UNSTEADY-STATE TEMPERATURE DISTRIBUTION IN A CONVECTING FIN OF CONSTANT AREA

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Abstract

A solution for the unsteady-state temperature distribution in a fin of constant area dissipating heat only by convection to an environment of constant temperature, is obtained. The partial differential equation is separated into an ordinary differential equation with position as the independent variable, and a partial differential equation with position and time as the independent variables. The problem is solved for either a step function in temperature or a step function in heat flow rate, for zero time, at one boundary while the other boundary is insulated. The initial condition is taken as an arbitrary constant. The unspecified boundary values (temperature or heat flow rate) are presented for both cases by utilizing dimensionless plots. Experimental verification is presented for the case of constant heat flow rate boundary condition.

Nomenclature

A	cross sectional area
Fo	Fourier number
h	convection coefficient
k	conduction coefficient
L	fin length
n	integer
Nu	Nusselt number
P	fin perimeter
q_0	heat flow rate at $x = 0$
$q(x, \tau)$	heat flow rate
t_0	base temperature ($x = 0$)
t_i	initial temperature
t_s	surrounding temperature
$t(x, \tau)$	temperature function

x	position variable
$X(x)$	functional in x
α	thermal diffusivity
$\Gamma(\tau)$	functional in τ
ζ	dimensionless position variable
$\theta(x, \tau)$	functional in x and τ
λ_n	eigenvalues
τ	time
$\phi(x, \tau)$	functional in x and τ

§ 1. Introduction

The subject of heat transfer from fins and extended surfaces has been studied analytically and experimentally for almost two centuries [1]. Ingenhouss [2] used fins to demonstrate the difference in thermal conductivities of various materials. J. B. Biot [3] and Despretz [4] formulated the mathematical model of the problem. Jacob [5] considered the fin problem in order to determine the corrections necessary for temperature measurements of thermocouples. Harper and Brown [6] dealt with the problem as encountered in the fin cooling of internal combustion engines. Harper and Brown also pioneered the analysis of variable geometry fins. Schmidt [7] extended the consideration of the variable geometry fin to the determination of the minimum mass profile fin. Many other investigators have also considered convecting fins.

Recently the subject of radiating fins has come under extensive study because of the interest in space. Shouman [8] has considered this problem in its general form and a biography on the subject is listed in this reference.

The above mentioned references deal with the steady state problem. However, the transient fin problem has as many practical applications as the steady-state problem. It is in this vein that this paper will deal with the transient convecting fin problem.

§ 2. Analysis

The governing partial differential equation is written as

$$\frac{\partial^2 t(x, \tau)}{\partial x^2} - \frac{hP}{kA}[t(x, \tau) - t_s] = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}(x, \tau). \tag{1}$$

The above equation is subject to the following boundary conditions:

Case I: For a step function in temperature at one boundary

B. C. 1. $t(0, \tau) = t_0$

2. $\frac{\partial t}{\partial x}(L, \tau) = 0$

I. C. $t(x, 0) = t_i$

Case II: For a step function in heat flow rate at one boundary:

B. C. 1. $\frac{\partial t}{\partial x}(0, \tau) = -q_0/kA$

2. $\frac{\partial t}{\partial x}(L, \tau) = 0$

I. C. $t(x, 0) = t_i$

Case I. We shall define:

$$\theta(x, \tau) = t(x, \tau) - t_0. \tag{2}$$

Utilizing (2), the boundary conditions are made homogeneous, giving:

$$\frac{\partial^2\theta(x, \tau)}{\partial x^2} - \frac{hP}{kA}\theta(x, \tau) - \frac{hP}{kA}(t_0 - t_s) = \frac{1}{\alpha}\frac{\partial\theta}{\partial\tau}(x, \tau) \tag{3}$$

with B. C. 1. $\theta(0, \tau) = 0$,

2. $\frac{\partial\theta}{\partial x}(L, \tau) = 0$, and

I. C. $\theta(x, 0) = t_i - t_0$

Now let

$$\theta(x, \tau) = X(x) + \phi(x, \tau). \tag{4}$$

Equation (4) yields a steady-state problem, and a transient problem. The steady-state problem and solution is

$$\frac{d^2X(x)}{dx^2} - \frac{hP}{kA}X(x) = \frac{hP}{kA}(t_0 - t_s) \tag{5}$$

B. C. 1. $X(0) = 0$

2. $\frac{dX}{dx}(L) = 0$

$$X(x) = (t_0 - t_s)\left[\frac{\cosh(hP/kA)^{\frac{1}{2}}(L - x)}{\cosh(hP/kA)^{\frac{1}{2}}L}\right] - 1. \tag{6}$$

