Algebraic Functions and an Analogue of the Geometry of Numbers: The Riemann-Roch Theorem

By

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1. Introduction. As is well known, the classical geometry of numbers has important applications in algebraic number theory. The purpose of this paper is to show that Mahler's analogue of the geometry of numbers in a field of formal power series [12] has equally important and closely related applications in the theory of algebraic functions of one variable. We show that the analogue of the theorem on successive minima is essentially the Riemann-Roch Theorem and in a subsequent paper we shall show that the theory of correspondences has a natural interpretation in the language of the geometry of numbers, which leads to a result on the minimum of the product of n linear forms and this in its turn gives the "Riemann Hypothesis" for function fields. This latter result may be regarded as the analogue of the application of the classical geometry of numbers to the problem of finding the minimum discriminant of an algebraic number field.

In a sense, there is nothing original about our proof of the Riemann-Roch Theorem. It is well known (cf. Eichler [8]) that the theorem is a consequence of a similar theorem for linear divisors and that the latter follows from an analogue of Minkowski's linear forms theorem. Again, the original proof given by Dedekind and Weber [6] uses an argument closely resembling our appeal to successive minima and the proof given in Hasse [10] uses a counting argument of a similar nature. Our main objects are to bring out the analogy with the classical geometry of numbers more clearly, to prepare the way for the proof of the Riemann Hypothesis and to show that the Riemann-Roch Theorem is already in Mahler's paper in a disguised form.

We review the necessary preliminaries from Mahler's paper in section 2 and prove what is essentially the Riemann-Roch Theorem for linear divisors. In section 3 we apply the results to function fields and obtain the Riemann-Roch Theorem in one of its forms[1]) in section 4. Finally in section 5 we sketch an easier approach for the case of a finite constant field.

I am grateful to Dr. Kiyek for his valuable criticisms and comments on an earlier version of this paper. In particular, my original proof of Theorem 2 applied only to

[1]) The inhomogeneous Riemann-Roch Theorem can be obtained from the methods of this paper. Of course the more sophisticated forms lie beyond its scope.

separable extensions with tame ramification at infinity. Dr. KIYEK suggested the much more general version given here, which gives the Riemann-Roch Theorem for arbitrary function fields.

2. The geometry of numbers in fields of power series. Most of the results in this section will be stated without proof. They are somewhat more general than those in MAHLER's paper, but the reader who is familiar with the applications of the classical geometry of numbers to number theory will recognize them immediately. Some details can be found in the author's London Ph.D. dissertation (University of London, 1956; this part unpublished).

Let $k = k_0(t)$ be a transcendental extension of the field k_0 and denote by $\mathfrak{o}$ the polynomial ring $k_0[t]$. For $\alpha = f/g$, $f, g \in \mathfrak{o}$, let

$$(1) \qquad \nu(\alpha) = \deg g - \deg f$$

be the "degree valuation" of $k_0(t)$. We define an absolute value $|\ |$ of $k_0(t)$ by

$$(2) \qquad |\alpha| = q^{-\nu(\alpha)}, \quad q > 1.$$

(If k_0 is a finite field, we take $q = \mathrm{Card}(k_0)$.) We denote by $\hat{k}$ the perfect completion of k with respect to this valuation and by P_n the n-dimensional space $\hat{k}^n$. If $\boldsymbol{x} \in P_n$ and $\boldsymbol{y} \in P_n$, then we define

$$(3) \qquad \|\boldsymbol{x} - \boldsymbol{y}\| = \max(|x_i - y_i|),$$

where $|\ |$ is the extension of the absolute value (2) to $\hat{k}$. With respect to the distance (3), P_n is an ultrametric space.

A *distance function* in P_n is a function $F\colon P_n \to \mathbf{R}$ such that

$$(4) \qquad F(\boldsymbol{o}) = 0, \quad F(\boldsymbol{x}) \neq 0 \quad \text{if} \quad \boldsymbol{x} \neq \boldsymbol{o},$$

$$(5) \qquad F(\lambda \boldsymbol{x}) = |\lambda|\, F(\boldsymbol{x}) \quad \text{for} \quad \lambda \in \hat{k},$$

$$(6) \qquad F(\boldsymbol{x} - \boldsymbol{y}) \leqq \max(F(\boldsymbol{x}), F(\boldsymbol{y})).$$

