On Common Transversals

By B. GRÜNBAUM in Jerusalem

1. Let $\mathfrak{P}$ denote a family of subsets of the plane. We shall say that $\mathfrak{P}$ has the property $T(n)$, n a natural number, if any n members of $\mathfrak{P}$ have a common transversal (i.e. are intersectable by a suitable straight line); we shall say that $\mathfrak{P}$ has property T, if all members of $\mathfrak{P}$ have a common transversal.

With these definitions we have the following theorem:

Theorem 1. *Let $\mathfrak{P}$ be a family of disjoint translates of a parallelogram. Then, if $\mathfrak{P}$ has property $T(5)$, it has property T.*

Proof. There is no loss of generality if we assume that the members of $\mathfrak{P}$ are squares. For simplicity of expression we shall call the directions determined by the edges of the squares horizontal resp. vertical.

We find it convenient to distinguish two possible cases.

Case 1. There exists a horizontal line H and a vertical one V, and two squares $P_1, P_2 \in \mathfrak{P}$, such that P_1 and P_2 are contained in different quadrants of a pair of opposite quadrants determined by H and V. Without restriction of generality, we may assume the case represented schematically in Fig. 1. Then any straight line intersecting both P_1 and P_2 is an "ascending" line, i. e. a line which is either parallel to H or V, or is contained in quadrants I and III except possibly for a finite interval. Therefore $T(5)$ implies that any 3 members of $\mathfrak{P}$ may be intersected by an ascending line. Theorem 1 is then a consequence of the following corollary of HELLY's theorem, due to HADWIGER and DEBRUNNER [3]:

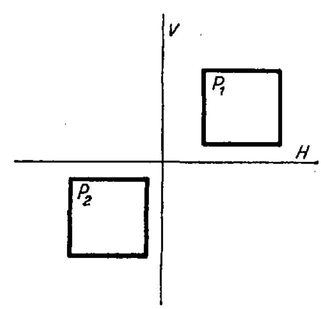

Fig. 1.

Given any family of parallelograms with parallel edges, such that any three can be intersected by an ascending line, there exists an ascending line intersecting all the parallelograms.

It may be mentioned parenthetically that this result of HADWIGER and DEBRUNNER obviously implies the following theorem due to SANTALÓ [4], which thus becomes a direct consequence of HELLY's theorem:

For any family of parallelograms with parallel edges, $T(6)$ implies T.

Case 2. Now we may assume that for any two squares of $\mathfrak{P}$ there exists a horizontal, or a vertical, line intersecting both, and therefore (since they are disjoint) a vertical, resp. horizontal, line separating them. Since $\mathfrak{P}$ contains at least 6 members*), it follows that there are at least three squares which are separated in pairs by parallel lines, either vertical or horizontal. Obviously, we may for definiteness assume the former case.

Let $\mathfrak{P}^*$ denote a subset of $\mathfrak{P}$, maximal with respect to the property that any two members of $\mathfrak{P}^*$ may be separated by a vertical line. If $\mathfrak{P}^* = \mathfrak{P}$, Theorem 1 follows from well-known results on common transversals of sets separated by parallel lines (see, e.g., HADWIGER-DEBRUNNER [3]). There remains therefore the case $\mathfrak{P}^* \neq \mathfrak{P}$. Using the assumptions we made, it is immediate that only the three cases represented schematically in Figs. 2—4 (and those derived from them by symmetry) are possible;

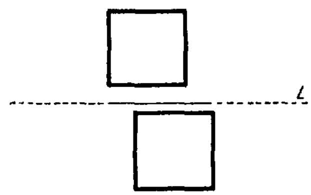

Fig. 2.

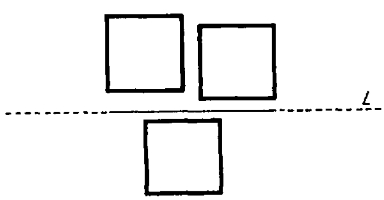

Fig. 3.

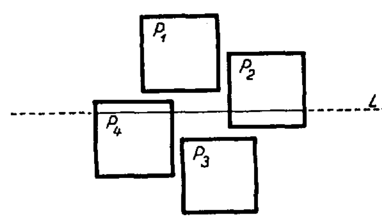

Fig. 4.

all the members of $\mathfrak{P}$ not shown in Figs. 2—4, intersect the dotted part of the line L. Let us call the two, three, resp. four, squares of $\mathfrak{P}$ corresponding to those of the schemes in Figs. 2—4, the principal squares of $\mathfrak{P}$.