The transient problem and solution by separation of variables is

$$\frac{\partial^2 \phi(x, \tau)}{\partial x^2} - \frac{hP}{kA} \phi(x, \tau) = \frac{1}{\alpha} \frac{\partial \phi}{\partial \tau} (x, \tau) \tag{7}$$

B. C. 1. $\phi(0, \tau) = 0$,

2. $\dfrac{\partial \phi}{\partial x} (L, \tau) = 0$,

I. C. $\phi(x, 0) = t_i - t_0 - X(x)$

$$\phi(x, \tau) = \sum_{n=1}^{\infty} B_n \exp\left[-\alpha\left(\frac{hP}{kA} + \lambda_n^2\right)\tau\right] \sin \lambda_n x \tag{8}$$

where the eigenequation is

$$\cos \lambda_n L = 0$$

with the eigenvalues $\lambda_n = (2n - 1)\, \pi/2L$, $n = 1, 2, 3, \ldots$ and

$$B_n = \frac{\int_0^L [t_i - t_0 - X(x)] \sin \lambda_n x \, \mathrm{d}x}{\int_0^L \sin^2 \lambda_n x \, \mathrm{d}x}$$

which becomes

$$B_n = \frac{2}{L} \left[\frac{(t_i - t_s)}{\lambda_n} - \frac{(t_0 - t_s)\, \lambda_n}{\left(\dfrac{hP}{kA} + \lambda_n^2\right)} \right]. \tag{9}$$

Now combining (6), (8) and (9) according to (2) and (4) gives

$$t(x, \tau) = t_0 + (t_0 - t_s) \left[\frac{\cosh(hP/kA)^{\frac{1}{2}} (L - x)}{\cosh(hP/kA)^{\frac{1}{2}} L} - 1 \right] +$$

$$+ \frac{2}{L} \sum_{n=1}^{\infty} \left\{ \left[\frac{(t_i - t_s)}{\lambda_n} - \frac{(t_0 - t_s)\, \lambda_n}{\left(\dfrac{hP}{kA} + \lambda_n^2\right)} \right] \times \right.$$

$$\left. \times \exp\left[-\alpha\left(\frac{hP}{kA} + \lambda_n^2\right)\tau\right] \sin \lambda_n x \right\} \tag{10}$$

which can be nondimensionalized to

$$\frac{t(\zeta, Fo) - t_s}{t_0 - t_s} = \frac{\cosh \sqrt{Nu}\,(1-\zeta)}{\cosh \sqrt{Nu}} + $$
$$+ 2 \sum_{n=1}^{\infty} \left\{ \left[\frac{2}{(2n-1)\pi} \left(1 - \frac{t_0 - t_i}{t_0 - t_s}\right) - \right.\right.$$
$$\left. - \frac{\frac{(2n-1)\pi}{2}}{Nu + \frac{(2n-1)^2 \pi^2}{4}} \right] \exp\left[-Fo\left(Nu + \frac{(2n-1)^2\pi^2}{4}\right)\right] \times$$
$$\left. \times \sin \frac{(2n-1)\pi}{2} \zeta \right\} \tag{11}$$

where $Nu = (hP/k)(L^2/A)$, $\zeta = x/L$ and $Fo = \alpha\tau/L^2$.