An inequality $F(\boldsymbol{x}) \leqq c$, $c > 0$, defines a *convex body*, $\mathscr{C}$. A convex body possesses the property that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathscr{C}$ then $\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \in \mathscr{C}$ for all $\lambda_1, \lambda_2 \in \hat{k}$ with $|\lambda_1| \leqq 1$, $|\lambda_2| \leqq 1$. Conversely, this property defines a convex body. MAHLER proved that every convex body is a parallelepiped, that is $F(\boldsymbol{x}) = |A\,\boldsymbol{x}|$ for some invertible (n, n)-matrix A with elements in $\hat{k}$ ([12], p. 498). A convex body $\mathscr{C}$ has a volume $V = V(\mathscr{C})$ which can be expressed in terms of the dimension of the k_0-module of points with coordinates in $k_0[t]$ inside a suitably expanded body. If $\mathscr{C}$ is given as above by the matrix A, then $V = (|\det A|)^{-1}$. In particular the convex body consisting of all points $\boldsymbol{x} = (x_1, \ldots, x_n)$ with

$$|x_1| \leqq q^{a_1}, \cdots, |x_n| \leqq q^{a_n}$$

has volume

$$q^{a_1 + \cdots + a_n}.$$

(For finite k_0 the volume is analogous to Jordan measure.) It is indeed this definition of volume which makes it possible to prove the Riemann-Roch Theorem and sections 7, 8 and 9 of MAHLER's paper contain most of the hard work necessary for the proof.

A *lattice* Λ in P_n is the image of $\mathfrak{o}^n$ under an invertible $\hat{k}$-linear mapping, λ, of the $\hat{k}$-vector space P_n into itself. The points of Λ will be called lattice points (of Λ). The absolute value (in the sense of (2)) of the determinant of λ will be denoted by $\Delta = \Delta(\Lambda)$.

Now let $\mathscr{C}$ be a convex body defined by $F(\boldsymbol{x}) \leqq 1$, of volume V and let Λ be a lattice of determinant Δ. MAHLER proved that there exist n $\hat{k}$-independent lattice points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ such that:

a) $F(\boldsymbol{x}_1)$ is the minimum of $F(\boldsymbol{x})$ in all lattice points $\boldsymbol{x} \neq \boldsymbol{o}$;

b) for $j \geqq 2$, $F(\boldsymbol{x}_j)$ is the minimum of $F(\boldsymbol{x})$ in all points of Λ independent of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}$;

c) the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are a basis for Λ over $\mathfrak{o} = k_0[t]$;

d) the numbers $\sigma_j = q^{\mu_j} = F(\boldsymbol{x}_j)$ $(1 \leqq j \leqq n)$, the *successive minima* of $\mathscr{C}$, depend only on $F(\boldsymbol{x})$ and Λ and satisfy

$$0 < \sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \sigma_n \tag{7}$$

and

$$\sigma_1 \sigma_2 \cdots \sigma_n = \frac{\Delta}{V}. \tag{8}$$

Now let $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ be a non-degenerate bilinear form on P_n. If $\mathscr{C}$ is the convex body defined by $F(\boldsymbol{x}) \leqq c$, and Λ a lattice with basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$, then the polar body $\mathscr{C}^*$ and the polar lattice Λ^* with respect to the bilinear form $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ are defined exactly as in ordinary number theory. Thus Λ^* is the lattice with basis $\boldsymbol{b}_1^*, \ldots, \boldsymbol{b}_n^*$, where $\langle \boldsymbol{b}_i, \boldsymbol{b}_i^* \rangle = 1$ and $\langle \boldsymbol{b}_i, \boldsymbol{b}_j^* \rangle = 0$ if $i \neq j$. We define the polar function to $F(\boldsymbol{x})$ by $G(\boldsymbol{o}) = 0$ and for $\boldsymbol{y} \neq \boldsymbol{o}$ by

$$G(\boldsymbol{y}) = \sup_{\boldsymbol{x} \neq \boldsymbol{o}} \frac{|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|}{F(\boldsymbol{x})}.$$

Then $G(\boldsymbol{y})$ is a distance function and $\mathscr{C}^*$ is the convex body defined by $G(\boldsymbol{y}) \leqq 1/c$. It is easy to see that $\mathscr{C}^*$ consists of all those points $\boldsymbol{y}$ of P_n for which $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leqq 1$ for all $\boldsymbol{x} \in \mathscr{C}$. Moreover, $\det \Lambda \det \Lambda^* = 1$.

Now let $c = 1$; then $\mathscr{C}^*$ has volume $1/V(\mathscr{C})$ and if $\tau_j = q^{\nu_j}$ $(1 \leqq j \leqq n)$ are the corresponding successive minima with respect to the polar lattice Λ^*, then

$$\sigma_j \tau_{n-j+1} = 1 \quad (1 \leqq j \leqq n). \tag{9}$$

Finally, the convex body $\mathscr{C}$ consists of the points $\boldsymbol{x} \in P_n$ such that

$$\boldsymbol{x} = y_1 t^{-\mu_1} \boldsymbol{x}_1 + \cdots y_n t^{-\mu_n} \boldsymbol{x}_n, \tag{10}$$

where $y_i \in \hat{k}$ and $|y_i| \leqq 1$. A similar result holds for $\mathscr{C}^*$. (Cf. [12], p. 509.)

