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BOOLEAN SKEW ALGEBRAS

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1. Introduction

In this paper we introduce a class of skew lattices which generalizes relatively complemented distributive lattices with a smallest element. A member $(A;\ \wedge, \vee, 0)$ of this class can be considered as an algebra of type $(2, 2, 0)$ satisfying the identities: $a\wedge a=a$, $a\wedge(b\wedge c)=(a\wedge b)\wedge c$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$, $(b\vee c)\wedge a= =(b\wedge a)\vee(c\wedge a)$, $(b\wedge a)\vee a=a$, and the condition: for all $a, b\in A$, there exists $c\in A$ such that $a=(a\wedge b)\vee c$ and $c\wedge b=0$. The element c in this last condition is uniquely determined by a and b and is denoted by $r(a, b)$. In this way, the class gives rise to a variety of algebras $(A;\ \wedge, \vee, r, 0)$ of type $(2, 2, 2, 0)$; we call it the variety of *Boolean skew algebras*. Consequences of our axioms are the identities $a\vee a=a$, $a\vee(b\vee c)=(a\vee b)\vee c$, $a=a\vee(a\wedge b)=(a\wedge b)\vee a=a\wedge(a\vee b)=a\wedge(b\vee a)=(b\vee a)\wedge \wedge a$, $(b\wedge c)\vee a=(b\vee a)\wedge(c\vee a)$, $a\wedge 0=0=0\wedge a$, and $a\vee 0=a=0\vee a$. Thus, we really are considering a class of skew lattices and it turns out that each of the identities: $a\wedge b=b\wedge a$, $a\vee b=b\vee a$, $a=a\vee(b\wedge a)$, $a=(a\vee b)\wedge a$, and $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$, defines the same proper subvariety.

On any Boolean skew algebra, the maps $x\to x\vee a$ and $x\to x\wedge a$ are actually endomorphisms. Using this observation it turns out that, up to isomorphism, there are two subdirectly irreducible Boolean skew algebras, viz. $\mathbf{3}=\{0, 1, 2\colon 1\wedge 2=1, 2\wedge 1=2, 1\vee 2=2, 2\vee 1=1$ are the non-trivial relations$\}$, which is the cogenerator of the variety, and its subalgebra $\mathbf{2}=\{0, 1;\ \wedge, \vee, 0\}$, which is the two element lattice considered as a relatively complemented distributive lattice. Thus, the lattice of subvarieties of Boolean skew algebras is the three-chain.

In the last section of the paper, we show how Boolean skew algebras arise from rings which possess central idempotent covers and from quasiprimal varieties of universal algebras.

2. Fundamentals

A *Boolean skew lattice* is an algebra $(A;\ \wedge, \vee, 0)$ of type $(2, 2, 0)$ satisfying the identities

(2.1) $$a\wedge a = a,$$

(2.2) $$a\wedge(b\wedge c) = (a\wedge b)\wedge c,$$

(2.3) $$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c),$$

(2.4) $$(b\vee c)\wedge a = (b\wedge a)\vee(c\wedge a),$$

(2.5) $$(b\wedge a)\vee a = a,$$

and the first order sentence

(2.6) for all $a, b \in A$, there exists $c \in A$ such that $a=(a\wedge b)\vee c$ and $c\wedge b=0$.

The laws (2.1) and (2.2) say that $(A;\ \wedge)$ is a *band* (idempotent semigroup). In section 3 we will see that the $\vee$-operation is associative so that a Boolean skew lattice is in effect a special type of semiring, i.e. an algebra $(A;\ \cdot, +)$ such that both $(A;\ \cdot)$ and $(A;\ +)$ are semigroups and $\cdot$ distributes over $+$ from both the left and the right. The skew lattice nature is indicated only by the vital absorption law (2.5). The precise relation of our algebras to skew lattices is considered in the next section; in the present section, we consider the properties of the $\wedge$-operation and the role of the element 0.

PROPOSITION 2.1 *A Boolean skew lattice satisfies the identities:* $a\vee a=a$; $a\wedge 0=0=0\wedge a$; $a\vee 0=a=0\vee a$.

PROOF. Because of identities (2.1) and (2.5), $a\vee a=(a\wedge a)\vee a=a$. By (2.6) $a=(a\wedge a)\vee t$ for some t such that $t\wedge a=0$. Hence, $a=a\wedge a=(a\vee t)\wedge a=(a\wedge a)\vee(t\wedge a)$, by 2.4, and so $a=a\vee 0$. Also, $0=(0\wedge a)\vee z$ for some z such that $z\wedge a=0$. Hence, $0\wedge a=(z\wedge a)\wedge a=z\wedge a=0$. This leads to $0\vee a=(0\wedge a)\vee a=a$ via (2.5). Finally, $a\wedge 0=(a\wedge 0)\vee 0=0$.

Thus, $(A;\ \wedge, 0)$ is a band with 0 as its zero element. In this connection we have the important:

LEMMA 2.2. *In a band* $(A;\ \wedge, 0)$ *with zero,* $a\wedge b=0$ *if and only if* $b\wedge a=0$. *Also,* $a\wedge b=0$ *implies* $a\wedge x\wedge b=0$ *for any* $x\in A$.

PROPOSITION 2.3. *For any elements a and b in a Boolean skew lattice* $(A;\ \wedge, \vee, 0)$, *the element c such that* $a=(a\wedge b)\vee c$ *and* $c\wedge b=0$, *arising from* (2.6), *is unique and is denoted by* $r(a, b)$.