Let $A(D)$ denote the ascending (descending) straight line intersecting all the principal squares of $\mathfrak{P}$ and enclosing a minimal angle with L. Obviously, if ascending (descending) lines intersecting the principal squares exist at all, $A(D)$ exists; one of them surely exists by $T(5)$.

If only one of the lines A and D exists, it must intersect also all the non-principal squares of $\mathfrak{P}$, since they intersect L and $T(5)$ holds.

Thus we may assume that both A and D exist. In cases represented by Figs. 2 and 3 we may establish the theorem as follows: Any non-principal square must, as above, intersect either A, or D, or both. If one of them does not meet A (or D), it must meet D (resp. A) and therefore $T(5)$ implies that any other such square is also intersected by D (resp. A); and if all the non-principal squares are intersected by both A and D, there is obviously nothing to be proved.

*) In view of SANTOLÓ's result cited above, we could obviously assume that $\mathfrak{P}$ contains exactly 6 members. We refrain from this assumption, since it would not simplify our proof.

Therefore, there remains only the case represented by Fig. 4. Now, A is obviously determined by P_1 and P_3, while D is determined by a pair of squares which is different from the pair P_2, P_4. Thus both A and D are determined by only three of the four principal squares, and therefore the reasoning applied above establishes that at least one of A, D intersects all the members of $\mathfrak{P}$.

This ends the proof of Theorem 1.

2. Remarks. $T(4)$ can not be substituted for $T(5)$ in Theorem 1, as is shown by the 5 central squares in Fig. 5. Moreover, for any natural $k > 4$ it is possible to construct similar examples, containing k squares, by modifying slightly the example of Fig. 5 (two additional squares are shown in Fig. 5). This may be contrasted with a result of HADWIGER [2]: For infinite families of disjoint, congruent convex bodies in the plane, $T(3)$ implies T. (See also below, section 3.)

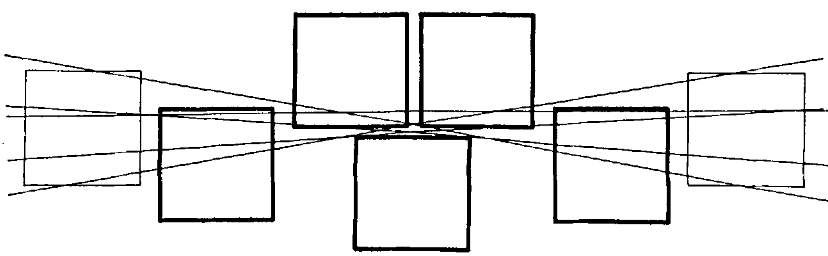

Fig. 5.

The other conditions of Theorem 1 do not seem to be equally necessary, although we were unable to weaken them. The following points seem worth mentioning:

(i) Theorem 1 probably holds also if $\mathfrak{P}$ denotes a family of disjoint translates of any convex set.

(ii) If, instead of disjoint translates of a set, families of congruent, disjoint sets are considered, $T(5)$ is not sufficient in order to ascertain T, as is shown by the example in Fig. 6 (obviously, suitable parallelograms or other sets may be substituted for the segments). It is possible that $T(6)$ implies T for any such family, and it seems very probable that $T(5)$ implies T for any family of congruent, disjoint squares.

(iii) The condition that the squares are disjoint may perhaps be dropped. The Theorem of SANTALÓ cited above implies that T is a consequence of $T(6)$ even in a more general situation. But the condition is necessary if rotations are allowed; indeed, if squares are constructed on the segments of Fig. 6, $T(5)$ holds while T does not hold.

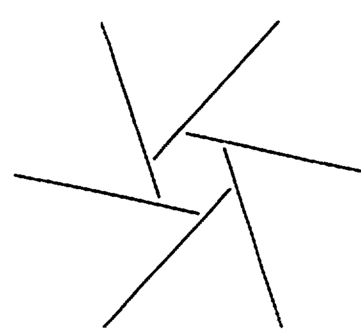

Fig. 6.

3. In a recent paper DANZER [1] proved that $T(5)$ implies T for families of congruent, disjoint circles. For such families $T(4)$ does not imply T as is shown by the example in Fig. 7 (due to HADWIGER-DEBRUNNER [3], where also other relevant examples may be found).