Our first theorem is simply a summary of all the foregoing.

Theorem 1. *Let $\mathscr{C}$ be a convex body in P_n and let $\mathscr{C}'$ be the convex body $t^{-2}\mathscr{C}^*$. Let l, l' respectively denote the dimensions of the k_0-modules of points of Λ, Λ^* in $\mathscr{C}, \mathscr{C}'$ respectively. Then*

$$q^{l-l'} = q^n (\sigma_1 \cdots \sigma_n)^{-1} = q^n \frac{V(\mathscr{C})}{\Delta(\Lambda)}.$$

Proof. The distance function of $\mathscr{C}'$ is $q^{-2}G(y)$, where $G(y)$ is the polar distance function defined above. Hence the successive minima $\tau_1', \ldots, \tau_n'$ of $\mathscr{C}'$ satisfy

$$\tau_1' \cdots \tau_n' = q^{-2n}\tau_1 \cdots \tau_n = q^{-2n}\frac{V(\mathscr{C})}{\Delta(\Lambda)}.$$

Now $\mathscr{C}$ consists of all those $x \in P_n$ such that (10) holds. For these to be in Λ, $y_j t^{-\mu_j} \in k_0[t]$. So if $-\mu_j \geqq 0$ there are $-\mu_j + 1$ k_0-independent choices for y_j.

For $\mathscr{C}'$ the result corresponding to (10) reads

$$y = y_1 t^{-\nu_1 - 2} x_1^* + \cdots + y_n t^{-\nu_n - 2} x_n^*, \quad |y_j| \leqq 1.$$

So if $-\nu_j \geqq 2$, there are $-\nu_j - 1$ k_0-independent choices for y_j.

It follows from (9) that

$$l - l' = \sum_{-\mu_j \geqq 0}(-\mu_j + 1) - \sum_{-\nu_j \geqq 2}(-\nu_j - 1).$$

Hence

$$q^{l-l'} = (\sigma_1 \cdots \sigma_n)^{-1} q^n = q^n \frac{V(\mathscr{C})}{\Delta(\Lambda)}.$$

3. Application to function fields. Let k_0, $k = k_0(t)$ and $\mathfrak{o} = k_0[t]$ be defined as in section 2 and let K be a finite algebraic extension of k of degree n.

Let ν be the valuation of k defined in (1) and let $\mathfrak{p}$ be the prime divisor of k corresponding to ν. Let $S = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_h\}$ be the set of extensions of $\mathfrak{p}$ to K. The corresponding normalized exponential valuations of K (cf. [13], p. 12) will be denoted by $\nu_1, \ldots, \nu_h$. Let e_i, f_i denote the ramification index and residue class degree respectively of $\mathfrak{P}_i$ over $\mathfrak{p}$. Let $\hat{k}$ be as in section 2 and let $\hat{K}_i$ denote the perfect completion of K with respect to ν_i, that is, at $\mathfrak{P}_i$. The unique extensions of $\mathfrak{P}_i$ and ν_i to $\hat{K}_i$ will be denoted by $\mathfrak{P}_i$ and ν_i. Set $K_\mathfrak{p} = \hat{k} \otimes_k K$. Then one has a canonical homomorphism, φ, of $\hat{k}$-algebras

$$\varphi : K_\mathfrak{p} \to \prod_{i=1}^{h} \hat{K}_i \tag{11}$$

defined by a continuous extension of the canonical diagonal imbedding

$$\psi : K \to \prod_{i=1}^{h} \hat{K}_i \tag{12}$$

(cf. [3], Chap. 6, § 8, No. 2). By projection onto the ith component, one obtains a map

$$\varphi_i : K_\mathfrak{p} \to \hat{K}_i \qquad (1 \leqq i \leqq h). \tag{13}$$

Write $[\hat{K}_i : \hat{k}] = n_i$. Then (cf. [2], Th. 3, p. 484) we have

$$e_i f_i = n_i, \quad \sum_{i=1}^{h} n_i = n. \tag{14}$$

It follows that ([3], Chap. 6, § 8, No. 5, Th. 2, Cor. 2) φ is an isomorphism of $\hat{k}$-algebras.

As is well known, there exists a $\mathfrak{P}_i$-integral basis for $\hat{K}_i/\hat{k}$ (cf. [13], p. 52, Th. 2.3.2).

