theorem 45], M_A is a p-ring; and P_A is a homomorphic image of M_A, so P_A is also a p-ring.

The statement for Boolean rings is a special case, since a Boolean ring is simply a 2-ring.

We note that Theorem 2 and its Corollary can be used to give an alternative proof that every extension of a Boolean ring by a Boolean ring is also a Boolean ring [2, Corollary 2]. Consider

$$(x, a)^2 = (x^2, -h(x\,|\,x) + \sigma_x a + a\,\sigma_x + a^2).$$

If A is Boolean, then so is M_A, and $\sigma_x a = a\,\sigma_x$ by Theorem 2; hence $\sigma_x a + a\,\sigma_x = 0$. Now $\nu h(x\,|\,x) = \sigma_{x^2} - \sigma_x^2$; so if Λ is also Boolean, $h(x\,|\,x) = 0$. Thus, (x, a) is idempotent when both A and Λ are Boolean.

Example. Let $\mathbf{R}$ denote the set of real numbers. Let A be the subalgebra of $C(\mathbf{R}, \mathbf{R})$ consisting of proper rational functions; that is, $f \in A$ if and only if $f = P/Q$, where P and Q are polynomials, $\deg P < \deg Q$, and Q has no real zeroes. By Theorem 2, M_A is isomorphic to the largest subalgebra of $C(\mathbf{R}, \mathbf{R})$ containing A as an ideal. It is clear that to meet this requirement, the algebra must consist of rational functions. One also sees quickly that the only rational functions by which one can multiply those functions in A such that the degrees of the numerator and denominator differ by one, and obtain an element of A, are those such that the degree of the numerator does not exceed that of the denominator. Thus, M_A is isomorphic to the subalgebra of $C(\mathbf{R}, \mathbf{R})$ consisting of rational functions with the degree of the numerator not greater than the degree of the denominator. This is also the subalgebra of $C(\mathbf{R}, \mathbf{R})$ generated by A and the identity element. Clearly P_A is isomorphic to $\mathbf{R}$.

Now consider an extension of A by $\mathbf{R}$. The induced homomorphism θ maps $\Lambda = \mathbf{R}$ into $P_A = \mathbf{R}$ and thus must either be the zero homomorphism or the identity homomorphism. Now by Proposition 2, inequivalent extensions induce different homomorphisms; so there are only two equivalence classes of extensions of A by $\mathbf{R}$. The direct sum induces the zero homomorphism, while M_A itself induces the identity homomorphism.

We now consider bimultiplication algebras of matrix algebras. Let $L_n(A)$ denote the K-algebra of all n by n matrices over the K-algebra A. Note that an element in $L_n(M_A)$ is a matrix of ordered pairs. When $C_A = 0$, the identification of $[\nu a_{ij}] \in L_n(M_A)$ with $[a_{ij}] \in L_n(A)$ permits one to multiply elements of $L_n(M_A)$ and $L_n(A)$; thus each element of $L_n(M_A)$ operates on $L_n(A)$ in a natural way. We shall use this tacitly. The element of $L_n(A)$ with $a \in A$ in the ij-th place and 0 elsewhere will be denoted by aE_{ij}. The symbol $S(a)$ will be used for the scalar matrix with $a \in A$ along the main diagonal.

Theorem 3. *Let A be a K-algebra with $C_A = 0$. Then $M_{L_n(A)}$ and $L_n(M_A)$ are K-isomorphic.*

Proof. For any element $[\sigma_{ij}]$ of $L_n(M_A)$, we define $\varrho([\sigma_{ij}])$ to be the element $\sigma \in M_{L_n(A)}$ such that $\sigma[a_{ij}] = [\sigma_{ij}][a_{ij}]$ and $[a_{ij}]\,\sigma = [a_{ij}][\sigma_{ij}]$ for all $[a_{ij}] \in L_n(A)$. It is easy to verify that ϱ is a K-monomorphism into $M_{L_n(A)}$.

To prove that ϱ is onto, consider any $\sigma \in M_{L_n(A)}$. For each $a \in A$, we define $\sigma_{ij}a$ and $a\,\sigma_{ij}$ to be the elements of A in the ij-th place of $\sigma(S(a))$ and $(S(a))\,\sigma$, respectively. It is straightforward to check that each σ_{ij} is in M_A, whence $[\sigma_{ij}] \in L_n(M_A)$. It will follow that $\varrho([\sigma_{ij}]) = \sigma$ once it is known that $\sigma[a_{ij}] = [\sigma_{ij}][a_{ij}]$ and $[a_{ij}]\,\sigma = [a_{ij}][\sigma_{ij}]$ for all $[a_{ij}] \in L_n(A)$. We verify only the first identity; the second can be obtained by similar considerations.

