Categorical Characterization of the MacNeille Completion

By

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Introduction. The MacNeille completion of a partially ordered set P was first introduced by means of a particular construction which generalizes Dedekind's construction of the totally ordered set of all real numbers from the rationals [2], [7]. Only later, characterizations were given in terms of order theoretic properties, determining the MacNeille completion of P up to isomorphism over P as an extension of P with specific properties [1], [3]. A natural problem arising in this context is that of describing the MacNeille completion in the much more confined language of order preserving mappings, i.e., in categorical terms. This we deal with in the present note, both, for the category of partially ordered sets and order preserving mappings, and for the category of Boolean lattices and Boolean homomorphisms.

One of the first problems concerning categories of concrete mathematical objects of the "structured set" type is to find a categorical description of the naturally given morphisms from subobjects to objects. In the case of partially ordered sets, Grothendieck's notion of strict monomorphism [5] provides the required description, and from this, a suitable categorical notion of essential extension furnishes the desired setting. The case of Boolean lattices is analogous, though somewhat simpler because all monomorphisms are embeddings there.

The results obtained here seem striking to us in that two categories rather dissimilar from the Abelian ones are shown to have, with respect to injectivity, exactly the same features as the latter, as may be seen by comparing this paper with [4]. This is all the more remarkable if one considers, as is proved at the end of the paper, that for lattices and lattice homomorphisms there are no non-trivial injectives at all.

Our thanks for useful advice on categorical matters go to J. W. Duskin.

1. Generalities. By a *partially ordered set* we mean, as usual, a set, called the underlying set, together with a partial order relation on it. The latter will always be denoted by "$\leqq$", and a subset, or element, of a partially ordered set P is taken to be a subset, or element of the underlying set. For two partially ordered sets P and Q, P is called a partially ordered subset of Q iff its underlying set is a subset of that of Q, and its partial order relation is the restriction of that of Q. All concepts relating to partially ordered sets, unless stated otherwise, will be used as in [2].

A *morphism* from a partially ordered set P to a partially ordered set Q is a triple (P, f, Q) where f is a mapping from the underlying set of P to the underlying set of Q

such that $x \leqq y$ implies $f(x) \leqq f(y)$. We do not distinguish in notation between f and (P, f, Q). That f is a morphism from P to Q will also be expressed as $f\colon P \to Q$. The partially ordered sets together with their morphisms clearly form a category which will be denoted by $\mathscr{P}$. All general categorical concepts will be used here as in [9].

A partially ordered set E will be called an *extension* of a partially ordered set P iff P is a partially ordered subset of E. The *natural morphism* $j\colon P \to E$ mapping the elements of P identically then has the property that $x \leqq y$ iff $j(x) \leqq j(y)$. In general, any morphism $j\colon P \to Q$ with this property will be called an *embedding*.

An extension E of a partially ordered set P is called *join dense (meet dense)* iff each element of E is the join (meet) of its predecessors in P. If E and E' are two extensions of a partially ordered set P then a morphism $E \to E'$ will be called *over* P iff it maps the elements of P identically.

A partially ordered set P is called a *retract* of a partially ordered set E iff there exist morphisms $j\colon P \to E$ and $f\colon E \to P$ such that $f \circ j$ is the identity on P [9]. A j for which such f exist will be called *retractable* and any such f a *retraction* of j. It is evident that a retractable morphism is in fact an embedding. P will be called a *retract of an extension* E of P iff the natural embedding $j\colon P \to E$ is retractable, and by a retraction $f\colon E \to P$ will be meant a retraction of j.

2. Special Morphisms. In this section, we give a *set theoretic* characterization of the epimorphisms and monomorphisms in $\mathscr{P}$ and a *categorical* characterization of embeddings and certain types of extensions.

Lemma 1. *The epimorphisms in $\mathscr{P}$ are exactly those morphisms given by onto mappings, and the monomorphisms are exactly those morphisms given by one-to-one mappings.*

Proof. Let $f\colon P \to Q$ be an epimorphism, and suppose f is not onto. Then, take any $a \in Q$ not an image under f and define a new partially ordered set as follows: S has the same elements as Q except for the element a which is replaced by two new elements, b and c, and the partial order of S is defined by putting $b \leqq c$, $c \leqq x$ for all $x \in Q$ above a in Q, $y \leqq b$ for all $y \in Q$ below a in Q, and $y \leqq x$ for all $x, y \in Q$ distinct from a iff this holds in Q. Then, take $g\colon Q \to S$ defined by $g(x) = x$ for all $x \neq a$ and $g(a) = b$, and $h\colon Q \to S$ defined by $h(x) = x$ for all $x \neq a$ and $h(a) = c$. Now one clearly has $g \circ f = h \circ f$ but $g \neq h$ which contradicts the assumption that f is an epimorphism. Hence f must be onto. The converse, of course, is obvious.