In particular, such a basis is given by

$$\omega_{i\varkappa}\pi_i^\lambda \qquad (1 \leqq \varkappa \leqq f_i;\ 0 \leqq \lambda \leqq e_i - 1) \tag{15}$$

where the $\omega_{i\varkappa}$ are integral elements at $\mathfrak{P}_i$, whose residue classes mod $\mathfrak{P}_i$ are linearly independent over the residue class field of k mod $\mathfrak{p}$, and π_i is a prime element for $\mathfrak{P}_i$, that is, $\nu_i(\pi_i) = 1$. Then we have

$$\nu_i\left(\sum_{\varkappa=1}^{f_i}\sum_{\lambda=0}^{e_i-1} x_{\varkappa\lambda}^{(i)}\omega_{i\varkappa}\pi_i^\lambda\right) = \min_{\varkappa,\lambda}\left(\nu_i(x_{\varkappa\lambda}^{(i)}\pi_i^\lambda\right) \tag{16}$$

and it follows from

$$\nu_i\left(\sum_{\varkappa=1}^{f_i}\sum_{\lambda=0}^{e_i-1} x_{\varkappa\lambda}^{(i)}\omega_{i\varkappa}\pi_i^\lambda\right) \geqq e_i m_i, \quad m_i \in \mathbf{Z}, \tag{17}$$

(cf. [5], p. 62) that

$$\nu(x_{\varkappa\lambda}^{(i)}) \geqq m_i. \tag{18}$$

For each of the completions $\hat{K}_i$, $1 \leqq i \leqq h$, let $\zeta_1^{(i)}, \ldots, \zeta_{n_i}^{(i)}$ be a fixed basis of the form (15). Then for $\alpha \in K$, we have

$$\alpha = \sum_{\mu=1}^{n_i} \alpha_\mu^{(i)}\zeta_\mu^{(i)} \qquad (\alpha_\mu^{(i)} \in \hat{k})$$

and we define a k-linear injection

$$\theta_i : K \to \hat{k}^{n_i} \qquad (1 \leqq i \leqq h)$$

by

$$\alpha \mapsto (\alpha_1^{(i)}, \ldots, \alpha_{n_i}^{(i)}). \tag{19}$$

These maps define a k-linear injection

$$\theta : K \to \hat{k}^n \tag{20}$$

in the obvious way. At the same time, one has a $\hat{k}$-linear isomorphism

$$\eta : \prod_{i=1}^{h} \hat{K}_i \to \hat{k}^n. \tag{21}$$

Let $\mathfrak{O}$ denote the integral closure of $\mathfrak{o}$ in K. Denote by $\mathscr{D}(K)$ the group of divisors of K and by $\mathscr{D}(k)$ the group of divisors of k. Let $\mathscr{S} = \mathscr{D}(K) - S$ be the set of "finite" prime divisors of K. A given divisor $\mathfrak{A} = \prod \mathfrak{P}^{\nu_{\mathfrak{P}}(\mathfrak{A})}$ of K can be written in the form

$$\mathfrak{A} = \mathfrak{A}_e \mathfrak{A}_u \tag{22}$$

with

$$\mathfrak{A}_e = \prod_{\mathfrak{P} \in \mathscr{S}} \mathfrak{P}^{\nu_{\mathfrak{P}}(\mathfrak{A})}, \quad \mathfrak{A}_u = \prod_{\mathfrak{P} \in S} \mathfrak{P}^{\nu_{\mathfrak{P}}(\mathfrak{A})}. \tag{23}$$

We shall show that the finite part $\mathfrak{A}_e$ corresponds to a lattice $\Lambda(\mathfrak{A})$ in $P_n = \hat{k}^n$ and that the infinite part $\mathfrak{A}_u$ corresponds to a convex body $\mathscr{C}(\mathfrak{A})$. The details are as follows.

We set (cf. [5], Chaps. I, II)

$$L(\mathfrak{A}) = \{\alpha \in K \mid \nu_{\mathfrak{P}}(\alpha) \geqq \nu_{\mathfrak{P}}(\mathfrak{A}) \text{ for } \mathfrak{P} \in \mathscr{D}(K)\}, \tag{24}$$

$$L(\mathfrak{A}_e) = L(\mathfrak{A}, \mathcal{S}) = \{\alpha \in K \mid \nu_{\mathfrak{P}}(\alpha) \geqq \nu_P(\mathfrak{A}),\ \mathfrak{P} \in \mathcal{S}\}, \tag{25}$$

$$L(\mathfrak{A}_u) = L(\mathfrak{A}, S) = \{\alpha \in K \mid \nu_{\mathfrak{P}}(\alpha) \geqq \nu_P(\mathfrak{A}),\ \mathfrak{P} \in S\}. \tag{26}$$

Now $L(\mathfrak{A}_e)$ is an $\mathfrak{O}$-ideal and has an $\mathfrak{o}$-basis of n elements ([14], p. 267, Th. 9). If $\alpha_1, \ldots, \alpha_n$ is such a basis, then the map $\theta(\alpha_\mu) = (\alpha_{\mu 1}, \ldots, \alpha_{\mu n})$ (see (19), (20)) defines a non-singular matrix $A = (\alpha_{\mu\lambda})$. The matrix A gives rise to an invertible linear mapping of $\hat{k}^n$ into itself, to which corresponds a lattice $\Lambda(\mathfrak{A})$. Every other $\mathfrak{o}$-basis of $L(\mathfrak{A}_e)$ defines the same lattice and we have $\theta(L(\mathfrak{A}_e)) = \Lambda(\mathfrak{A})$.