For fixed $a \in A$, we examine $\sigma(aE_{ij})$. Write $\sigma(aE_{ij}) = [c_{lm}]$, let $b \in A$ be arbitrary, and choose any integer $k \neq j$, $1 \leqq k \leqq n$. Then

$$\sum_l c_{lk}\, b\, E_{lk} = (\sigma(a\,E_{ij}))(b\,E_{kk}) = \sigma((a\,E_{ij})(b\,E_{kk})) = \sigma(0) = 0;$$

thus $c_{lk} b = 0$. Also $[b c_{lm}] = (S(b))(\sigma(aE_{ij})) = ((S(b))\,\sigma)(aE_{ij})$, a matrix with all entries not in the j-th column equal to zero; thus $b c_{lk} = 0$. Since $C_A = 0$, it follows that $c_{lk} = 0$, for $k \neq j$ and $1 \leqq l \leqq n$. Therefore only the j-th column of $\sigma(aE_{ij})$ can be nonzero.

Now

$$[\sigma_{ij}\, a] = \sigma(S(a)) = \sigma\left(\sum_k a\,E_{kk}\right) = \sum_k \sigma(a\,E_{kk}).$$

In view of what has just been shown, the j-th columns of $[\sigma_{ij} a]$ and $\sigma(aE_{jj})$ must coincide, that is, $\sigma(a\,E_{jj}) = \sum\limits_l \sigma_{lj}\, a\, E_{lj}$.

We next consider $\sigma(aE_{ij})$ when $i \neq j$. Now for any $b \in A$,

$$(\sigma(a\,E_{ii} - a\,E_{ij}))(b\,E_{ii} + b\,E_{ji}) = \sigma((a\,E_{ii} - a\,E_{ij})(b\,E_{ii} + b\,E_{ji})) = \sigma(0) = 0.$$

But this expression is also

$$(\sigma(a\,E_{ii}) - \sigma(a\,E_{ij}))(b\,E_{ii} + b\,E_{ji}) =$$
$$= \left(\sum_l \sigma_{li}\, a\, E_{li} - \sum_l c_{lj}\, E_{lj}\right)(b\,E_{ii} + b\,E_{ji}) = \sum_l (\sigma_{li}\, a - c_{lj})\, b\, E_{li}.$$

Thus, $(\sigma_{li} a - c_{lj})\, b = 0$ for all $b \in A$. Furthermore,

$$(S(b))(\sigma(a\,E_{ii} - a\,E_{ij})) =$$
$$= (S(b))\left(\sum_l \sigma_{li}\, a\, E_{li} - \sum_l c_{lj}\, E_{lj}\right) = \sum_l b\,(\sigma_{li}\, a)\, E_{li} - \sum_l b\, c_{lj}\, E_{lj},$$

which can also be written

$$((S(b))\,\sigma)(a\,E_{ii} - a\,E_{ij}) = [b\,\sigma_{lm}](a\,E_{ii} - a\,E_{ij}) = \sum_l (b\,\sigma_{li})\, a E_{li} - \sum_l (b\,\sigma_{li})\, a E_{lj}.$$

Hence $b(c_{lj} - \sigma_{li} a) = 0$ for all $b \in A$. Since $C_A = 0$, we have $c_{lj} = \sigma_{li} a$, $1 \leqq l \leqq n$. It follows that $\sigma(a\,E_{ij}) = \sum\limits_l \sigma_{li} a E_{lj}$. Combining this with our earlier result, we have

$$\sigma(a\,E_{ij}) = \sum_l \sigma_{li}\, a\, E_{lj}$$

in all cases.

Finally, for all $[a_{ij}] \in L_n(A)$, these equations and the additivity of σ yield $\sigma[a_{ij}] = [\sigma_{ij}][a_{ij}]$, as we wished to prove.

References

[1] C. J. Everett Jr., An extension theory for rings. Amer. J. Math. **64**, 363–370 (1942).
[2] C. W. Kohls, Properties inherited by ring extensions. Michigan Math. J. **12**, 399–404 (1965).
[3] C. W. Kohls and L. J. Lardy, Extensions and retractions of algebras of continuous functions. Illinois J. Math. **12**, 539–549 (1968).
[4] S. MacLane, Extensions and obstructions for rings. Illinois J. Math. **2**, 316–345 (1958).

[5] N. H. McCoy, Rings and ideals. Buffalo 1948.
[6] U. SHUKLA, Cohomologie des algèbres associatives. Ann. Sci. École Norm. Sup. 78, 163–209 (1961).

Eingegangen am 30. 7. 1968

Anschrift der Autoren:
C. W. Kohls
L. J. Lardy
Department of Mathematics
Syracuse University
Syracuse (N.Y.), USA