PROOF. Suppose $a=(a\wedge b)\vee c=(a\wedge b)\vee d$ and $c\wedge b=0=d\wedge b$. Using (2.3) and Proposition 2.1, and Lemma 2.2, $c\wedge a=(c\wedge a\wedge b)\vee(c\wedge c)=0\vee c=c$. But $c\wedge a=c\wedge\big((a\wedge b)\vee d\big)=(c\wedge a\wedge b)\vee(c\wedge d)=c\wedge d$. Also, $a\wedge d=(a\wedge b\wedge d)\vee(d\wedge d)=d$ and $a\wedge d=(a\wedge b\wedge d)\vee(c\wedge d)=c\wedge d$. Hence, $c=c\wedge a=c\wedge d=a\wedge d=d$, as required.

Because of the proposition, we can introduce a new binary operation r on any Boolean skew lattice, obtain a variety and yet not affect homomorphisms and congruences. More precisely, a *Boolean skew algebra* $(A;\ \wedge, \vee, r, 0)$ is an algebra of type (2, 2, 2, 0) such that the reduct $(A;\ \wedge, \vee, 0)$ satisfies the identities (2.1)—(2.5) and (2.6) is replaced by the identities

$$(2.6)' \qquad a=(a\wedge b)\vee r(a, b), \quad r(a, b)\wedge b=0.$$

Also, if $(A_1;\ \wedge, \vee, r, 0)$ and $(A_2;\ \wedge, \vee, r, 0)$ are Boolean skew algebras and $f\colon (A_1;\ \wedge, \vee, 0)\to(A_2;\ \wedge, \vee, 0)$ is a homomorphism between the underlying Boolean skew lattices then for any $a, b\in A_1$, $f(a)=f\big((a\wedge b)\vee r(a, b)\big)=\big(f(a)\wedge f(b)\big)\vee f\big(r(a, b)\big)$ and $f\big(r(a, b)\big)\wedge f(b)=f\big(r(a, b)\wedge b\big)=f(0)=0$. By Proposition 2.3, $f\big(r(a, b)\big)=r\big(f(a), f(b)\big)$ and so f is a homomorphism of Boolean skew algebras. It follows that if $(A;\ \vee, \wedge, r, 0)$ is a Boolean skew algebra and Θ is a congruence on the underlying Boolean skew lattice $(A;\ \wedge, \vee, 0)$ then the quotient A/Θ is a Boolean skew algebra, Θ has the substitution property for the r-operation, and the associated projection of A onto A/Θ is homomorphism of Boolean skew algebras. These observations will be used whenever we consider congruences and homomorphisms in the variety of Boolean skew algebras, which will be henceforth denoted by **BSA**. For the sake of brevity, we will refer to a Boolean skew algebra as a **BSA**-algebra.

The next result summarizes the most important properties of the $\wedge$-operation.

PROPOSITION 2.4. *Any* **BSA**-*algebra satisfies the identities:*

(i) $a\wedge b\wedge a = a\wedge b$, (ii) $(a\wedge b)\wedge c = (a\wedge c)\wedge(b\wedge c)$, (iii) $a\wedge b\wedge c = a\wedge c\wedge b$,

(iv) $c\wedge(a\wedge b) = (c\wedge a)\wedge(c\wedge b)$, (v) $(a\wedge b)\wedge(c\wedge d) = (a\wedge c)\wedge(b\wedge d)$.

PROOF. (i) $a\wedge b\wedge a=(a\wedge b)\wedge((a\wedge b)\vee r(a, b))=((a\wedge b)\wedge(a\wedge b))\vee((a\wedge b)\wedge r(a, b))= a\wedge b$, by Proposition 2.1 and Lemma 2.2.

(ii) $a=(a\wedge c)\vee r(a, c)$ and so $a\wedge(b\wedge c)=((a\wedge c)\wedge(b\wedge c))\vee(r(a, c)\wedge b\wedge c)=(a\wedge c)\wedge \wedge(b\wedge c)$, by Lemma 2.2.

(iii) $a\wedge b\wedge c=a\wedge b\wedge c\wedge b$ (by (i))$=(a\wedge b)\wedge(c\wedge b)=(a\wedge(c\wedge b))\wedge((b\wedge(c\wedge b))$ (by (ii))$=a\wedge(c\wedge b)\wedge b\wedge(c\wedge b)=a\wedge(c\wedge b)\wedge b$ (by (i))$=a\wedge c\wedge b$.

(iv) $c\wedge(a\wedge b)=c\wedge(b\wedge a)$ (by (iii))$=c\wedge(b\wedge a)\wedge c$ (by (i))$=(c\wedge(c\wedge b))\wedge(a\wedge c)= =(c\wedge a\wedge c)\wedge(c\wedge b)$ (by (iii))$=(c\wedge a)\wedge(c\wedge b)$ (by (i)).

(v) $(a\wedge b)\wedge(c\wedge d)=(a\wedge b\wedge c)\wedge d=(a\wedge c\wedge b)\wedge d$ (by (iii))$=(a\wedge c)\wedge(b\wedge d)$.