The dimensionless heat flow rate can be expressed as

$$\frac{q(\zeta, Fo)}{\frac{kA}{L}(t_0 - t_s)} = \frac{\sqrt{Nu}\sinh \sqrt{Nu}\,(1-\zeta)}{\cosh \sqrt{Nu}} + $$
$$+ 2 \sum_{n=1}^{\infty} \left\{ \left[\left(1 - \frac{t_0 - t_i}{t_0 - t_s}\right) - \right.\right.$$
$$\left. - \left(\frac{\frac{(2n-1)^2\pi^2}{4}}{Nu + \frac{(2n-1)^2 \pi^2}{4}} \right) \right] \exp\left[-Fo\left(Nu + \frac{(2n-1)^2\pi^2}{4}\right)\right] \times$$
$$\left. \times \cos \frac{(2n-1)}{2} \pi\zeta \right\}. \tag{12}$$

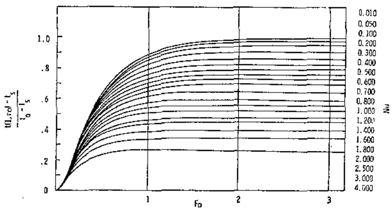

Fig. 1

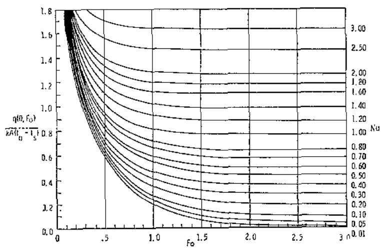

Fig. 2

Fig. 1. Graphical representation of (11) for $\zeta = 1$ and $t_i = t_s$.
Fig. 2. Graphical representation of (12) for $\zeta = 0$ and $t_i = t_s$.

Graphs giving

$$\frac{t(1, Fo) - t_s}{t_0 - t_s} \quad \text{and} \quad \frac{q(0, Fo)}{kA(t_0 - t_s)}$$

for $t_i = t_s$ are shown by Figs. 1 and 2 respectively. The results obtained using (10) were compared with the answers obtained by a finite difference numerical solution. For the finite difference solution, 40 nodes were used with time intervals of three seconds. The maximum deviation between the two results was 1.5% for typical values of the parameters.

Case II. For this situation we shall define:

$$t(x, \tau) = X(x) + \theta(x, \tau). \tag{13}$$

Equation (13) is used to separate the problem into a steady-state and transient part. The steady-state problem and solution is

$$\frac{d^2X(x)}{dx^2} - \frac{hP}{kA} X(x) = -\left(\frac{hP}{kA}\right) t_s \tag{14}$$

B. C. 1. $\frac{dX}{dx}(0, \tau) = -q_0/kA$

2. $\frac{dX}{dx}(L, \tau) = 0$

$$X(x) = \left[\left(\frac{q_0}{kA} \Big/ \sqrt{\frac{hP}{kA}}\right) \frac{\cosh(hP/kA)^{\frac{1}{2}}(L - x)}{\sinh(hP/kA)^{\frac{1}{2}} L} + t_s\right]. \tag{15}$$

The transient problem and solution by separation of variables is

$$\frac{\partial^2\theta(x,\tau)}{\partial x^2} - \frac{hP}{kA}\,\theta(x,\tau) = \frac{1}{\alpha}\,\frac{\partial\theta}{\partial\tau}\,(x,\tau) \tag{16}$$

B. C. 1. $\dfrac{\partial\theta}{\partial x}\,(0,\tau) = 0$

2. $\dfrac{\partial\theta}{\partial x}\,(L,\tau) = 0$

I. C. $\theta(x,0) = t_i - X(x)$

(Note: The validity of the division used for B. C. 1 between (14) and (16) can be shown mathematically by using the substitution

$$\psi(x,\tau) = t(x,\tau) - \frac{q_0}{kA}\,\frac{(L-x)^2}{2L}$$

to make the B. C. presented in Case II homogeneous. The final result will be exactly the same.)

$$\theta(x,\tau) = \sum_0^\infty D_n \exp\left[-\alpha\left(\frac{hP}{kA} + \lambda_n^2\right)\tau\right]\cos\lambda_n x \tag{17}$$

where the eigenequation is

$$\sin\lambda_n L = 0$$

with the eigenvalues

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 0, 1, 2, \ldots .$$

and

$$D_0 = \frac{1}{L}\int_0^L [t_i - X(x)]\,dx = (t_i - t_s) - \frac{q_0/kA}{L\left(\dfrac{hP}{kA}\right)} \tag{18}$$

$$\underset{(n\neq 0)}{D_n} = \frac{\int_0^L [t_i - X(x)]\cos\lambda_n x\,dx}{\int_0^L \cos^2\lambda_n x\,dx} = \frac{-\left(\dfrac{2}{L}\right)q_0/kA}{\left(\dfrac{hP}{kA} + \lambda_n^2\right)}.$$

