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Superlinear elliptic boundary value problems with rotational symmetry

By

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In this paper we extend and sharpen an earlier existence result [3] for superlinear elliptic boundary value problems on balls $\Omega = B_R = \{x \mid |x| < R\} \subset \mathbb{R}^N$.

Let

(1) $$Lu = -r^{1-N}\partial_r(a(r)\,r^{N-1}\,\partial_r u)$$

be a uniformly elliptic radial differential operator on Ω with $0 < a_0 \leqq a \in L^\infty$, and let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carthéodory function satisfying the conditions

(2) $$g(x, u) = g(|x|, u),$$

g is Lipschitz continuously differentiable with respect to u a.e. in Ω and there exist constants p, s, t

$$2 < p < q := \frac{2N}{N-2}\,{}^{2)}, \quad s > q/(q-p) > \tfrac{1}{2}N, \quad t > \tfrac{1}{2}N$$

and functions $\sigma \in L^s$, $\tau \in L^t$ such that

(3) $$\begin{aligned} |g(x,u)| &\leqq \sigma(x)\,|u|^{p-1} + \tau(x), \\ |g_u(x,u)| &\leqq \sigma(x)\,|u|^{p-2} + \tau(x), \\ |g_u(x,u) - g_u(x,v)| &\leqq (\sigma(x)(|u|^{p-2} + |v|^{p-2}) + \tau(x))\,|u-v|. \end{aligned}$$

There exists a function $\varrho\colon \mathbb{R} \to \mathbb{R}$ such that $\varrho(t)/t^2 \to \infty$ $(t \to \infty)$ and constants $\alpha, A, \beta, B, \Gamma$, $0 < \alpha < 1$, $2 < A$, $0 < B$, $0 \leqq \Gamma$, such that for any $u \in L^p$ with $\|u\|_p \geqq \Gamma$ the estimate holds:

(4) $$\varrho(\|u\|_p) \leqq A \int_\Omega G(u)\,dx \leqq \int_\Omega g(u)\,u\,dx \leqq \alpha \int_\Omega g_u(u)\,u^2\,dx \leqq B\,\|u\|_p^p.$$

Here, $G(u) = \int_0^u g(v)\,dv$ is a primitive of g.

With the above assumptions on g and L we prove

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[2]) $p < \infty$, $s > 1$ if $N = 2$.

Theorem. *There exists $k_0 \in \mathbb{N}$ such that for any $k \geqq k_0$ there exists a pair of radial solutions u^+, u^-, $u^\pm(x) = u^\pm(|x|)$, of the boundary value problem*

$$(5) \qquad Lu = g(x, u) \quad in \quad \Omega, \qquad u \,|\, \partial\Omega = 0$$

with the following properties:

(i) $u^-(0) < 0 < u^+(0)$.

(ii) u^+, u^- *both posses exactly k nodes* $r_1^\pm$, $u^+(r_1^+) = 0$, $u^-(r_1^-) = 0$, *in* $]0, R]$.

The above Theorem gives a more delicate description of the solution set than our earlier result [2]; also the growth assumptions on g are slightly weaker. In particular, the existence of infinitely many radial solutions with prescribed sign at $r = 0$ had been controversial. An a priori estimate for positive solutions of equations (5) due to Gidas and Spruck seemed to incidate the existence of functions g depending only on u such that (3) and (4) are satisfied but there exists only one solution of $\Delta u + g(u) = 0$ in B_R with $u(R) = 0$, $u(0) < 0$ (cp. [1]). Hopefully, the existence result presented here may help to clarify the situation.

The proof of our Theorem above largely uses ideas from [2]. However, we apply lower semi-continuity type arguments to obtain solutions of (5) with the behavior prescribed in the Theorem. In fact, the use of Lusternik-Schnirelman theory in [2] to obtain solutions was not justified since the manifolds K_k defined there need not be differentiable and the Lusternik-Schnirelman deformation hence need not be defined. I thank J.-M. Coron and H. Berestycki for having pointed out this mistake. Incidentally, this "regularity gap" in our original proof seems to be reflected in a slightly strengthened hypothesis on the integrability of the "free term" τ in (3), as compared with the assumptions made in [2].