It may be worthwhile to make some remarks about the above identities. Firstly, a band $(A;\ \wedge)$ satisfying the identity (iii) of Proposition 2.4 necessarily satisfies (i) and (ii) and hence all of the identities of the proposition. Moreover, it is not hard to see that if the law $a\wedge b\wedge a=a\wedge b$ holds on $(A;\wedge)$ then the laws (ii), (iii) and (v) are equivalent. Bands satisfying (iii) and (v) have been studied extensively by semigroup theorists, see PETRICH [9] for detailed information. The identity $a\wedge b\wedge a=a\wedge b$ holds in any skew lattice $(A;\ \wedge,\ \vee)$, see for example JORDAN [6] and GERHARDTS [4]; the role of (iii) was also considered by JORDAN in the same paper and GERHARDTS in another paper [5].

3. Skew lattices

Here we will take implicit advantage of the results of the previous section.

PROPOSITION 3.1. *Any* **BSA**-*algebra satisfies the identities:*

(i) $a\vee(a\wedge b) = a$, (ii) $(a\wedge b)\vee a = a$, (iii) $a\wedge(a\vee b) = a$, (iv) $a\wedge(b\vee a) = a$.

PROOF. Clearly (i) and (iii) are equivalent, as are (ii) and (iv).

(iii) $a\wedge(a\vee b) = ((a\wedge b)\vee r(a, b))\wedge(a\vee b) = (a\wedge b\wedge(a\vee b))\vee(r(a, b)\wedge(a\vee b)) =$

$= ((a\wedge b\wedge a)\vee(a\wedge b))\wedge((r(a, b)\wedge a)\vee(r(a, b)\wedge b)) = ((a\wedge b)\wedge(a\wedge b))\vee((r(a, b)\wedge a)\vee$

$\vee 0) = (a\wedge b)\vee(r(a, b)\wedge a) = (a\wedge b\wedge a)\vee(r(a, b)\wedge a) = ((a\wedge b)\vee r(a, b))\wedge a = a\wedge a = a.$

(iv) $a\wedge(b\vee a) = ((a\wedge b)\vee r(a, b))\wedge(b\vee a) = ((a\wedge b)\wedge(b\vee a))\vee(r(a, b)\wedge(b\vee a)) =$

$= (a\wedge b\wedge b)\vee(a\wedge b\wedge a)\vee(r(a, b)\wedge a) = (a\wedge b)\vee((r(a, b)\wedge a) = a$, as above.

PROPOSITION 3.2. *Any* **BSA**-*ylgebra satisfies the associative law:*

$$a\vee(b\vee c)=(a\vee b)\vee c.$$

PROOF. Let $x=a\vee(b\vee c)$ and $y=(a\vee b)\vee c$. Then, $x\wedge c=(a\wedge c)\vee((b\vee c)\wedge c)=(a\wedge c)\vee c=c$, by repeated application of the identity (2.5). Hence, $x=c\vee r(x,c)$. Also, $y\wedge c=c$, by (2.5) and (2.4) and so $y=c\vee r(y,c)$.

But $x\wedge r(x,c)=r(x,c)$ i.e. $(a\vee(b\vee c))\wedge r(x,c)=r(x,c)$. Expanding and simplifying, we obtain $(a\vee b)\wedge r(x,c)=r(x,c)$.

Also, $a\wedge x=a\wedge(a\vee(b\vee c))=a$ and $b\wedge x=b\wedge(a\vee(b\vee c))=(b\wedge a)\vee b=b$, by the previous proposition. Hence, $(a\vee b)\wedge x=a\vee b$. Using the previous paragraphs, we obtain: $a\vee b=(a\vee b)\wedge x=(a\vee b)\wedge(c\vee r(x,c))=((a\vee b)\wedge c)\vee((a\vee b)\wedge r(x,c))=((a\vee b)\wedge c)\vee r(x,c)$.

On the other hand, $y\wedge r(y,c)=r(y,c)$, i.e. $((a\vee b)\vee c)\wedge r(y,c)=r(y,c)$ and hence $(a\vee b)\wedge r(y,c)=r(y,c)$.

But $(a\vee b)\wedge y=(a\vee b)\wedge((a\vee b)\vee c)=a\vee b$, by Proposition 3.1. Thus $a\vee b=(a\vee b)\wedge y=(a\vee b)\wedge((y\wedge c)\vee r(y,c))=(a\vee b)\wedge(c\vee r(y,c))=((a\vee b)\wedge c)\vee((a\vee b)\wedge r(y,c))=((a\vee b)\wedge c)\vee r(y,c)$.

Hence, $a\vee b=((a\vee b)\wedge c)\vee r(x,c)=((a\vee b)\wedge c)\vee r(y,c)$ and $r(x,c)\wedge c=0=r(y,c)\wedge c$. By Proposition 2.3, $r(x,c)=r(y,c)$. Thus, $x=c\vee r(x,c)=c\vee r(y,c)=y$, as required.

PROPOSITION 3.3. *A* **BSA**-*algebra satisfies the distributive law:*

$$(b\wedge c)\vee a=(b\vee a)\wedge(c\vee a).$$

PROOF. $(b\vee a)\wedge(c\vee a)=(b\wedge(c\vee a))\vee(a\wedge(c\vee a))=((b\wedge c)\vee(b\wedge a))\vee a$ (by Proposition 3.1) $=(b\wedge c)\vee((b\wedge a)\vee a)$ (by Proposition 3.2) $=(b\wedge c)\vee a$ (by identity (2.5).

We are finally in a position to describe **BSA**-algebras as skew lattices.