Now, let $f\colon P \to Q$ be a monomorphism, and suppose f is not one-to-one, i.e., $f(a) = f(b)$ for distinct $a, b \in P$. Then let S be the discrete partially ordered set, i.e., its partial order is the equality relation, with underlying set $\{a, b\}$, and consider $g\colon S \to P$ with $g(a) = g(b) = a$, $h\colon S \to P$ with $h(a) = h(b) = b$. Then, $f \circ g = f \circ h$ whereas $g \neq h$ which contradicts the assumption that f is a monomorphism. Hence f must be one-to-one. The converse, again, is obvious.

Note that a monomorphism, though one-to-one, need not be an embedding. In this regard, the category $\mathscr{P}$ is rather like the category of topological spaces and continuous maps, and unlike categories of algebraic structures and their homomorphisms.

Generally, in a category, a monomorphism $f\colon P \to Q$ is called *strict* [5] iff every morphism $g\colon T \to Q$ which equalizes every pair of morphisms $u, v\colon Q \to S$ equalized by f factors through f, i.e., if $u \circ f = v \circ f$ always implies $u \circ g = v \circ g$ then there exists an $h\colon T \to P$ such that $g = f \circ h$. An object in a category will be called *strictly injective* iff it satisfies the usual injectivity condition with respect to *strict* monomorphisms.

Lemma 2. *A morphism $f\colon P \to Q$ is an embedding iff it is a strict monomorphism.*

Proof. Let $f\colon P \to Q$ be an embedding; then it is a monomorphism, and it remains to be shown that it is strict. For this, take any $g\colon T \to Q$ which equalizes every pair of morphisms from Q equalized by f, and assume that $g(T) \nsubseteq f(P)$. Then, by the proof of Lemma 1, there exists a partially ordered set S and $u, v\colon Q \to S$ which differ *only at one* point a in $g(T)$, not in $f(P)$. Hence, $u \circ f = v \circ f$ whereas $u \circ g \neq v \circ g$, a contradiction. It thus follows that $g(T) \subseteqq f(P)$, and $h = j \circ g$, j the inverse of f on $f(P)$, provides the factorization.

Conversely, let $f\colon P \to Q$ be a strict monomorphism, and take any $a, b \in P$ with $f(a) < f(b)$. Then, let T be the partially ordered subset of Q determined by $\{f(a), f(b)\}$, and $g\colon T \to Q$ the natural embedding. Now, clearly, g equalizes any pair $u, v\colon Q \to S$ equalized by f, hence there exists an $h\colon T \to P$ such that $g = f \circ h$. Since $g(f(a)) = f(a)$, one has $h(f(a)) = a$, and the same for b, and $f(a) < f(b)$ then implies $a < b$. This shows that f is an embedding.

We shall call a monomorphism $f\colon P \to E$ *essential* iff it is strict, and if any morphism $g\colon E \to Q$ such that $g \circ f$ is a strict monomorphism is itself a strict monomorphism. An extension E of P will, analogously, be called essential iff the natural embedding $P \to E$ is essential. Note that in the category of all left modules over a ring, all monomorphisms are strict, and hence the counterpart of the notion of essential extension defined here coincides with what one usually means by essential extension of a module.

Essential extensions of essential extensions are again essential extensions, and essential extensions are again essential extensions of any smaller extension. This is readily proved in general, although here it becomes obvious in view of the following characterization of essential extensions.

Lemma 3. *An extension E of a partially ordered set P is essential iff it is both, meet and join dense.*

Proof. Let E be a meet and join dense extension of P, and $f\colon E \to Q$ such that $f \mid P$ is a strict monomorphism, i.e., an embedding. Now, for any $a, b \in E$ such that $a \nleqq b$ there then exists an $x \leqq a$, such that $x \nleqq b$ and $x \in P$ (join density) and hence also a $y \geqq b$ such that $y \nleqq x$ and $y \in P$ (meet density). Now, $f \mid P$ being an embedding, one has $f(x) \nleqq f(y)$, and this implies $f(a) \nleqq f(b)$ since $f(x) \leqq f(a)$ and $f(b) \leqq f(y)$. Thus f is an embedding.

Conversely, assume that E fails to be, say, a join dense extension of P. Then there exists an $a \in E$ which is not the join of all $x \leqq a$, $x \in P$, and hence there exists a $b \ngeqq a$ in E such that every lower bound of a in P is also a lower bound of b. Now let $M \supseteqq E$ be any completion of E, and consider $f\colon E \to M$ defined by $f(x) = \bigvee p\,(x \geqq p \in P)$.