We turn now to the definition of $\mathscr{C}(\mathfrak{A})$. For $\mathfrak{P}_i \in S$, write $\nu_i(\mathfrak{A}) = a_i$. For some $b_i \in \mathbf{Z}$, we have $a_i = b_i e_i + r_i$, $0 \leqq r_i < e_i$. If $\pi \in \underline{k}$ is a prime element for $\mathfrak{p}$, in particular if $\pi = t^{-1}$, then the condition

$$\nu_i\left(\sum_{\varkappa=0}^{f_i} \sum_{\lambda=0}^{e_i-1} x_{\varkappa\lambda}^{(i)} \omega_{i\varkappa} \pi_i^\lambda\right) \geqq a_i \tag{27}$$

is equivalent to

$$\nu_i\left(\sum_{\varkappa=0}^{f_i} \sum_{\lambda=0}^{e_i-1} t^{b_i} x_{\varkappa\lambda}^{(i)} \omega_{i\varkappa} \pi_i^\lambda\right) \geqq r_i .$$

Now it follows from (16), (17) and (18) that this last inequality is equivalent to

$$\left.\begin{array}{ll} \nu(t^{b_i} x_{\varkappa\mu}^{(i)}) \geqq 1 & (0 \leqq \mu \leqq r_i - 1) \\ \nu(t^{b_i} x_{\varkappa\mu}^{(i)}) \geqq 0 & (r_i \leqq \mu \leqq e_i - 1) \end{array}\right\} (1 \leqq \varkappa \leqq f_i). \tag{28}$$

On taking into consideration the formula for the special convex body given in section 2, we obtain:

Lemma 1. *The totality of n-tuples* $(x_{\varkappa\lambda}^{(i)})$ $(1 \leqq i \leqq h;\ 1 \leqq \varkappa \leqq f_i;\ 0 \leqq \lambda \leqq e_i - 1)$ *for which conditions* (27) *hold is a convex body* $\mathscr{C}(\mathfrak{A})$ *in* P_n *with volume*[2])

$$V(\mathfrak{A}) = q^{-\sum_{i=1}^{h} f_i \nu_i(\mathfrak{A})}. \tag{29}$$

In order to calculate the determinant of the given lattice $\Lambda(\mathfrak{A})$, we must make some further computations.

Let $\sigma: K \to k$ be a pseudo-spur (cf. [8], p. 45) which will be kept fixed in what follows. It follows from [11], Rule 3, p. 417 that there exists just one proper linear map

$$\sigma_{\mathfrak{p}}: K_{\mathfrak{p}} \to \hat{k} \tag{30}$$

which coincides with σ on K. (Note that φ is an isomorphism.) The map $\sigma_{\mathfrak{p}}$ induces a

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[2]) If

$$\hat{L}(\mathfrak{A}, S) = \left\{\alpha \in \prod_{i=1}^{h} \hat{K}_i \mid \nu_i(\alpha) \geqq a_i\right\}$$

then

$$\eta(\hat{L}(\mathfrak{A}, S)) = \mathscr{C}(\mathfrak{A}), \quad \text{cf. (21).}$$

pseudo-spur

(31) $$\sigma_{\mathfrak{P}_i} = \sigma_i : \hat{K}_i \to \hat{k} \qquad (1 \leqq i \leqq h),$$

and for $\alpha \in K_{\mathfrak{p}}$ we have

(32) $$\sigma_{\mathfrak{p}}(\alpha) = \sum_{i=1}^{h} \sigma_i(\varphi_i(\alpha)).$$

We refer to [8] for the notion of a pseudo-discriminant, D_σ. The elements $\zeta_\gamma^{(i)}$ $(1 \leqq \lambda \leqq n_i;\ 1 \leqq i \leqq h)$ form a basis of $\prod_{i=1}^{h} \hat{K}_i$ over $\hat{k}$ and, denoting the whole basis by $\zeta_1, \ldots, \zeta_n$ we have (see [11], proof of Lemma 3)

(33) $$\det(\sigma_{\mathfrak{p}}(\varphi^{-1}(\zeta_\lambda)\,\varphi^{-1}(\zeta_\mu))) = \prod_{i=1}^{h} D_{\sigma_i}(\zeta_1^{(i)}, \ldots, \zeta_{n_i}^{(i)}).$$