According to GERHARDTS [4] and SLAVIK [10], [11], a *skew lattice* is an algebra $(A;\ \wedge,\vee)$ of type $(2,2)$ satisfying the identities:

(3.1) $$a\wedge(b\wedge c)=(a\wedge b)\wedge c \quad\text{and}\quad a\vee(b\vee c)=(a\vee b)\vee c,$$

(3.2) $$a\wedge(b\vee a)=a \quad\text{and}\quad (a\wedge b)\vee a=a$$

(3.3) $$a\wedge(a\vee b)=a \quad\text{and}\quad (b\wedge a)\vee a=a.$$

It should be noted that with these identities, the usual lattice-duality between $\wedge$ and $\vee$ is extended by the dualities: $a\wedge b\to b\vee a$ and $a\vee b\to b\wedge a$. This duality was built into the subject by its founder P. JORDAN, see [6] for an extensive bibliography. According to GERHARDTS [4] a skew lattice $(A;\ \wedge,\vee)$ is *distributive* if the dual identities

(3.4) $$(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c) \quad\text{and}\quad a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c),$$

are satisfied. Also, in [10], [11], SLAVIK gives the necessary and sufficient condition for the reflection (maximal homomorphic image) of a skew lattice in the variety of lattices to be a distributive lattice; it is the satisfaction of the identity $(a\wedge(b\vee c))\wedge((a\wedge b)\vee(a\wedge c))=a\wedge(b\vee c)$. Thus, when it comes to distributivity, it is not clear what the appropriate notion of a "distributive skew lattice" should be; certainly, our identities (2.3) and (2.4) offer an alternative which is consistent with the work of Slavik even if they do not conform with Jordan's inbuilt notion of duality. Bearing in mind (3.1)—(3.3) and our propositions so far, we can summarize as follows:

THEOREM 3.4. *Let* $(A;\ \wedge, \vee, r, 0)$ *be a Boolean skew algebra. Then,* $(A;\ \wedge, \vee)$ *is a skew lattice satisfying:*

(i) *the additional absorption law* $a\wedge(a\vee b)=a$, *and*

(ii) *the distributive laws* $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$, $(a\wedge b)\vee c=(a\vee c)\wedge(b\vee c)$.

Of all possible absorption laws and distributive laws, we are missing: $a\vee(b\wedge a)=a=(a\vee b)\wedge a$ and $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$; we shall decide their occurrence in the next section.

4. Lattice of subvarieties

We begin this section with a technical result; of course, the associativity of the $\vee$-operation is presumed from now on.

LEMMA 4.1. *In any* **BSA**-*algebra,* $a\wedge b=0$ *implies* $a\vee b=b\vee a$. *In particular,* $a=r(a,b)\vee(a\wedge b)$ *is an identity on any* **BSA**-*algebra. More generally,* $b\vee a=a\vee b\vee(a\wedge b)$ *is an identity.*

PROOF. Suppose $a\wedge b=0$. Then, $(a\vee b)\wedge(b\vee a)=((a\vee b)\wedge b)\vee((a\vee b)\wedge a)=b\vee a$ (by identities 2.4, 2.5 and Lemma 2.2). Because of Lemma 2.2, we may validly interchange the roles of a and b to obtain $(b\vee a)\wedge(a\vee b)=a\vee b$. But Proposition 2.4(i) implies that $(a\vee b)\wedge(b\vee a)=(a\vee b)\wedge((b\vee a)\wedge(a\vee b))$, and so $(a\vee b)\wedge(b\vee a)=(a\vee b)\wedge(a\vee b)=a\vee b$. Hence, $a\vee b=b\vee a$.

Now $a=(a\wedge b)\vee r(a,b)$ and so $b\vee a=b\vee((a\wedge b)\vee r(a,b))=b\vee(a\wedge b)\vee r(a,b)$. In addition $a\vee b=((a\wedge b)\vee r(a,b))\vee b=(a\wedge b)\vee b\vee r(a,b)$ (as $r(a,b)\wedge b=0)=b\vee r(a,b)$. Hence, $b\vee a=b\vee(r(a,b)\vee(a\wedge b))=(b\vee r(a,b))\vee(a\wedge b)=(a\vee b)\vee(a\wedge b)=a\vee b\vee(a\wedge b)$, as required.

We also need a well-known consequence of Theorem 3.4:

LEMMA 4.2. *In any skew lattice,* $a\vee b=b\vee a\vee b$ *is an identity.*

PROOF. $b\vee a\vee b=b\vee(a\vee b)=(b\wedge(a\vee b))\vee(a\vee b)$ (by identity (3.2), of Proposition 3.1) $=a\vee b$ (by identity (3.3)).

PROPOSITION 4.3. *The following conditions on any two fixed elements* a *and* b *of a* **BSA**-*algebra are equivalent.*

(i) $a\wedge b=b\wedge a$,

(ii) $a\vee b=b\vee a$,

(iii) $a\vee b=a\vee b\vee a$,

(iv) $b\vee a=b\vee a\vee b$,

(v) $a\vee(b\wedge a)=a$ *and* $b\vee(a\wedge b)=b$,

(vi) $(a\vee b)\wedge a=a$ *and* $(b\vee a)\wedge b=b$.

PROOF. (i)⇒(ii) By (i) and Lemma 4.1, $a\vee b=b\vee a\vee(b\wedge a)=b\vee a\vee(a\wedge b)=b\vee(a\vee(a\wedge b))=b\vee a$.