Proof. As in [2] we interpret (5) as the Euler equations of a functional $E\colon H \to \mathbb{R}$ where

$$H = \{u \in H_0^{1,2}(\Omega) \,|\, u(x) = u(|x|)\}.$$

Indeed, E is given by

$$E(u) = a(u) - \int_\Omega G(u)\,dx$$

with

$$a(u) = \tfrac{1}{2} \int_\Omega a\,|\nabla u|^2\,dx.$$

Also, for $u, v \in H$ denote by

$$b(u, v) = \int_\Omega a\,\nabla u \cdot \nabla v\,dx$$

the bilinear pairing associated to L.

Then weak solutions u for (5) equivalently may be characterized as critical points of E satisfying

$$b(u, v) = \int_\Omega g(u)\,v\,dx$$

for all $v \in H$.

By Lemma 1 of [2] H is continuously embedded into $C^0(\Omega\setminus\{0\})$. Thus, we may define for $k \in \mathbb{N}$

$$M_k^\pm = \{u \in H \mid \exists 0 < r_1 < \cdots < r_k = R : u(r_l) = 0,\ 1 \leqq l \leqq k,$$
$$\pm(-1)^l u(x) \geqq 0 \quad \text{in} \quad \Omega_l := \{x \mid r_{l-1} < |x| < r_l\},$$
$$\|u_l\|_p \geqq \Gamma;\ b(u_l, u_l) = \int_{\Omega_l} g(u_l)\, u_l\, dx,\ 1 \leqq l \leqq k\}.$$

Here, $u_l = u$ in Ω_l, $u_l = 0$ outside Ω_l; $r_0 := 0$. From (4) it is easy to see that $M_k^\pm \neq \emptyset$ for large k. Indeed, assume for the moment that $R > 1$. Consider u, $u(r) = 1$, $r < 1$,

$$u(r) = \cos((2k+1)\pi(r-1)/2(R-1)), \quad r \geqq 1.$$

Since $b(u_l, u_l) \to \infty\ (k \to \infty)$, $1 \leqq l \leqq k$, by (4) we can find $t_l > 0$ such that

$$b(t_l u_l, t_l u_l) = \int_{\Omega_l} g(t_l u_l)\, t_l u_l\, dx, \quad \|t_l u_l\|_p \geqq \Gamma,$$
$$1 \leqq l \leqq k, \quad \text{if} \quad k \geqq k_0.$$

For such k set

$$c_k^\pm = \inf\{E(u) \mid u \in M_k^\pm\}.$$

By (4) we have the estimate (cp. Lemma 2 of [2])

$$c\|u\|_{1,2}^2 \geqq a(u) \geqq E(u) = a(u) - \int_\Omega G(u)\, dx \geqq c \int_\Omega g(u)\, u\, dx$$
$$= c\, b(u, u) \geqq c\|u\|_{1,2}^2 \tag{6}$$

for $u \in M_k^\pm$. Thus the numbers $c_k^\pm$ are well-defined for k sufficiently large.

Lemma 1. *There exists $k_0 \in \mathbb{N}$ such that for $k \geqq k_0$ $c_k^\pm$ is attained in $M_k^\pm$.*