Let us denote by $m_i = m_i(\sigma)$ the σ-differential exponent for $\mathfrak{P}_i$ (cf. [11], § 4). Then it follows from the proof of [11] Lemma 3, that

(34) $$\nu(\det(\sigma_{\mathfrak{p}}(\varphi^{-1}(\zeta_\lambda)\,\varphi^{-1}(\zeta_\mu)))) = \sum_{i=1}^{h} f_i m_i.$$

Now let $\alpha_1, \ldots, \alpha_n$ be a basis for K/k and suppose that

$$\alpha_\mu = \sum_{\lambda=1}^{n} \alpha_{\lambda\mu}\,\varphi^{-1}(\zeta_\lambda), \quad \alpha_{\lambda\mu} \in \hat{k}, \quad 1 \leqq \mu \leqq n.$$

Then we have

(35) $$D_\sigma(\alpha_1, \ldots, \alpha_n) = (\det(\alpha_{\lambda\mu}))^2 \det(\sigma_{\mathfrak{p}}(\varphi^{-1}(\zeta_\lambda)\,\varphi^{-1}(\zeta_\mu))).$$

From this and from (33) and (34), we obtain

(36) $$\nu(D_\sigma(\alpha_1, \ldots, \alpha_n)) = 2\,\nu(\det(\alpha_{\lambda\mu})) + \sum_{i=1}^{h} f_i m_i.$$

Denote by $D_\sigma(\mathfrak{A}, \mathscr{S})$ the $\mathfrak{o}$-ideal generated by the pseudo-discriminant of an $\mathfrak{o}$-basis of the $\mathfrak{O}$-ideal $L(\mathfrak{A}_e)$. It follows from [8], p. 85, formula (6), that[3])

(37) $$D_\sigma(\mathfrak{A}, \mathscr{S}) = (N_{K/k}(\mathfrak{A}_e))^2 D_\sigma(\mathfrak{E}, \mathscr{S})$$

where $\mathfrak{E}$ is the identity in $\mathscr{D}(K)$.

We can now state and prove:

Lemma 2. *Let $\Lambda(\mathfrak{A})$ be the lattice defined by $L(\mathfrak{A}_e)$. Then if $\Delta(\mathfrak{A}) = |\det \Lambda(\mathfrak{A})|$ we have*

(38) $$\Delta(\mathfrak{A}) = q^{\delta(\mathfrak{A}) + a}.$$

Here

(39) $$\delta(\mathfrak{A}) = \sum_{\mathfrak{P} \in \mathscr{S}} \deg(\mathfrak{P})\,\nu_{\mathfrak{P}}(\mathfrak{A})$$

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[3]) The condition on p. 84 is satisfied because $\mathfrak{O}$ is a free $\mathfrak{o}$-module of rank n.

and the number

$$a = \tfrac{1}{2}\sum_{i=1}^{h} f_i m_i - \tfrac{1}{2}\nu(D_\sigma) \tag{40}$$

where D_σ is a generator of the $\mathfrak{o}$-ideal $D(\mathfrak{E}, \mathscr{S})$, is independent of $\mathfrak{A}$.

Proof. The lattice $\Lambda(\mathfrak{A})$ is given by the matrix $A = (\alpha_{\lambda\mu})$ where the $\alpha_{\lambda\mu}$ are defined as above. We have

$$\Delta(\mathfrak{A}) = |\det(\alpha_{\lambda\mu})| = q^{-\nu(\det(\alpha_{\lambda\mu}))}.$$

Now it follows from (36) that

$$\nu(\det(\alpha_{\lambda\mu})) = \tfrac{1}{2}\nu(D_\sigma(\alpha_1, \ldots, \alpha_n)) - \tfrac{1}{2}\sum_{i=1}^{h} f_i m_i.$$

For every prime divisor $\mathfrak{q} \neq \mathfrak{p}$ in $\mathscr{D}(k)$ we have, by (37),

$$\nu_{\mathfrak{q}}(D_\sigma(\alpha_1, \ldots, \alpha_n)) = 2\sum_{\mathfrak{P}|\mathfrak{q}} f(\mathfrak{P}/\mathfrak{q})\,\nu_{\mathfrak{P}}(\mathfrak{A}) + \nu_{\mathfrak{q}}(D_\sigma),$$

where $f(\mathfrak{P}/\mathfrak{q})$ is the relative degree of the prime divisor $\mathfrak{P}$ of K over $\mathfrak{q}$. On multiplying this equation by $\deg \mathfrak{q}$ and using the relation

$$\sum_{\mathfrak{q} \neq \mathfrak{p}} \deg(\mathfrak{q})\,\nu_{\mathfrak{q}}(\alpha) + \nu_{\mathfrak{q}}(\alpha) = 0 \qquad (\alpha \in k,\ \alpha \neq 0)$$

summation over all $\mathfrak{q} \neq \mathfrak{p}$ gives

$$\nu(\det(\alpha_{\lambda\mu})) = -\sum_{\mathfrak{P} \in \mathscr{S}} \deg(\mathfrak{P})\,\nu_{\mathfrak{P}}(\mathfrak{A}) + \tfrac{1}{2}\nu(D_\sigma) - \tfrac{1}{2}\sum_{i=1}^{h} f_i m_i.$$

This completes the proof of the lemma.