(ii)⇒(iii) By (iii) $a\vee b=a\vee b\vee a$. But $a\vee b\vee a=b\vee a$ due to Lemma 4.2. Hence (ii) holds.

(iv)$\Rightarrow$(ii) follows in a similar fashion.

Of course, (v) and (vi) are equivalent and it is easy to see that (ii) implies (v).

It remains to establish (v)$\Rightarrow$(i). Assume that $a\vee(b\wedge a)=a$ and $b\vee(a\wedge b)=b$. Hence, $(a\vee(b\wedge a))\wedge b=a\wedge b$, i.e. $(a\wedge b)\vee(b\wedge a)=a\wedge b$. Also, $(b\vee(a\wedge b))\wedge a=b\wedge a$, i.e. $(b\wedge a)\vee(a\wedge b)=b\wedge a$. By Lemma 4.2, $a\wedge b=(a\wedge b)\vee(b\wedge a)=((b\wedge a)\vee(a\wedge b))\vee \vee(b\wedge a)=(b\wedge a)\vee(b\wedge a)=(b\wedge a)$, as required.

From this we obtain

THEOREM 4.4. *Each of the following identities defines the same subvariety of the variety* **BSA**

(i) $a\wedge b=b\wedge a$

(ii) $a\vee b=b\vee a$

(iii) $a\vee b=a\vee b\vee a$

(iv) $a\vee(b\wedge a)==a$

(v) $(a\vee b)\wedge a=a$

(vi) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$

(vii) $a\vee(b\vee c)=(a\vee b)\vee(a\vee c)$.

PROOF. Because of Proposition 4.3, we can assume that (i)—(v) are equivalent.

Of course, (ii)$\Rightarrow$(vii). But (vii) implies (iii). Indeed, in (vii) put $c=0$, to obtain $a\vee b=a\vee b\vee a$.

Due to Proposition 3.3, (ii) implies (vi). But (vi) implies (iv). Indeed, put $c=0$ in (vi) to obtain $a=a\vee 0=(a\vee b)\wedge a$.

The subvariety defined by Theorem 4.4 is nothing more than the variety of generalized Boolean algebras (relatively complemented distributive lattices with 0), wherein the relative complement of $a\wedge b$ in the interval $[0, a]$ is taken as a fundamental operation, namely $r(a, b)$. In this context, we may also describe Boolean algebras.

COROLLARY 4.5. *Let* $(A;\ \wedge, \vee, r, 0)$ *be a* **BSA**-*algebra and suppose there exists an element* $1\in A$ *such that* $1\wedge a=a$ *for each* $a\in A$. *Then,* $1\vee a=a$ *for all* $a\in A$, *both* $\wedge$ *and* $\vee$ *are commutative, and* $(A;\ \wedge, \vee, ', 0, 1)$ *is a Boolean algebra, wherein* $a'=r(1, a)$ *and* $r(a, b)=a\wedge b'$ *for all* $a, b\in A$.

PROOF. Firstly, $1\vee a=1\vee(1\wedge a)=1$. Secondly, $a\vee(b\wedge a)=(1\wedge a)\vee(b\wedge a)= =(1\vee b)\wedge a=1\wedge a=a$. The remainder follows from Theorem 4.4.

As a contrast to (vii) of Theorem 4.4, we have the following positive consequence of Lemma 4.1.

PROROSITION 4.6. *Any skew lattice satisfies the identity* $(b\vee c)\vee a=(b\vee a)\vee(c\vee a)$.

PROOF. This is an easy consequence of Lemma 4.2.

Combining this proposition with (ii) of Proposition 2.4 and Proposition 3.3, we are led to the following important results.

THEOREM 4.7. *Let a be a fixed element of a* **BSA**-*algebra* $(A;\ \wedge, \vee, r, 0)$. *Then, the maps* $x\to x\vee a$ *and* $x\to x\wedge a$ *are endomorphisms of the algebra. Moreover, if*

Θ_a and Ψ_a denote the respective associated congruences, whereby for $x, y \in A$

$$x \equiv y(\Theta_a) \text{ if and only if } x \vee a = y \vee a, \text{ and}$$

$$x \equiv y(\Psi_a) \text{ if and only if } x \wedge a = y \wedge a,$$

then $\Theta_a \cap \Psi_a = \omega$, the smallest congruence on A and $\Theta_a \vee \Psi_a = \iota$, the largest congruence on A.

PROOF. It remains to prove our claims about the congruences.

Let $x, y \in A$ be arbitrary. Then, $x \equiv x \wedge a$ and $y \wedge a \equiv y(\Psi_a)$. Also, $(x \wedge a) \vee a = a = (y \wedge a) \vee a$ so that $x \wedge a \equiv y \wedge a(\Theta_a)$. It follows that $x \equiv y(\Theta_a \vee \Psi_a)$ and $\Theta_a \vee \Psi_a = \iota$.