Proof. Consider a minimizing sequence $u^n \in M_k^\pm$ $E(u^n) \to c_k^\pm$. By (6) we find that the sequence u^n is uniformly bounded in H. Hence we may extract a weakly convergent subsequence $u^n \to u$. Moreover, $u^n \to u$ strongly in L^π for any $\pi < q$ and also $g(u^n) \to g(u)$ in $L^{\pi'}$ for some $\pi' > q/(q-1)$. Also by Lemma 3 below we may extract a further subsequence still denoted by u^n such that for any l, $1 \leqq l \leqq k$ $r_l^n \to r_l$ $(u \to \infty)$ and $|r_l - r_m| \geqq c(k)$ if $l \neq m$. Collecting these facts shows that u satisfies the following:

$$\exists 0 < r_1 < \cdots < r_k = R\colon u(r_l) = 0, \quad \pm(-1)^l u(r) \geqq 0 \quad \text{in} \quad \Omega_l$$
$$\text{and} \quad \|u_l\|_p \geqq \Gamma, \quad 1 \leqq l \leqq k, \quad \int_{\Omega_l} g(u_l)\, u_l\, dx = \lim_{n\to\infty} \int_{\Omega^n_l} g(u_l^n)\, u_l^n\, dx.$$

Finally, by weak lower semicontinuity of $b(u, u)$:

$$b(u_l, u_l) \leqq \int_{\Omega_l} g(u_l)\, u_l\, dx, \quad 1 \leqq l \leqq k.$$

Assume that in the latter inequality equality holds for all l. Then

$$b(u, u) = \lim b(u^n, u^n) \quad \text{and} \quad u^n \to u \ \text{strongly in } H.$$

Thus $u \in M_k^{\pm}$ and $E(u) = c_k^{\pm}$, proving the lemma in this case. Assume that for some l,

$$b(u_l, u_l) < \int_{\Omega_l} g(u_l) u_l \, dx .$$

We then derive a contradiction as follows:

Consider the function $f\colon \{t > 0\} \to \mathbb{R}$ given by

$$t \mapsto b(t u_l, t u_l) - \int_{\Omega} g(t u_l) t u_l \, dx .$$

By (4) if $t_0 \geqq 1$ and $f(t_0) < 0$ then $f(t) < 0$ for all $t \geqq t_0$. Assume now there exists $t_l > 0$ such that

$$b(t_l u_l, t_l u_l) = \int_{\Omega_l} g(t_l u_l) t_l u_l \, dx , \quad \| t_l u_l \|_p \geqq \Gamma, \quad 1 \leqq l \leqq k . \tag{7}$$

By the above $t_l \leqq 1$. Set $v(x) = t_l u_l(x)$, $x \in \Omega_l$. Then $v \in M_k^{\pm}$ and by the equalities

$$c_k^{\pm} = \lim E(u^n) = \lim \left[\tfrac{1}{2} \int_{\Omega} g(u^n) u^n \, dx - \int_{\Omega} G(u^n) \, dx\right]$$
$$= \tfrac{1}{2} \int_{\Omega} g(u) u \, dx - \int_{\Omega} G(u) \, dx ,$$
$$E(v) = \sum_{l=1}^{k} \left[\tfrac{1}{2} \int_{\Omega_l} g(t_l u_l) t_l u_l \, dx - \int_{\Omega_l} G(t_l u_l) \, dx\right],$$

and the estimate for $u \in H$, $\| u \|_p \geqq \Gamma$,

$$\frac{d}{dt} \left[\tfrac{1}{2} \int_{\Omega} g(t u) t u - \int_{\Omega} G(t u) \, dx\right]$$
$$= \frac{1}{2t} \left[\int_{\Omega} g_u(t u) t^2 u^2 \, dx - \int_{\Omega} g(t u) t u \, dx\right] > 0$$

we obtain as a contradiction

$$E(v) < c_k^{\pm} .$$

It remains to verify that there exists k_0 such that (7) can always be achieved for $k \geqq k_0$. Assume there exists a sequence $k \to \infty$ such that for some $l = l(k)$

$$b(t_l u_l, t_l u_l) < \int_{\Omega} g(t_l u_l) t_l u_l \, dx , \quad \| t_l u_l \|_p = \Gamma$$

where u is obtained as above as the weak limit of a minimizing sequence in $M_k^{\pm}$. Clearly, then $\{t_l u_l\}_k$ is bounded in H; whence $t_l u_l \to u^*$ weakly in H as $(k \to \infty)$ and $t_l u_l \to u^*$ strongly in L^p. Thus, $\| u^* \|_p = \Gamma$ but from Lemma 2 below we conclude that $\operatorname{supp}(u^*) = \emptyset$ and $u^* \equiv 0$. This concludes the proof of Lemma 1.