Now let $l(\mathfrak{A})$ be the dimension of the k_0-module $L(\mathfrak{A}) = L(\mathfrak{A}_{\mathrm{e}}) \cap L(\mathfrak{A}_{\mathrm{u}})$. By construction, $l(\mathfrak{A})$ is also the number of k_0-independent points of the lattice $\Lambda = \Lambda(\mathfrak{A})$ in the convex body $\mathscr{C}(\mathfrak{A})$. So we can apply Theorem 1 once we are able to identify the body $\mathscr{C}'$ and the lattice Λ^*.

Now the pseudo-spur σ defines a non-degenerate bilinear form on the k-vector space K and, by means of φ and η (see (12) and (21)), induces a non-degenerate bilinear form on P_n. It is with respect to these bilinear forms that we speak of complementary ideals in K and polar lattices and polar bodies in P_n, respectively.

Let $\mathfrak{o}_{\mathfrak{p}}$ be the valuation ring of the prime divisor $\mathfrak{p}$ and $\mathfrak{O}_{\mathfrak{p}}$ its integral closure in K. Furthermore, let $\mathfrak{D}_{\mathrm{e}}$ be the pseudo-different of $\mathfrak{O}$ over $\mathfrak{o}$, $\mathfrak{D}_{\mathrm{u}}$ the pseudo-different of $\mathfrak{O}_{\mathfrak{p}}$ over $\mathfrak{o}_{\mathfrak{p}}$. If $\mathfrak{A}$ is a divisor of K, put $\mathfrak{A}^* = \mathfrak{A}^{-1}\mathfrak{D}^{-1}$, where $\mathfrak{D} = \mathfrak{D}_{\mathrm{e}}\mathfrak{D}_{\mathrm{u}}$. Then the complementary ideal to the $\mathfrak{O}$-ideal $L(\mathfrak{A}_{\mathrm{e}})$ is $L((\mathfrak{A}\mathfrak{D})_{\mathrm{e}}^{-1})$, and the complementary ideal to the $\mathfrak{O}_{\mathfrak{p}}$-ideal $L(\mathfrak{A}_{\mathrm{u}})$ is $L((\mathfrak{A}\mathfrak{D})_{\mathrm{u}}^{-1})$. By definition of the polar lattice and taking into account (cf. section 2) that the polar body of $\mathscr{C}$ consists of all those points $\boldsymbol{y}$ of P_n for which $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leqq 1$ for all $\boldsymbol{x} \in \mathscr{C}$, i.e. $\langle \boldsymbol{x}, \boldsymbol{y}\rangle \in \hat{\mathfrak{o}}_{\mathfrak{p}}$, the completion of $\mathfrak{o}_{\mathfrak{p}}$ in $\tilde{k}$, we get

$$\Lambda^*(\mathfrak{A}) = \Lambda(\mathfrak{A}^*), \quad \mathscr{C}^*(\mathfrak{A}) = \mathscr{C}(\mathfrak{A}^*).$$

Write

$$\mathfrak{A}' = \mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{U}^2 \tag{41}$$

where $\mathfrak{U} = \prod_{i=1}^{h} \mathfrak{P}_i^{e_i}$ is the divisor of the denominator of t in K. Then $\Lambda(\mathfrak{A}') = \Lambda^*(\mathfrak{A})$ and $\mathscr{C}(\mathfrak{A}')$ is the body $\mathscr{C}'$ of Theorem 1 corresponding to $\mathscr{C}^* = \mathscr{C}^*(\mathfrak{A})$.

Collecting together all our results, we see that Lemmas 1 and 2 yield the following formula for $l(\mathfrak{A}) - l(\mathfrak{A}')$.