Nevertheless, in contrast to the situation in case of parallelograms, we have:

Theorem 2. *For families of disjoint, congruent circles containing at least six members, $T(4)$ implies T.*

In view of DANZER's result, Theorem 2 is obviously a consequence of the following: *Given a family $\mathfrak{C}$ of five disjoint, congruent circles, satisfying $T(4)$ but not $T(5)$, it is impossible to enlarge it by a sixth circle, congruent to those of $\mathfrak{C}$ and disjoined from them, in such a way that $T(4)$ holds in the enlarged family.*

We sketch here the proof of the last statement.

Let C_1 and C_3 denote two members of $\mathfrak{C}$ which are at maximal distance. Let A and B denote the two closed segments contained in the boundary of the convex hull of $C_1 \cup C_3$. Discarding the other a priori possible cases it is easily shown that, with suitable notations, we must have $A \cap C_2 \neq \varnothing$, $B \cap C_4 \neq \varnothing$ and $B \cap C_5 \neq \varnothing$. A completely elementary (but somewhat lengthy and tedious) examination of the possible types of families, differing in the order in which quadruples of circles are intersected by straight lines, reveals that in each of the cases no sixth circle may be added to the family under the assumed conditions. (Figs. 7 and 8 represent two of the possible types; in fact, these are the two extremal types, the other being intermediate between them. It is obvious that only relatively small changes in the mutual positions of the circles are possible.) This proves our assertion, and with it Theorem 2.

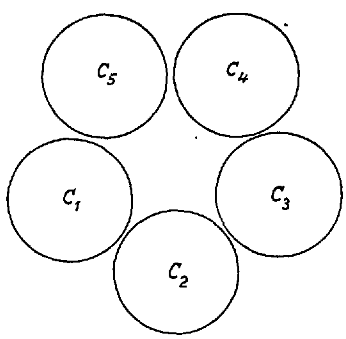

Fig. 7.

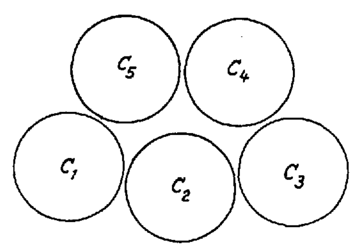

Fig. 8.

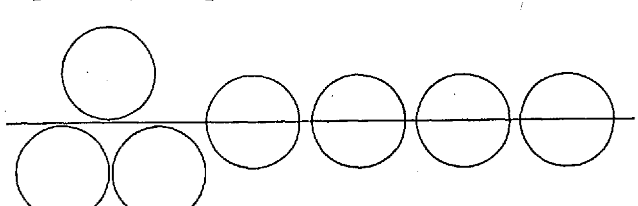

Fig. 9.

Remark. For finite families of congruent, disjoint circles, $T(3)$ does not imply T. The example in Fig. 9 may obviously be modified in such a way as to contain any given number of circles.

References

[1] L. DANZER, Über ein Problem aus der kombinatorischen Geometrie. Arch. Math. **8**, 347–351 (1957).
[2] H. HADWIGER, Über einen Satz Hellyscher Art. Arch. Math. **7**, 377–379 (1956).
[3] H. HADWIGER – H. DEBRUNNER, Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene. L'enseignement mathématique **1**, 56–89 (1955).
[4] L. A. SANTALÓ, Un teorema sobre conjuntos de paralelepipedos de aristas paralelas. Publ. Inst. Mat. Univ. Nac. Litoral **2**, 49–60 (1940); **3**, 202–210 (1942).

Added in proof. Remark (iii) on p. 467 may be sharpened as follows: Without the assumption that the sets are disjoint, $T(5)$ does not imply T even if only families of translates of a given parallelogram are considered. This follows from the following example:

Let $\mathfrak{P}$ consist of 6 squares with sides of length 20 parallel to the coordinate axes, and centers $(-22;4)$, $(0;-15)$, $(12;11)$, $(22;4)$, $(12;-11)$ and $(0;15)$, respectively. Then $T(5)$ holds but T does not hold; obviously the size of the family may be increased to any number > 6.

Similarly, if the sets are disjoint but instead of translations only we allow translations and homotheties of the given parallelogram, $T(5)$ does not imply T. Examples to that effect, containing any number of sets, are easily constructed.

Eingegangen am 23. 4. 1958