For $k \in \mathbb{N}$ let u^n denote a minimizing sequence in $M_k^{\pm}$ as before and let r_l^n denote the nodes of u^n. Ω_l^n, u_l^n are then defined as before. Let μ denote Lebesgue measure.

Lemma 2. $\liminf\limits_{n \to \infty} \left(\inf\limits_{1 \leq l \leq k} \| u_l^n \|_p \right) \to \infty \quad (k \to \infty)$

$$\limsup_{n \to \infty} \left(\sup_{1 \leq l \leq k} \mu(\Omega_l^n) \right) \to 0 \quad (k \to \infty) .$$

Proof. Note that for any given $s > 0$ any $k \in \mathbb{N}$ $E(u^n) < c_k^{\pm} + s$ for large u. Hence Lemma 2 may be proved exactly as Lemmas 7 and 8 in [3].

Lemma 3. *For any* $k \in \mathbb{N}$ *there exists* $c(k) > 0$ *such that for any pair* l, m, $1 \leqq l < m \leqq k$,

$$\liminf_{n \to \infty} |r_l^n - r_m^n| \geqq c(k).$$

Proof. This follows from Lemma 7 in [3].

From Lemma 1 we conclude that for sufficiently large k a minimizing sequence in M converges in $M_k^{\pm}$. Let u denote the limit of such a sequence, and let r_l be the zeroes of u which we obtained in the proof of Lemma 1.

Also let Ω_l always correspond to such a $u \in M_k^{\pm}$. Quantities related to comparison functions w will recieve an apostrophe, e. g. r_l'.

For later use we note the following technical lemma.

Lemma 4. *Let* $k = k(m)$ *be a sequence of integers* $k \geqq k_0$. *Let* $w = w(m)$ *be a sequence in* H *such that* $w(r_l') = 0$ *at* r_l', $0 < r_l' < \cdots < r_k' = R$, *and* $\pm (-1)^l w_l \geqq 0$, $\|w_l\|_p \geqq \Gamma$, $1 \geqq l \geqq k$. *Assume there exist numbers* $\delta_l = \delta_l(m) \leqq 1$, $\varepsilon_l = \varepsilon_l(m) \leqq 1$, $1 \leqq l \leqq k$, $\varepsilon = \varepsilon(m) > 0$ *such that*

$$\Big|\int_{\Omega_l} g(u_l)\, u_l\, dx - \int_{\Omega_l'} g(w_l)\, w_l\, dx\Big| \leqq \delta_l, \quad 1 \leqq l \leqq k,$$

$$\Big|\int_{\Omega_l} g_u(u_l)\, u_l^2\, dx - \int_{\Omega_l'} g_u(w_l)\, w_l^2\, dx\Big| \leqq \delta_l, \quad 1 \leqq l \leqq k,$$

$$|b(u_l, u_l) - b(w_l, w_l)| \leqq \varepsilon_l, \qquad 1 \leqq l \leqq k,$$

$$1 \geqq a(u) - a(w) \geqq \varepsilon > 0.$$

(Note that the index m *has been suppressed in the notation.)*

Also assume that

$$\varepsilon^{-1} \Big| \sum_l (\delta_l + \varepsilon_l)^2 \Big/ \int_{\Omega_l} g(u_l)\, u_l\, dx \Big| \to 0 \qquad (m \to \infty),$$

$$\varepsilon^{-1} \sum_l \delta_l \to 0 \qquad (m \to \infty).$$

Then there exists m_0 *such that for* $m \geqq m_0$ *there exists* $v = v(m) \in M_k^{\pm}$ *with the property*