Let $c, d \in A$ be such that $c \equiv d(\Theta_a \cap \Psi_a)$. In other words $c \vee a = d \vee a$ and $c \wedge a = d \wedge a$. Then $(c \vee a) \wedge r(c, a) = (d \vee a) \wedge r(d, a)$ and so $c \wedge r(c, a) = d \wedge r(c, a)$. But, $c \wedge r(c, a) = r(c, a)$ and so $r(c, a) = d \wedge r(c, a) = ((d \wedge a) \vee r(d, a)) \wedge r(c, a) = r(d, a) \wedge r(c, a)$. Similarly, $r(d, a) = r(c, a) \wedge r(d, a)$. Hence, $r(d, a) = r(c, a) \wedge r(d, a) = (r(d, a) \wedge r(c, a)) \wedge r(d, a) = r(d, a) \wedge r(c, a)$ (by Proposition 2.4(i)) $= r(c, a)$, i.e. $r(c, a) = r(d, a)$. Then, $c = (c \wedge a) \vee r(c, a) = (d \wedge a) \vee r(d, a) = d$. Hence, $\Theta_a \cap \Psi_a = \omega$.

We are now in a position to determine the subdirectly irreducible members of **BSA**. To do this, we introduce an important subclass of **BSA**-algebras and briefly study them.

Let B be any non-empty set and 0 be an element which is not in B. Put $A = B \cup \{0\}$ and endow A with the operations $\wedge, \vee$ and r defined as follows:

$$a \wedge b = \begin{cases} a & \text{if } b \neq 0 \\ 0 & \text{if } b = 0, \end{cases} \quad a \vee b = \begin{cases} b & \text{if } b \neq 0 \\ a & \text{if } b = 0, \end{cases} \quad r(a, b) = \begin{cases} 0 & \text{if } b \neq 0 \\ a & \text{if } b = 0. \end{cases}$$

Also, treat 0 as the constant associated with a nullary operation on A. Then, it is readily verifiable that $(A;\ \wedge, \vee, r, 0)$ is a **BSA**-algebra; it will be called the *smooth* **BSA**-algebra generated by the set $B = A \setminus \{0\}$.

LEMMA 4.8. *Let $(A;\ \wedge, \vee, r, 0)$ be a smooth* **BSA**-*algebra generated by the set $B = A \setminus \{0\}$. Then,*

i. *for $b_1, b_2 \in B$ with $b_1 \neq b_2$, the smallest congruence on A which identifies b_1 and b_2 is given by $x \equiv y(\Theta(b_1, b_2))(x, y \in A)$ if and only if $x = y$ or $\{x, y\} = \{b_1, b_2\}$; in other words, $\Theta(b_1, b_2)$ is the smallest equivalence relation on A which identifies b_1 and b_2;*

ii. *for $b \in B$, i.e. $b \neq 0$, the smallest congruence on A which identifies b and 0 is $\Theta(b, 0) = \iota$;*

iii. *the congruence lattice of A is isomorphic to the lattice of equivalence relations of the set $B = A \setminus \{0\}$, together with a new largest element $1 = \Theta(b, 0)$ for any $b \in B$, adjoined.*

PROOF. (i) An examination of the possibilities shows that whenever $x \equiv y(\Theta(b_1, b_2))$, where $\Theta(b_1, b_2)$ is as claimed, $x \wedge t \equiv y \wedge t$, $t \wedge x \equiv t \wedge y$, $x \vee t \equiv y \vee t$ and $t \vee x \equiv t \vee y(\Theta(b_1, b_2))$ for any $t \in A$. It follows that $\Theta(b_1, b_2)$ is a congruence, and it must have the desired properties.

(ii) Let $a \in A$ and $b \neq 0$. As $b \equiv 0(\Theta(b, 0))$, $a = a \wedge b \equiv a \wedge 0 = 0(\Theta(b, 0))$. It follows that $\Theta(b, 0) = \iota$.

(iii) is an immediate consequence of (i) and (ii).

Because of Theorem 1 of WHITMAN [12], any lattice is isomorphic to a sublattice of the lattice of equivalence relations on a suitable chosen set. Hence, part (iii) of Lemma 4.8 shows

COROLLARY 4.9. *The congruence lattices of all (smooth)* **BSA**-*algebras do not satisfy any particular lattice-identity.*

On the other hand, we have

THEOREM 4.10. *Up to isomorphism, the only subdirectly irreducible* **BSA**-*algebras are the two-element and three-element smooth algebras* **2** *and* **3**, *described in Section* 1.

PROOF. Suppose $(A;\ \wedge, \vee, r, 0)$ is subdirectly irreducible. Let $b\in A$ be such that $b\neq 0$. As $0\vee b=b=b\vee b$, $0\equiv b(\Theta_b)$ and so $\Theta_b\neq\omega$. Because of Theorem 4.7, $\Psi_b=\omega$. But for any $a\in A$, $a\wedge b\equiv a(\Psi_b)$ and consequently $a\wedge b=a$. By absorption, $a\vee b=(a\wedge b)\vee b=b$. Also, $r(a,b)\wedge b=0$, so $r(a,b)=0$. When $b=0$, the results $a\wedge b=0=b$, $a\vee b=a\vee 0=a$ and $a=(a\wedge b)\vee r(a,b)=0\vee r(a,b)=r(a,b)$ are forced. Hence, $(A;\ \wedge, \vee, r, 0)$ is smooth and generated by $B=A\setminus\{0\}$; B is not empty as $(A;\ \wedge, \vee, r, 0)$ is subdirectly irreducible and so A has at least two elements. But, if B possessed at least three distinct elements b_1, b_2 and b_3, Lemma 4.8 (i) would produce the impossibility $\Theta(b_1,b_2)\wedge\Theta(b_1,b_3)=\omega$, yet $\Theta(b_1,b_2)\neq\omega\neq\Theta(b_1,b_3)$. Thus, there are at most two non-zero elements; an easy computation shows that **2** and **3** are subdirectly irreducible.