Then $f|P$ is the natural embedding $P \to M$, but $f(b) \geqq f(a)$ and hence f is not an embedding. The dual case works out dually.

3. Injectivity. The injective partially ordered sets turn out to be neatly characterized, both by certain categorical conditions resembling the situation in other categories, and by internal properties.

Proposition 1. *The following are equivalent for a partially ordered set P:*

(1) *P is complete,*

(2) *P is strictly injective,*

(3) *P is a retract of every extension,*

(4) *P has no proper essential extensions.*

Proof. (1) $\Rightarrow$ (2). Given a homomorphism $f\colon A \to P$ and any strict monomorphism $g\colon A \to B$. Then define $h\colon B \to P$ by

$$h(b) = \bigvee_{g(a) \leqq b,\, a \in A} f(a).$$

This is clearly a morphism, and for $b = g(a_0)$, $a_0 \in A$, one has $h(b) = f(a_0)$ since $g(a) \leqq g(a_0)$ implies $a \leqq a_0$ and hence $f(a) \leqq f(a_0)$. Thus $f = h \circ g$.

(2) $\Rightarrow$ (3). If P is injective with respect to strict monomorphisms and $E \supseteqq P$ any extension of P then, the natural embedding $j\colon P \to E$ being a strict monomorphism, there exists an $f\colon E \to P$ such that $f \circ j$ is the identity on P.

(3) $\Rightarrow$ (4). If $E \supseteqq P$ is an essential extension and, by hypothesis on P, $f\colon E \to P$ a retraction then, for the natural injection $j\colon P \to E$, $f \circ j$ is the identity on P, thus a strict morphism, and the same holds then for f by hypothesis on E. Therefore, f is one-to-one, but since it is also a retraction one has $E = P$.

(4) $\Rightarrow$ (1). By Lemma 3, the MacNeille completions of P are essential extensions, and if no proper such extension of P exists then P is already complete.

Corollary. *Any retract of a complete partially ordered set is complete.*

Proof. It is evident that retracts of strict injectives are strict injectives; hence the assertion.

Remark. In (2), the restriction to *strict* monomorphisms cannot be dropped, as the following consideration shows: Let A, B, and P have the same underlying set, A discrete, B arbitrary, and P complete (e.g. well-ordered with last element); then, take $f\colon A \to P$ and $g\colon A \to B$ to be given by the identity mapping of the underlying set. Clearly, any $h\colon B \to P$ such that $f = h \circ g$ must also be given by the identity mapping, and B may obviously be chosen in such a way that the latter fails to determine a morphism.

4. MacNeille completions. We recall that a MacNeille completion M of a partially ordered set P is a complete extension of P which is both meet and join dense, the existence of such extensions being given by the well-known construction of MACNEILLE [2], [8].

Proposition 2. *The following are equivalent for an extension E of a partially ordered set P:*

(1) *E is a MacNeille completion of P;*

(2) *E is an essential, strictly injective extension of P;*

(3) *E is a strictly injective extension of P not containing any properly smaller such extension of P;*

(4) *E is an essential extension of P not contained in any properly larger such extension of P.*

Proof. (1) $\Rightarrow$ (2). This follows directly from the definition, in view of Lemma 3 and Proposition 1.

(2) $\Rightarrow$ (3). Let $E' \subseteqq E$ be another strictly injective extension of P, $f\colon E' \to E$ the natural embedding, and $g\colon E' \to E'$ the identity morphism. Then, there exists $h\colon E \to E'$ such that $h \circ f = g$, i. e., h is a retraction. Now, E is also an essential extension of E', and hence h is an embedding. It follows from this that $E' = E$.

(3) $\Rightarrow$ (4). If $E' \supseteqq P$ is an essential extension of P such that $E' \supseteqq E$ then E' is also an essential extension of E, and by Proposition 1 $E' = E$. To see that E is itself an essential extension, consider a MacNeille completion M of P. The natural embedding $P \to E$ then extends to a mapping $f\colon M \to E$ which must be an embedding since M is an essential extension of P by (2); by hypothesis on E and, again, (2) it follows that $f(M) = E$, i.e., f is an isomorphism, and therefore E an essential extension.

(4) $\Rightarrow$ (1). Consider a MacNeille completion M of E; since E is an essential extension of P, M is also an essential extension of P, and hence $M = E$. This shows E is complete and therefore a MacNeille completion of P.