Theorem 2. *Let* $\mathfrak{A}$ *be a divisor of* K *and let* $\mathfrak{A}'$ *be the divisor defined in* (41). *Then*

$$l(\mathfrak{A}) - l(\mathfrak{A}') = n - \tfrac{1}{2}\sum_{i=1}^{h} f_i m_i - \deg(\mathfrak{A}) + \tfrac{1}{2}\nu(D_\sigma). \tag{42}$$

4. The Riemann-Roch Theorem. Two divisors $\mathfrak{A}$ and $\mathfrak{B}$ of K are said to be in the same class if $\mathfrak{A}\mathfrak{B}^{-1}$ is a principal divisor. We obtain the Riemann-Roch Theorem in the form:

Theorem 3. *There exists a class* $\mathfrak{k}$ *and a non-negative integer* g, *which depends only on* K, *such that if* $\mathfrak{A}\mathfrak{A}' \in \mathfrak{k}$ *then*

$$l(\mathfrak{A}) = -\deg(\mathfrak{A}) - g + 1 + l(\mathfrak{A}'). \tag{43}$$

Proof. Let $\mathfrak{k}$ be the class containing $\mathfrak{D}\mathfrak{U}^{-2}$. Set

$$g = 1 - n + \tfrac{1}{2}\left(-\nu(D_\sigma) + \sum_{i=1}^{h} f_i m_i\right). \tag{44}$$

By Theorem 2 we have

$$l(\mathfrak{A}) - l(\mathfrak{A}') = -g - \deg(\mathfrak{A}) + 1. \tag{45}$$

If we take $\mathfrak{A} = \mathfrak{E}$ in (45), then we have

$$l(\mathfrak{E}) - l(\mathfrak{E}') = -g - \deg(\mathfrak{E}) + 1 = -g + 1.$$

Now, by the product formula[4]) (cf. [5], Chap. 1 or [13], Chap. 5), $l(\mathfrak{E}) = 1$. Hence

$$g = l(\mathfrak{D}^{-1}\mathfrak{U}^2) \geqq 0.$$

Again, if $\deg(\mathfrak{A}) \geqq 1$, then the product formula gives $l(\mathfrak{A}) = 0$. Moreover, $\deg(\mathfrak{A}^{-1}) > 2g - 2$ implies $l(\mathfrak{A}') = 0$, that is, there are no points of Λ^* in $\mathscr{C}'$. For $\deg(\mathfrak{A}^{-1}) > 2g - 2$ implies $l(\mathfrak{A}) - l(\mathfrak{A}') > g - 1$ and this implies

$$\frac{\Lambda^*}{V'} > q^{n+g-1}.$$

Now let α' be an element of K corresponding to a point of Λ^* in $\mathscr{C}'$. Then by Lemmas 1 and 2

$$\prod_{\mathfrak{P}} q^{\nu_{\mathfrak{P}}(\alpha')} \geqq \frac{\Lambda^*}{V'} q^{-(n+g-1)}.$$

But then $\Lambda^*/V' > q^{n+g-1}$ contradicts the product formula. Whence the result.

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[4]) The product formula can be interpreted in terms of lattice points, corresponding to elements of K, inside a cube.

This completes the proof of Theorem 3 and our outline of familiar properties which can be obtained from it (cf. [8], pp. 148—151), except for the independence of g. To prove this, we observe that if $\deg(\mathfrak{A}^{-1}) > 2g - 2$, then

$$g = -l(\mathfrak{A}) - \deg(\mathfrak{A}) + 1 .$$

But both $l(\mathfrak{A})$ and $\deg(\mathfrak{A})$ depend only on K and not on t; so g depends only on K ([8], p. 151).

5. The case of finite k_0. Let $q = \mathrm{Card}(k_0)$ and let $\mathscr{C}$ be a convex body in P_n of volume V. Let l denote the dimension of the k_0-module of points of a lattice Λ with determinant Δ in $\mathscr{C}$. Then $\mathscr{C}$ contains q^l points of Λ. Consider the set of all possible translates of $\mathscr{C}$ by lattice points. Since P_n is an ultrametric space, two such translates either do not overlap or they are identical. By applying a linear transformation of determinant Δ^{-1} we can transform this situation to the following one. We have a system of congruent convex bodies each of which contains q^l points with coordinates in $k_0[t]$ and which do not overlap.

Now consider a large "sphere" (or cube):

$$\| \boldsymbol{x} \| \leqq q^N.$$

This contains $q^{n(N+1)}$ points with coordinates in $k_0[t]$. Hence

$$\frac{q^{n(N+1)}}{q^l} \cdot \frac{V}{\Delta} \leqq q^{nN} .$$

Now from (44)

$$\frac{V}{\Delta} = q^{(-n-g-\deg(\mathfrak{A})+1)} .$$

So

$$q^{(l+nN)} \geqq q^{(-n-g-\deg(\mathfrak{A})+1)} q^{n(N+1)} .$$

Therefore

$$q^l \geqq q^{(-g-\deg(\mathfrak{A})+1)} .$$

Hence

(46) $$l \geqq -g - \deg(\mathfrak{A}) + 1 .$$

Moreover, equality holds if V/Δ is large enough. So (46) holds for all divisors $\mathfrak{A}$, with equality if $-\deg(\mathfrak{A})$ is large enough. This is Riemann's Theorem.

The proof of the Riemann-Roch Theorem can now be completed as in [9].

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