COROLLARY 4.11. *The lattice of varieties of* **BSA**-*algebras is the three-chain. The only non-trivial variety of* **BSA**-*algebras is the variety of generalised Boolean algebras, which is described by any of the identities of Theorem* 4.4.

5. The occurrence of Boolean skew lattices

(1) *Rings with a central idempotent covers*

Let R be an associative ring. Let $E(R)$ be its generalized Boolean algebra of central idempotents. The order on $E(R)$ is given by: $e\leqq f$ $(e, f\in E(R))$ if and only if $e=ef$. The infimum and supremum of $e, f\in E(R)$ are $e\wedge f=ef$ and $e\vee f=e+f-ef$, respectively. Moreover, $r(e,f)=e-ef$ for any $e, f\in E(R)$. An element $e\in E(R)$ is called a central idempotent cover of $a\in R$ if $a=ae$ and e is the smallest element in $E(R)$ with this property, i.e. if $a=af$ for $f\in E(R)$ also, then $e\leqq f$. A ring R is a *ring with central idempotent covers*, or more briefly a *C-ring* if each element $a\in R$ possesses a central idempotent cover denoted by $C(a)$. This class of rings was briefly considered by the author in Section 4.1.2 of [3]. However, PENNING [8] seems to have been the first to have explicitly discussed these rings; they are his "minimal duplicator rings".

LEMMA 5.1. *In any C-ring R both* $C(C(a)b)=C(a)C(b)$ *and* $C(a+b-aC(b))=C(a)\vee C(b)(=C(a)+C(b)-C(a)C(b))$ *hold for any* $a, b\in R$.

PROOF. The first assertion is well known and vital to the study of C-rings; it is Lemma 2.13 of PENNING [8]. However, we will include a proof.

Firstly, if $e\in E(R)$ and $xe=0$ then $C(x)e=0$. Indeed, $x(C(x)-C(x)e)=x$ and $C(x)-C(x)e\in E(R)$ and so $C(x)\leqq C(x)-C(x)e$, i.e. $C(x)(C(x)-C(x)e)=$ $=C(x)-C(x)e$, and so $C(x)e=0$.

Secondly, let $x\in R$ and $e\in E(R)$ be arbitrary. Then, $x(e-eC(ex))=0$ and $e-C(ex)\in E(R)$. From the previous paragraph, we can infer that $C(x)(e-eC(ex))=0$. Hence, $C(x)e=eC(ex)$. But $e\in E(R)$ and $(ex)e=ex$ so $C(ex)\leqq e$. Hence, $C(x)e=$ $=eC(ex)=C(ex)$. Finally, for any $a, b\in R, C(a)\in E(R)$ and so $C(C(a)b)=$ $=C(C(a))C(b)=C(a)C(b)$, as required.

We now turn to the second identity. Now, $(a+b-aC(b))(C(a)\vee C(b))=$ $=a(C(a)\vee C(b))+b(C(a)\vee C(b))-aC(b)(C(a)\vee C(b))=a+b-aC(b)$ since $C(a)$, $C(b)\leqq C(a)\vee C(b)$. Hence, $C(a+b-aC(b))\leqq C(a)\vee C(b)$.

On the other hand, let $e\in E(R)$ be any central idempotent such that $(a+b-aC(b))e=a+b-aC(b)$. Multiply both sides by $C(b)$ and simplify to obtain $be=b$. Then, we must have $C(b)e=C(b)$. But $ae+be-aC(b)e=a+b-aC(b)$. Hence, $ae+b-aC(b)=a+b-aC(b)$ and so $a=ae$ and $C(a)\leqq e$. But we already know that $C(b)\leqq e$. Hence, $C(a)\vee C(b)\leqq e$. It now follows that $C(a+b-aC(b))=$ $=C(a)\vee C(b)$.

Using the first identity of Lemma 5.1 it is not hard to see that by introducing a new unary operation C, it is possible to turn the class of C-rings into a quasivariety of algebras. It is the quasivariety **CR** of algebras $(R;\ +,\ -,\ \cdot, 0, c)$ of type $(2, 1, 2, 0, 1)$, whose defining relations are

(i) the identities saying that the reduct $(R;\ +,\ -,\ \cdot, 0)$ is a ring,
(ii) the identities $C(a)b=bC(a)$, $aC(a)=a$, $C(C(a)b)=C(a)C(b)$, and
(iii) the quasi-identity (universal Horn sentence)

$$(a^2 = a)\&(ab = ba) \Rightarrow a = C(a).$$

We now turn to the relationship with Boolean skew algebra.

