Bisected Chords of a Convex Body

By

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1. Preliminaries. A *plane convex body* is a compact, convex subset of the plane with nonvoid interior. A *convex curve* is the boundary of a plane convex body. If S is any Lebesgue measurable subset of the plane, we shall denote its measure by $|S|$.

For each point p in a plane convex body K, let $n(p)$ denote the number of chords of K bisected by p. If C is the boundary of K, then it is readily seen that for each interior point p, $n(p)$ is half the number of points of intersection (if finite) of C with $2p - C$. For $p \in C$, we take $n(p) = 0$. It is known that $n(p)$ is almost everywhere finite, and integrable over K. In this paper, we shall be primarily concerned with the properties of the sets M_k, F_k, $k = 0, 1, 2, \ldots$, defined as follows:

$$M_k = \{p \in K : n(p) = k\}, \quad F_k = \bigcup_{r \geqq k} M_r. \tag{1.1}$$

In section 2 it will be shown that the quantity $\lambda(K)$, defined by

$$\lambda(K) = \frac{|M_1|}{|K|} \tag{1.2}$$

is a measure of symmetry, and sharp bounds will be derived for $\lambda(K)$ (Theorem 1). Sharp bounds are also given in case K ranges over all curves of constant width (Theorem 2).

Many questions can be asked concerning these F_k; for example: is F_k connected, or even convex? Easy examples show that F_3 ist not in general convex, but Ceder [3] has proved that F_3 is always connected. The results of the present paper are metric in nature. For example, it follows as an immediate corollary of Theorem 1 that

$$0 \leqq |F_3| \leqq \tfrac{1}{4}|K|, \tag{1.3}$$

where equality holds on the left if and only if K is centrally symmetric, and on the right if and only if K is a triangle.

In section 3 we also consider the structure of the set F_3 in case K is a polygon.

The *difference body*, $DK = K + (-K)$, is the main tool used here. It is well-known that

$$4|K| \leqq |DK| \leqq 6|K|, \tag{1.4}$$

where equality holds on the left if and only if K is centrally symmetric, and on the right if and only if K is a triangle (see [2, p. 105]). Denoting by $M(K_1, K_2)$ the *mixed area* of K_1 and K_2 [1, p. 34], one has

$$|DK| = 2(|K| + M(K, -K)). \tag{1.5}$$

Thus (1.4) is equivalent to

$$|K| \leqq M(K, -K) \leqq 2|K|, \tag{1.6}$$

with equality holding in the same cases.

Setting $q = 2p$, one has from [1, p. 35]

$$\int 2n(p)\,dq = 4(M(K, -K) + M(K, K)), \tag{1.7}$$

where the integration is over all positions of q (to compare (1.7) with Blaschke's formula, set $K = K_0 = K_2$, $K_1 = -K$; so $F_{01} = M(K, -K)$, $F_{02} = M(K, K)$). Using the fact that $dq = 4dp$ and $M(K, K) = |K|$, it follows from (1.5), (1.6), and (1.7), that

$$|K| \leqq \int_K n(p)\,dp \leqq \tfrac{3}{2}|K|, \tag{1.8}$$

with equality holding in the same cases as in (1.4).

Let C be the boundary of the plane convex body K, parametrized by its arc-length s. For each regular point of C (point having a unique support line) let $b(s)$ denote the distance between the support line through that point and the opposite parallel support line. Since, except for a countable set, each point of C is a regular point, the integral of $b(s)$ around C is well-defined, and one has

$$\int_C b(s)\,ds = 4\int_K n(p)\,dp = |DK|. \tag{1.9}$$

This is proved in [1, p. 35] under the assumption that C is smooth, but is not difficult to establish without smoothness restrictions.

A more general problem is obtained by letting $n(p)$ be the number of chords of K divided by p in the ratio $1 : \lambda$, $\lambda > 1$. It is possible to generalize many of our results to this more general case.

2. Some properties of M_k.

Lemma 1. *If k is even, then* $|M_k| = 0$.

Proof. We first note that $n(p) \geqq 1$ for each interior point of K, so M_0 is the boundary of K and $|M_0| = 0$. We shall show that if $p \in M_k$, k even and $\geqq 2$, then p is the midpoint of a diameter of K (a diameter is a chord joining points of K lying in opposite parallel support lines). Since the set of midpoints of diameters of K has measure zero, the lemma will follow.

So, let C denote the boundary of K. If C and $2p - C$ have a support line in common at one of their points of intersection, then the corresponding chord of C is a diameter bisected by p, and we are through. If C and $2p - C$ never have a support line in common at a point where they intersect, then they cross at each intersection. Consider a pair q and q' of points of intersection, where p bisects qq'. If C, traversed in the positive direction, crosses from the inside to the outside of $2p - C$

at q, then it crosses from the outside to the inside at q'. The number of intermediate crossings, going from q to q', must therefore be even. But the number of intermediate crossings plus 1 is exactly k, so k is odd. Thus if k is even, C and $2p - C$ must have a common support line at some point of intersection, so p is the midpoint of a diameter. This completes the proof.

Theorem 1. *Let K be a plane convex body, and let $\lambda(K)$ be defined by* (1.2). *Then*

$$\tfrac{3}{4} \leqq \lambda(K) \leqq 1,$$

and equality holds on the left if and only if K is a triangle, and on the right if and only if K is centrally symmetric.