In the above proof, only the *existence* of MacNeille completions was used but no more about them; since strict monomorphisms which are epimorphisms are in fact isomorphisms here one has:

Corollary. *Any two MacNeille completions of a partially ordered set P are isomorphic over P.*

Condition (3) in Proposition 2 might be expressed by saying that E is a *minimal* strictly injective extension of P. A formally different condition would be that E is a *least* such extension in the sense that it can be embedded, over P, in any other strictly injective extension of P. These two conditions are, however, also equivalent: If E is minimal and $E' \supseteqq P$ any strictly injective extension then the fact that E is an essential extension already implies it can be embedded in E' over P. Conversely, if E is least, then it can be embedded over P in a MacNeille completion of P and must then be isomorphic to it.

In a similar way, (4) states that E is a *maximal* essential extension of P, and this turns out to be equivalent to being a *largest* essential extension of P in the sense that any other essential extension of P can be embedded in it. If E is maximal and E' any essential extension of P then the strict injectivity of E shows that E' can be embedded in E over P. Conversely, if E is largest then any MacNeille completion can be embedded in E over P, and E must then be equal to the image, hence isomorphic to it.

The characterizations which arise from (3) and (4) by replacing strict injectivity by completeness and essential extension by meet and join dense extension were first given in [3].

Since strict injectivity is the same as injectivity in the category of all left modules over a ring, the MacNeille completions of partially ordered sets correspond exactly to the injective hulls of modules. There are, however, aspects in which these two concepts do differ: Both categories admit products, and for modules, the product of injective hulls of two modules A and B is an injective hull of $A \times B$, but the analogous statement for MacNeille completions of partially ordered sets is false. For instance, the closed unit interval is a MacNeille completion of the open unit interval $]0, 1[$, but $(]0, 1[\times]0, 1[) \cup \{(0, 0), (1, 1)\}$ is a MacNeille completion of $]0, 1[\times]0, 1[$, which is not the closed unit square.

5. Boolean lattices. The MacNeille completions of a Boolean lattice B are, as is well known, again Boolean lattices [2], characterized as the complete Boolean lattices C containing B as Boolean sublattice and as join (or, equivalently, meet) dense subset. It therefore seems natural to investigate whether they have categorical properties analogous to those of the MacNeille completions of partially ordered sets. Some results on *injective* Boolean lattices are given in [6], [7], but the relation between these and essential extensions is not discussed there. Our considerations here closely parallel those of the preceding sections, with completely analogous results, and we therefore restrict ourselves to a more condensed presentation.

In the following, $\mathscr{B}$ denotes the category of Boolean lattices and Boolean lattice homomorphisms. The same type of notation as before is used in the present context.

Lemma 4. *The epimorphisms in $\mathscr{B}$ are exactly the onto homomorphisms, and the monomorphisms in $\mathscr{B}$ exactly the one-to-one homomorphisms; moreover, all monomorphisms are embeddings.*

Proof. That onto homomorphisms are epimorphisms is, as usual, clear. In order to prove the converse it is sufficient to show that, for any Boolean lattice B and a proper Boolean sublattice A of B, there exist two distinct Boolean lattice homomorphisms from B into a two-element Boolean lattice which coincide on A, and this amounts to saying that there exist distinct ultrafilters $U, V \subseteqq B$ such that $U \cap A = V \cap A$. In order to see this, let B be the field of all open-closed subsets of a compact, zero-dimensional, Hausdorff space Ω, and assume that every ultrafilter in A is contained in *only one* ultrafilter in B. Now, take any $\xi \in \Omega$ and consider the ultrafilter $W \subseteqq A$ of all members of A containing ξ. Since W is contained in only one ultrafilter in B, it follows that the intersection of all members of W is $\{\xi\}$, and by a well-known theorem about compact spaces, W is therefore a basis for the neighbourhood filter of ξ. This shows that A is a basis for the topology of Ω. Hence every member of B is the union of members of A, but then, by compactness, also the union of finitely many members of A, and therefore $B = A$, a contradiction.

Next, let $f\colon A \to B$ be a monomorphism, but $f(a) = f(c)$ for two distinct a, $c \in A$. Then, a Boolean lattice with two free generators can be used here in the same way a discretely ordered two-element set was used in the proof of Lemma 1 to obtain the desired contradiction. The converse is, again, obvious.

Finally, to see that every monomorphism $f: A \to B$ is an embedding, let $f(a) \leqq f(c)$ for any $a, c \in A$. Then $f(a) = f(a) \wedge f(c) = f(a \wedge c)$, hence $a = a \wedge c$, and thus $a \leqq c$.

Remark. The above statement about epimorphisms and monomorphisms is given in [10] but without proof.