Now combining (15), (17) and (18) according to (13) gives

$$t(x,\tau) = \left[\left(\frac{q_0}{kA}\Big/\sqrt{\frac{hP}{kA}}\right)\frac{\cosh(hP/kA)^{\frac{1}{2}}\,(L-x)}{\sinh(hP/kA)^{\frac{1}{2}}\,L} + t_s\right] +$$

$$+\left[(t_i - t_s) - \frac{\dfrac{q_0}{kA}}{\dfrac{hP}{kA}L}\right]\exp\left[-\alpha\frac{hP}{kA}\tau\right] -$$

$$-\frac{2}{L}\left(\frac{q_0}{kA}\right)\sum_{n=1}^{\infty}\frac{\exp\left[-\alpha\left(\dfrac{hP}{kA}+\lambda_n^2\right)\tau\right]\cos(\lambda_n x)}{\dfrac{hP}{kA}+\lambda_n^2}$$

which can be nondimensionalized to

$$\frac{t(\zeta, Fo) - t_s}{\dfrac{q_0}{kA}\Big/\dfrac{hP}{kA}L} = \sqrt{Nu}\,\frac{\cosh\sqrt{Nu}\,(1-\zeta)}{\sinh\sqrt{Nu}} +$$

$$+\left[\frac{t_i - t_s}{\left(\dfrac{q_0}{kA}\right)\Big/\left(\dfrac{hP}{kA}L\right)} - 1\right]\exp[-Nu\,Fo] -$$

$$-\,2Nu\sum_{n=1}^{\infty}\frac{\exp[-Fo(Nu + n^2\pi^2)]}{Nu + n^2\pi^2}\cos(n\pi\zeta) \qquad (20)$$

where dimensionless parameters are as defined for (11). The heat flow can be expressed as

$$\frac{q(\zeta, Fo)}{q_0} = \frac{\sinh\sqrt{Nu}\,(1-\zeta)}{\sinh\sqrt{Nu}} -$$

$$-\,2\pi\sum_{n=1}^{\infty} n\,\frac{\exp[-Fo(Nu + n^2\pi^2)]\sin(n\pi\zeta)}{Nu + n^2\pi^2}. \qquad (21)$$

Graphs giving

$$\frac{t(0, Fo) - t_s}{(q_0/hPL)} \quad \text{and} \quad \frac{t(1, Fo) - t_s}{(q_0/hPL)}$$

for $t_i = t_s$ are shown by Figs. 3 and 4 respectively.

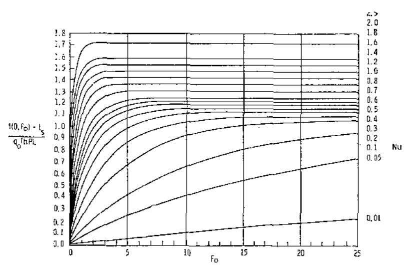

Fig. 3. Graphical representation of (20) at $\zeta = 0$ and $t_i = t_s$.

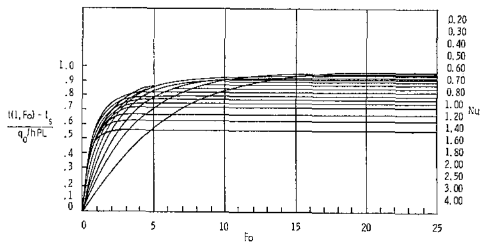

Fig. 4. Graphical representation of (20) at $\zeta = 1$ and $t_i = t_s$.

The results obtained using (19) were compared with the answers obtained by a finite difference numerical solution. For the finite difference solution 40 nodes were used with time intervals of 0.6 seconds. The maximum deviation between the two results was 2% for typical values of the parameters.