Proof. We first prove the left-hand inequality. Using (1.8) and Lemma 1, we have

$$\tfrac{3}{2}|K| \geqq \int_K n(p)\,dp = \int_{M_1} n(p)\,dp + \int_{K \sim M_1} n(p)\,dp \geqq |M_1| + 3\{|K| - |M_1|\},$$

so

$$|M_1| \geqq \tfrac{3}{4}|K|,$$

establishing the left-hand inequality. If equality holds, then it must hold on the right hand side of (1.8), so K is a triangle.

The right-hand inequality is trivial. If equality holds, then $|M_1| = |K|$, so $|M_3| = |M_5| = \cdots = 0$, and

$$\int_K n(p)\,dp = |M_1| = |K|,$$

so equality holds on the left-hand side of (1.8) and K is centrally symmetric. This completes the proof.

Remark. The last theorem shows that $\lambda(K)$ is a measure of symmetry for plane convex bodies. It is clear that $\lambda(K)$ is also an affine invariant of K. Theorem 1 implies that K is centrally symmetric if and only if the set of points of K bisecting more than one chord of K has measure zero. This generalizes the result of Viet [4] that a plane convex body is centrally symmetric if there exists at most one point which bisects more than one chord. The inequality (1.3) follows from the fact that $|K| = |M_1| + |F_3|$.

Theorem 2. *Let K be a plane convex body of constant width. Then*

$$1 \geqq \lambda(K) \geqq \lambda(R) \sim .943,$$

where R is a Reuleaux triangle. Equality holds on the right if and only if K is a Reuleaux triangle, and on the left if and only if K is a circle.

Proof. Assume that K has constant width 1. Then $|DK| = \pi$, since DK is a circle of radius 1. Using (1.9), we have

$$\begin{aligned} |M_1| + 3\{|K| - |M_1|\} &= |M_1| + 3\{|M_3| + |M_5| + \cdots\} \leqq \\ &\leqq |M_1| + 3|M_3| + 5|M_5| + \cdots = \\ &= \int_K n(p)\,dp = \frac{\pi}{4} = \frac{\pi}{2(\pi - \sqrt{3})}|R|, \end{aligned}$$

where R is a Reuleaux triangle of breadth 1. It is easy to see that the boundary of any translate of $-R$ intersects the boundary of R in at most 6 points; hence if $K = R$, then $M_5 = M_7 = \cdots = \emptyset$, and equality holds above throughout. Hence if $K = R$,

$$|M_1| = \frac{5\pi - 6\sqrt{3}}{4\pi - 4\sqrt{3}}|R| = \lambda(R)|R| \sim .943|R|.$$

In general, by the Blaschke-Lebesgue Theorem [2, p. 132], $|K| \geqq |R|$, so

$$3|K| - 2|M_1| \leqq \frac{\pi}{2(\pi - \sqrt{3})}|R| \leqq \frac{\pi}{2(\pi - \sqrt{3})}|K|,$$

and

$$|M_1| \geqq \frac{5\pi - 6\sqrt{3}}{4\pi - 4\sqrt{3}}|K| = \lambda(R)|K|.$$

If equality holds, then $|K| = |R|$, and by the Blaschke-Lebesgue Theorem, K is a Reuleaux triangle.

The left-hand inequality is trivial. By Theorem 1, equality can hold only if K is centrally symmetric, hence a circle. This completes the proof.

3. Chord bisectors of polygons. Let K be a convex polygon with boundary C, and let $C_1, C_2, \ldots, C_r$ be the segments forming C. Let l_j be the length of C_j, and let θ_{ij} be the smaller of the angles between the lines carrying C_i and C_j. For a general convex curve, equation (1.9) can be put in the form

$$\int_K n(p)\,dp = \tfrac{1}{4}\int_C b(s)\,ds = \tfrac{1}{8}\int\int_{C \times C} \sin\theta(s, s')\,ds\,ds', \tag{3.1}$$

where $\theta(s, s')$ is the smaller angle between the support lines at the points with parameter s and s' respectively (uniquely defined except for a countable number of points). For our convex polygon, (3.1) reduces to,

$$\int_K n(p)\,dp = \tfrac{1}{8}\sum_{1 \leq i, j \leq r} l_i l_j \sin\theta_{ij}. \tag{3.2}$$

The formula (3.2) can be derived directly, as follows. Those points of K which bisect chords having endpoints respectively on C_i and C_j form the parallelogram $P_{ij} = \tfrac{1}{2}(C_i + C_j)$. Since

$$|P_{ij}| = \tfrac{1}{4} l_i l_j \sin\theta_{ij},$$

(3.2) follows from

$$\int_K n(p)\,dp = \tfrac{1}{2}\sum_{1 \leq i, j \leq r} |P_{ij}|. \tag{3.3}$$

For a convex r-gon, there are $r(r-1)/2$ distinct (possibly degenerate) P_{ij}. Those points which are interior to three or more P_{ij} lie in the set F_3 (defined by (1.1)). Points belonging to three or more P_{ij}, but not necessarily interior points of those P_{ij}, may or may not lie in F_3. In case K is a convex quadrilateral with no two sides parallel, F_3 consists precisely of those points interior to three P_{ij}, and is readily seen to be the interior of a "concave" quadrilateral.

References

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