$$E(v) < E(u).$$

Proof. By the estimate

$$\begin{aligned} &\frac{d}{dt}\Big[b(tw_l, tw_l) - \int_{\Omega_l'} g(tw_l)\, tw_l\, dx\Big]\Big|_{t=1} \\ &= \frac{d}{dt}\Big[b(tu_l, tu_l) - \int_{\Omega_l} g(tu_l)\, tu_l\, dx\Big]\Big|_{t=1} \\ &\quad + \mathcal{O}(\varepsilon_l, \delta_l) = \Big[\int_{\Omega_l} g(u_l)\, u_l\, dx - \int_{\Omega_l} g_u(u_l)\, u_l^2\, dx\Big] + \mathcal{O}(\varepsilon_l, \delta_l) \\ &\leqq -c \int_{\Omega_l} g(u_l)\, u_l\, dx \end{aligned}$$

for $k \geqq k_0$, it follows that there exists $t_l > 0$,

$$|1 - t_l| \leqq c(\delta_l + \varepsilon_l) / \int_{\Omega_l} g(u_l) u_l \, dx,$$

such that

$$b(t_l w_l, t_l w_l) = \int_{\Omega_l'} g(t_l w_l) t_l w_l \, dx, \quad 1 \leqq l \leqq k.$$

Moreover, for t between t_l and 1 we have

$$|b(tw_l, tw_l) - \int_{\Omega_l'} g(tw_l) t w_l \, dx| \leqq \delta_l + \varepsilon_l.$$

Letting $v_l = t_l w_l$ we obtain $v \in M_k^{\pm}$. By the estimate

$$\begin{aligned} E(v) &= E(w) + \sum_l \int_1^{t_l} \frac{d}{dt} E(tw_l) \\ &\leqq E(u) - \varepsilon + c \cdot \sum_l \delta_l + c \cdot \sum_l |1 - t_l| \sup |b(tw_l, tw_l) \\ &\quad - \int_{\Omega_l} g(tw_l) t w_l \, dx| \\ &\leqq E(u) - \varepsilon + c \cdot \sum_l \delta_l + c \cdot \sum_l (\delta_l + \varepsilon_l)^2 / \int_{\Omega_l} g(u_l) u_l \, dx \\ &< E(u), \quad \text{if} \quad m \geqq m_0, \end{aligned}$$

the lemma follows.

Let $\tilde{\Omega}_l = \{x \in \Omega_l \mid u(x) \neq 0\}$, $1 \leqq l \leqq k$.

Lemma 5. *$Lu = g(u)$ in $\tilde{\Omega}_l$, $1 \leqq l \leqq k$, $k \geqq k_0$.*

Proof. For fixed l consider

$$\begin{aligned} H_l &= \{w \in H \cap H_0^{1,2}(\tilde{\Omega}_l) \mid \|w\|_p \geqq \Gamma\}, \\ K_l &= \{w \in H_l \mid b(w, w) = \int_{\Omega} g(w) w \, dx\}. \end{aligned}$$

By the estimates of Lemma 3 in [2] 0 is a regular value of the C^1-function $k_l : H_l \to \mathbb{R}$ given by

$$k_l(w) = b(w, w) - \int_{\Omega} g(w) w \, dx.$$

Thus, K_l is a C^1-manifold in a neighborhood of u_l and as in the proof of Lemma 3 in [2] the tangential space at u_l in H_l is spanned by the tangential space of u_l in K_l and the vector u_l. Now, by definition of $c_k^{\pm}$ and Lemma 1 u_l is critical for E in K_l. By the condition $b(u_l, u_l) = \int_{\Omega_l} g(u_l) u_l dx$ also the derivative of E in the direction u_l vanishes. Thus, u_l is critical for E on H_l which is equivalent to the assertion of the lemma.