THEOREM 5.2. *Let $(R;\ +,\ -,\ \cdot, 0, C)$ be member of the quasivariety* **CR** *of C-rings. Then, the algebra $(R;\ \wedge, \vee, r, 0)$, whose operations are defined by:*

$$a\wedge b = aC(b),\ a\vee b = a+b-aC(b),\ r(a, b) = a-aC(b)$$

is a Boolean skew algebra. Moreover, the map $a\mapsto C(a)$ is a **BSA**-*retraction of $(R;\ \wedge, \vee, r, 0)$ onto the generalized Boolean algebra $(E(R);\ \wedge, \vee, r, 0)$ of central idempotents of the ring $(R;\ +,\ -,\ \cdot, 0)$.*

PROOF. Identity (2.1) holds as $aC(a)=a$. Identity (2.5) holds because $C(a)^2=$ $=C(a)$. Identities (2.2) and (2.4) are easy consequences of $C(C(a)b)=C(a)C(b)$. Identity (2.3) holds because of $C(a\vee b)=C(a)\vee C(b)$ (the second identity of Lemma 5.1). As $C(b)\ (a-aC(b))=0, C(b)C(a-aC(b))=0$. Hence $(a\wedge b)C(r(a, b))=$ $=aC(b)C(a-aC(b))=0$. It follows that $a=(a\wedge b)\vee r(a, b)$. Of course, $r(a, b)\wedge b=$ $=(a-aC(b))C(b)=0$, so the identity (2.6) also holds. The final assertion is clear.

Of course, Theorem 5.2 yields a faithful functor $\mathscr{F}$: **CR**$\to$**BSA** which preserves products. If Z_2 and Z_3 denote the fields with two and three elements, respectively, viewed as **CR**-algebras then the subdirectly irreducible Boolean skew algebras are $\mathbf{2}=F(Z_2)$ and $\mathbf{3}=F(Z_3)$. Hence, Theorem 4.10 says that each **BSA**-algebra is isomorphic to a **BSA**-subalgebra of $F(R)$ for some suitable C-ring R.

(2) *Quasiprimal algebras*

Let A be a non-empty set. Then, the functions $t\colon A^3 \to A$ and $q\colon A^4 \to A$, defined by

$$t(a, b, c) = \begin{cases} a & \text{if } a \neq b \\ c & \text{if } a = b, \end{cases} \quad q(a, b, c, d) = \begin{cases} c & \text{if } a = b \\ d & \text{if } a \neq b \end{cases}$$

are respectively called the *ternary* and *quaternary discriminators* on A. These functions are related by

$$t(a, b, c) = q(a, b, c, a) \text{ and } q(a, b, c, d) = t(t(a, b, c), t(a, b, d), d).$$

A universal algebra A is called *quasiprimal* if it is finite, not trivial and the ternary (quaternary) discriminator on the underlying set is a polynomial over A. A variety $\mathbf{V}$ of universal algebras is called quasiprimal if it is generated by a finite set of quasiprimal algebras such the ternary (quaternary) discriminator is represented by a common polynomial on each of these generators. Quasiprimal varieties abound; an excellent survey is contained in BULMAN-FLEMING and WERNER [2].

Let A be a set with at least two elements, 0 be any element of A and $B = A \setminus \{0\}$. In terms of the discriminator functions on A, we may define binary operations by

$$a \wedge b = q(0, b, 0, a) = t(0, t(0, b, a), a), \quad a \vee b = q(0, b, a, b) = t(b, 0, a)$$

and

$$r(a, b) = q(0, b, a, 0) = t(0, b, a).$$

Then, the resulting algebra $(A;\ \wedge, \vee, r, 0)$ is nothing more than the smooth BSA-algebra generated by B. From this it follows that there is a faithful functor from any quasiprimal variety into the variety of Boolean skew algebras.

It should be mentioned that on page 64 of [7], KEIMEL and WERNER define the derived operations $\wedge$, $\vee$, and r (their notation for $r(a, b)$ is $a \setminus b$) on any algebra in a quasiprimal variety. Thus, our remarks provide a characterization of their derived algebra.

On any non-empty set A it is possible to define other functions of interest besides t, q, $\wedge$, $\vee$, and r. Some authors, e.g. KEIMEL and WERNER [7] and BIGNALL [1] prefer to replace t by the function $d\colon A^3 \to A$, defined by

$$d(a, b, c) = \begin{cases} c & \text{if } b \neq c \\ a & \text{if } b = c. \end{cases}$$

It is a matter of choice; the two functions are related by $t(a, b, c) = d(c, b, a)$. In [1], BIGNALL introduced the function $/\colon A^2 \to A$ given by

$$a/b = d(0, a, b) = q(a, b, 0, b) = \begin{cases} 0 & \text{if } a = b \\ b & \text{if } a \neq b. \end{cases}$$

Here, as before, 0 is a fixed element of A. Of course, each of $\wedge$, $\vee$, r and $/$ are given in terms of d (and t and q). The importance of $/$ is that d can be put in terms of $\wedge$, $\vee$ and $/$. Bignall's equation is $d(a, b, c) = ((a \wedge b)/a) \vee (a \wedge c) \vee (b/c)$. Notice also that $r(a, b) = b/(a \vee b)$. In this way Bignall showed that the variety of algebras

(B: $d, 0$) of type (3, 0) generated by any algebra (A; $d, 0$) where A is an infinite set and d is the above discriminator on A is definitionally equivalent to a variety (his variety of quasi-Boolean skew lattices or **QBSL**'s) of algebras (B; $\vee, \wedge, /, 0$) of type (2, 2, 2, 0), where (B; $\wedge, \vee$) is a certain skew lattice. His work has important applications to quasiprimal varieties and is to be published elsewhere. His equational base for **QBSL** contains twelve identities which we will not state explicitly. As each **QBSL** yields a derived Boolean skew algebra, some of his axioms are redundant, for example it follows that there is no need to postulate the associativity of the $\vee$-operation. In this way, we obtain applications of our work, which needless to say was greatly inspired by my student Bignall.

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