We now turn to the concept of essential extension, formally defined as before, but the condition of strictness of the monomorphisms may, of course, be dropped in view of Lemma 4.

Lemma 5. *An extension E of a Boolean lattice B is essential iff it is join dense.*

Proof. Let E be join dense and $f: E \to C$ such that $f | B$ is one-to-one. In order to show that f itself is one-to-one it suffices to show that $f(x) = 0$ implies $x = 0$ for any $x \in E$. Given $f(x) = 0$ one has that $f(b) = 0$ for all $b \leqq x$ in B and hence $b = 0$ for these b; since x is the join of these b this implies $x = 0$.

Conversely, assume that the extension E of B is not join dense. Then, as is easily seen, there exist elements $y < x$ in E which have the same lower bounds in B.

Now consider the ideal $J = [0, x - y]$ in E. If $b \leqq x - y$, $b \in B$, then $b \leqq x$ and $b \leqq -y$, but also $b \leqq y$, and hence $b = 0$. Thus $J \cap A$ is zero, and this implies that the natural homomorphism $E \to E/J$ is one-to-one on A whereas it is not so on E since $y < x$. It follows that E is not an essential extension of B.

Concerning injectivity, the situation here is exactly analogous to that in the case of partially ordered sets:

Proposition 3. *The following are equivalent for a Boolean lattice B:*

(1) *B is complete,*

(2) *B is injective,*

(3) *B is a retract of every extension,*

(4) *B has no proper essential extensions.*

The equivalence of (1), (2), and (3) was shown in [6], and the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are obtained in the same way as their counterparts in Proposition 3, (4) $\Rightarrow$ (1) in view of Lemma 5.

Finally, we have the following characterization of the MacNeille completions of Boolean lattices:

Proposition 4. *The following are equivalent for an extension E of a Boolean lattice B:*

(1) *E is a MacNeille completion of B,*

(2) *E is an essential, injective extension of B,*

(3) *E is an injective extension of B not containing any properly smaller such extension of B,*

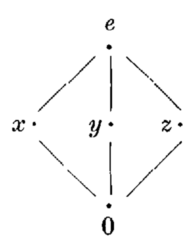
(4) *E is an essential extension of B not contained in any properly larger such extension of B.*

The proof of this proceeds almost verbatim like that of Proposition 2, and there is no need for giving it here. The same applies to the remark following Proposition 2 regarding minimal versus least injective extensions and maximal versus largest essential extensions.

6. Concluding remarks. In closing this paper it may be of interest to contrast the situation found here with that in some other categories which are, in some sense, not too far removed from the categories discussed here.

Thus, the category $\mathscr{K}$ of compact zero-dimensional Hausdorff spaces and continuous mappings is dually isomorphic to $\mathscr{B}$, but in $\mathscr{K}$ the following is known to hold (or easy to prove): The two-point spaces and hence their products are injective; therefore, every $X \in \mathscr{K}$ can be embedded (= mapped by a monomorphism) into an injective $Y \in \mathscr{K}$, and $\mathscr{K}$ thus has a large supply of injectives. On the other hand, however, *no* $X \in \mathscr{K}$ has non-trivial essential extensions.

Now, for the category $\mathscr{L}$ of all the lattices and lattice homomorphisms (for which, of course, $\mathscr{B} \subseteqq \mathscr{L} \subseteqq \mathscr{P}$) the proof of Lemma 2 shows that all join-meet dense extensions are essential, and thus there are many essential extensions in $\mathscr{L}$. However, *$\mathscr{L}$ has no non-trivial injectives:* Take a lattice K of the type



e.g. the lattice of all subgroups of the Klein four-group, and let K_0 be the sublattice $\{0, x, y, e\}$ of K. Then there exists, for *any* lattice L with more than one element, a lattice homomorphism $f_0\colon K_0 \to L$ which cannot be extended to a lattice homomorphism $f\colon K \to L$. To see this, let $a, b \in L$ be distinct and assume, which may be done, that $a < b$. Then put $f_0(0) = f_0(x) = a$ and $f_0(y) = f_0(e) = b$. Now, if $f\colon K \to L$ were a lattice homomorphism extending f_0 one would have $a \leqq f(z) \leqq b$, and hence $b = f(e) = f(x \vee z) = f(x) \vee f(z) = f(z)$, but also $a = f(0) = f(y \wedge z) = f(y) \wedge \wedge f(z) = f(z)$, a contradiction.

By similar, though somewhat more extensive arguments, we can prove that *every lattice containing at least two elements has arbitrarily large essential extensions* which is stronger since it implies the above result.

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