§ 3. Experimental verification

In order to gain confidence in the degree of agreement between the analytical solution and the actual problem, it was decided to carry out experimental verification of the results. Since it is rather difficult to simulate experimentally a step function in temperature, it was decided to attempt to simulate experimentally the case of a step function in heat flow rate.

A copper rod, 2 feet long and one half inch in diameter, was insulated at one end. A heating tape was wrapped around the rod at the other end for a length of eight inches. This heater will be referred to here as the primary heater. Glass wool insulation $\frac{3}{4}$ inch thick was wrapped around the primary heater. A secondary heater made of the same heating tape was wrapped on top of the insu-

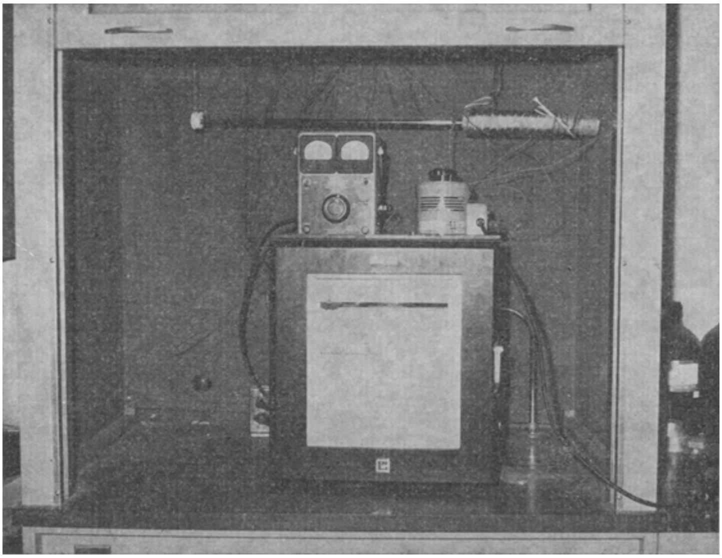

Fig. 5. Experimental set-up.

lation. Both the primary and secondary heaters were supplied independently from separate variable power supplies. Copper-Constantan thermocouples made of No. 30 gage wires were attached to the rod at three inch intervals. A hole $\frac{3}{16}$ inch deep was drilled to accommodate the thermocouples and aluminum epoxy was used to mount the thermocouples in place. The thermocouples were connected to a Leeds and Northrup multi-channel continuous recorder. A photograph of the experimental setup is shown in Fig. 5.

A fixed amount of power was supplied to the primary heater while the secondary heater power supply was manually regulated as a function of time. The purpose of the secondary heater is to compensate for the energy absorbed by the insulation and the goal is to approach as closely as possible a constant heat flow rate into the rod at $x = 0$. A plot of the heat flux ratio as a function of time for $x = 0$ with and without the use of the secondary heater is shown in Fig. 6. It can be seen from the figure that constant heat flow rate case was not completely simulated. However, it can be seen that the use of the secondary heater leads to a better approximation to the constant heat flow rate case than without the use of the secondary heater. Fig. 7 shows the variation with time of the

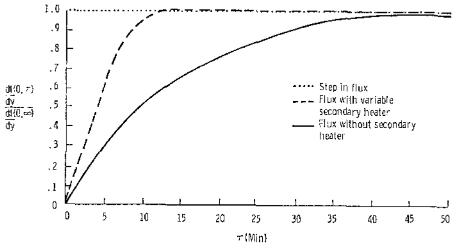

Fig. 6

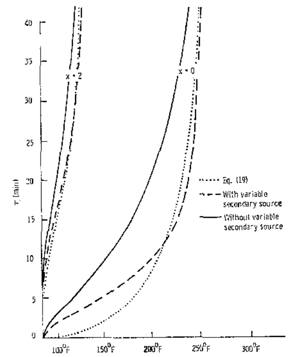

Fig. 7

Fig. 6. Comparison of experimental flux with and without variable secondary heater to step function.

Fig. 7. Comparison of temperature histories at $x = 0$ and $x = 2$ between (19) and experimental runs with and without variable secondary source

$$t_s = t_i = 81°\text{F}.$$

temperatures at both ends of the rod. It can be seen from this figure that as the experimental heat flow rate approaches the step function, the experimental temperature histories approach those predicted analytically by (19).

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