Lemma 6. *$a(r) \partial_r u(r)$ is continuous in $\Omega \setminus \{0\}$, if $k \geqq k_0$.*

Proof. Indeed, by Lemma 5 $a\,\partial_r u$ is continuous in $\tilde{\Omega} = \bigcup_{l=1}^{k} \tilde{\Omega}_l$[1]. Since $\partial_r u = 0$ almost everywhere in $\Omega\setminus\tilde{\Omega}$ we may choose a piecewise continuous representative of $a\,\partial_r u$ on $\Omega\setminus\{0\}$. Assume there exists $x^0 \in \Omega$, $|x^0| = r^0 > 0$, such that $a\,\partial r u$ is not continuous at r^0. Since $a\,\partial_r u$ is continuous to the left and to the right of r^0 the left and right limits $a\,\partial_r u^-$ and $a\,\partial_r u^+$ both exist. Assume i) that both limits differ from zero. If $r^0 = r_l$ for some l and if $\nu > 0$ let $r^- < r^0$ be maximal such that $\pm(-1)^l u(r^-) = \nu$[2] and let $r^+ > r^0$ be minimal such that $\pm(-1)^l u(r^+) = -\nu$. If $x^0 \in \Omega_l$, let $r^- < r^0$, be maximal and let $r^+ > r^0$ be minimal such that $\pm(-1)^l u(r^\pm) = \nu$. If ii) $a\,\partial_r u^- = 0$ let $r^- = r^0 - \nu$, r^+ as before. If iii) $a\,\partial_r u^+ = 0$ let $r^+ = r^0 + \nu$, and let r^- be defined as in i). Obviously, if both limits equal zero there is nothing to prove. In all cases now set $\Omega_\nu = \{x \mid r^- < |x| < r^+\}$. Clearly, $\mu(\Omega_\nu) \to 0$ as $\nu \to 0$. Define $w = w(\nu) \equiv u$ outside Ω_ν, and let w be the unique solution of

$$\begin{aligned} Lw &= 0 \quad \text{in} \quad \Omega_\nu, \\ w &= u \quad \text{on} \quad \partial\Omega_\nu \end{aligned}$$

inside Ω_ν. Then $w \in H$. Also, if $x^0 \in \Omega_l$ for some l, $w(r_m) = 0$ for all m, and we may let $\Omega_l' = \Omega_l$, $1 \leqq l \leqq k$. Whereas, if $r^0 = r_l$ for some l, then $w(r_m) = 0$ only at $m \neq l$ and $w(r_l') = 0$ at some r_l', $r^- < r_l' < r^+$. In any event we obtain

$$\left| \int_{\Omega_m} g(u_m)\, u_m\, dx - \int_{\Omega_m'} g(w_m)\, w_m\, dx \right| \leqq c \cdot \nu\, \mu(\Omega_\nu)^{1-1/t} = \delta(\nu)$$

for $m = l, l+1$, $|\cdots| = 0$ else. Also

$$\left| \int_{\Omega_m} g_u(u_m)\, u_m^2\, dx - \int_{\Omega_m'} g_u(w_m)\, w_m^2\, dx \right| \leqq c \cdot \nu\, \mu(\Omega_\nu)^{1-1/t}$$

for $m = l, l+1$, $|\cdots| = 0$ else. Moreover, by piecewise continuity of $a\,\partial_r u(r)$ and continuity of $a\,\partial_r w$ for small $\nu > 0$:

$$|b(u_m, u_m) - b(w_m, w_m)| \leqq c^* \nu, \qquad m = l, l+1,$$

with a constant c^* depending on u and r^0. Finally, since $a\,\partial_r u$ is discontinuous at r_l^0, for sufficiently small $\nu > 0$ we obtain by a piecewise partial integration that

$$b(u, u) - b(w, w) = \int_{\Omega} a\, |\nabla(u - w)|^2\, dx \geqq c' \nu,$$

with a constant $c' > 0$ depending on u, r^0, and the discontinuity. Thus, for small $\nu > 0$ and $k \geqq k_0$ Lemma 4 yields a comparison function $v \in M_k^\pm$ such that $E(v) < E(u)$. A contradiction results proving the lemma.

Lemma 7. *$\pm(-1)^l u_l > 0$ in Ω_l, $1 \leqq l \leqq k$, $k \geqq k_0$.*

Proof. Assume there exists $x^0 \in \Omega_l$, $|x^0| = r^0$, such that $u(x^0) = 0$. By Lemma 6 also $\partial_r u(r^0) = 0$. Now we claim: There exists $\varepsilon > 0$, $\gamma > 0$ such that for $r \geqq r^0/2$

[1] More precisely: $a\partial_r u$ is uniformly continuous on $\tilde{\Omega} \cap B_\varepsilon(0)$ for any $\varepsilon > 0$.

[2] The sign is determined by the membership of u in M_k^+ or M_k^- resp.

and satisfying $r^0 - \varepsilon < r < r^0 + \varepsilon$ we have

(9) $$|u(r) - u(r^0)| \leqq c\,|r^\gamma - r^{0\gamma}|\,.$$

Indeed, letting $\tilde{g}(u) = g(u)$ if $u \neq 0$, $\tilde{g}(u) = 0$ if $u = 0$, by Lemmas 5 and 6 almost everywhere in $]0, R]$

$$-\partial_r(r^{N-1}a(r)\,\partial_r u) = \tilde{g}(u)\,r^{N-1}.$$

Integrating between r^0 and r we thus obtain, assuming $r \geqq r^0/2$ and $|u| \leqq 1$ on $\Omega_r = \{x \in \Omega_l \mid r^0 < |x| < r \text{ or } r < |x| < r^0\}$:

$$|a(r)\,\partial_r u(r)| \leqq r^{1-n} \int_{\Omega_r} |g(u)|\,dx \leqq c r^{1-n}\,\mu(\Omega_r)^{1-1/t} \leqq c r^{1-n/t}.$$ Hence for such r:

$$|u(r) - u(r^0)| \leqq c\,|r^\gamma - r^{0\gamma}|$$

with $\gamma = 2 - n/t > 0$. In particular $|u(r)| \leqq 1$ for $r \geqq r^0/2$, $r^0 - \varepsilon < r < r^0 + \varepsilon$, if $\varepsilon > 0$ is chosen sufficiently small, and our above assumption is justified for r in this range. This proves (9).

Now, for k large, by Lemma 2 $r_{l+1} - r_l < \varepsilon$ if $r_l < R/2$, resp. $r_{l-2} > r_l - \varepsilon$ if $r_l \geqq R/2$. Hence $|u| \leqq 1$ on Ω_{l+1} or Ω_{l-1}, resp. if $u(x^0) = 0$ at some $x^0 \in \Omega_l$. By Lemma 2 this is impossible for large k, proving the assertion of this lemma.

Proof of the Theorem. It remains to verify that u solves (5), the remaining assertions being a consequence of Lemma 7. By Lemmas 5 and 7 $Lu = g(u)$ in Ω_l, $1 \leqq l \leqq k$, $k \geqq k_0$. Let $v \in H$. Using Lemma 6 an integration by parts gives

$$\begin{aligned} 0 = \sum_l \int_{\Omega_l} (Lu - g(u))\,v\,dx &= \sum_l \Big[b(u_l, v) - \int_{\Omega_l} g(u_l)\,v\,dx\Big] \\ &- \sum_l \int_{\partial\Omega_l} v\,a\,n \cdot \nabla u\,do = b(u, v) - \int_{\Omega} g(u)\,v\,dx\,. \end{aligned}$$

Here, n denotes the exterior normal and do the measure on $\partial\Omega_l$.

Thus, u weakly solves (5). By standard regularity results, moreover, $u \in H^{2,t}$ and (5) is satisfied a.e. This concludes the proof.

The list of references below is by no means complete. For more detailed bibliographical references confer e.g. [2] or [3].

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