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ON THE SUM OF DISTANCES BETWEEN n POINTS ON A SPHERE

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§1. Introduction

If $p_1, p_2, \ldots, p_n$ are variable points on a unit sphere, let $S(n)$ denote the maximum value of the function $\sum_{i<j} |p_i - p_j|$. The circle and the Hilbert sphere are nicely treated by FEJES TÓTH [3]; the respective extremal configurations are the regular polygons and the regular inscribed simplices. The respective $S(n)$ are $n \cot\left(\frac{\pi}{2n}\right) = \frac{2}{\pi} n^2 - \frac{\pi}{6} + O\left(\frac{1}{n^2}\right)$ and $n\sqrt{\frac{1}{2} n(n-1)}$. The exact determination of $S(n)$ for higher dimensional Euclidean spheres seems very difficult. Fejes Tóth conjectured that $S(n) < \alpha n^2$ where α is the "constant of uniform distribution" for the sphere.

The paper of BJÖRCK [2], employing the elegant methods of potential theory, proves the conjecture. Let μ vary over all positive Borel measures of mass n on the sphere. Björck's method shows that $2\alpha n^2$ is the maximum value attained by the energy integral $\iint |p-q| \, d\mu(p) d\mu(q)$; the uniform distribution uniquely maximizes.

Various results of STOLARSKY and the author [1] show that $\alpha n(n-1) < S(n) < \alpha n^2 - \beta n^{-\gamma}$, $\beta > 0$, $\gamma > 1$. The right inequality is obtained by a difficult method using spherical harmonics. In this article we introduce new methods for sharpening this inequality. We concentrate on the ordinary 2-sphere although our methods (with more computation) would apply to spheres of all dimensions. In light of Björck's results, $S(n)$ may be viewed as a measure of how nearly n points may be uniformly distributed on the sphere.

PROPOSITION 1.1. *For the 2-sphere we have*

$$\frac{2}{3} n^2 - 10 n^{\frac{1}{2}} < S(n) < \frac{2}{3} n^2 - \frac{1}{2}. \tag{1}$$

We will comment on the accuracy of our method at the end.

§2. Proof of Proposition 1.1

LEMMA 2.1. *Let $p_1 \leqq p_2 \leqq \ldots \leqq p_n$ and $q_1 \leqq q_2 \leqq \ldots \leqq q_n$ be any two sets, each consisting of n real numbers. Then*

$$(2) \qquad \sum_{i,j} |p_i - q_j| \geqq \sum_{i<j} |p_i - p_j| + \sum_{i<j} |q_i - q_j| + \sum_{i=1}^{n} |p_i - q_i| .$$

PROOF. First we prove (2) assuming that each p_i and q_i is an integer. We use double induction, on n and on the quantity $\sum_{i=1}^{n} |p_i - q_i|$.

If $\sum_{i=1}^{n} |p_i - q_i| = 0$, the two sets are identical and (2) is an equality. If $n = 1$, (2) is certainly an equality. Suppose (2) is true if $n < k$, and if $n = k$, providing $\sum_{i=1}^{k} |p_i - q_i| < K$. Let $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ satisfy $\sum_{i=1}^{k} |p_i - q_i| = K$. If $p_k = q_k$, terms involving the index k make the same contribution to both sides of (2). Hence we may suppress these points and apply an induction hypothesis for $n = k - 1$ to assert the validity of (2). Otherwise, assume without loss of generality that $p_k > q_k$. Let r be the least value of i for which $p_i = p_k$; $p_r \geqq q_i + 1$ for each i. Replace p_r by $p_r' = p_r - 1$ so that $\Sigma' |p_i - q_i| = K - 1$. In this modified situation (2) is valid by an induction hypothesis. If we replace the modified situation by the original, note that we add $k(= n)$ to the left side of (2) while we add at most k to the right. This completes the double induction.

If the p_i and q_i are rational numbers, we may multiply each by the least common multiple of all denominators, and then apply the result about integers. The general result follows at once from the fact that real numbers may be approximated by rationals.

For our purposes a somewhat weaker result will be more tractable.

COROLLARY 2.1. *Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be as before. Then*

$$(3) \qquad \sum_{i,j} |p_i - q_j| \geqq \sum_{i<j} |p_i - p_j| + \sum_{i<j} |q_i - q_j| + \sum_{i=1}^{n} \varrho(p_i) .$$

Here $\varrho(p_i) = \min_j |p_i - q_j|$.

PROOF. Since $\varrho(p_i) \leqq |p_i - q_i|$, the result follows. The most important thing about (3) is that it is valid when the numbers are not indexed according to size.

If $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are arbitrary points in a Euclidean space, the work of I. J. SCHOENBERG [4] shows that the quadratic form

$$\sum_{i<j} |p_i - p_j| x_i y_j + \sum_{i<j} |q_i - q_j| y_i y_j + \sum_{i,j} |p_i - q_j| x_i y_j$$

is negative semidefinite on the hyperplane $\Sigma x_i + \Sigma y_i = 0$. Setting each $x_i = 1$

and $y_i = -1$ gives

$$\sum_{i,j} |p_i - q_j| \geqq \sum_{i<j} |p_i - p_j| + \sum_{i<j} |q_i - q_j|.$$

The inequalities (2), (3) are sharpened one dimensional versions. We remark that the higher dimensional inequalities can be sharpened by using these results together with (6) below.

LEMMA 2.2. *Let $q_1, q_2, \ldots, q_n$ be numbers in the interval $[-1, 1]$. Let $\rho(x) = \min_i |x - q_i|$. Then*

$$\int_{-1}^{1} \rho(x)dx \geqq \frac{1}{n}. \tag{4}$$

PROOF. For convenience suppose $q_i \leqq q_{i+1}$. We use a simple convexity argument. If $1 < i < n$ and $q_i \neq \frac{1}{2}(q_{i-1} + q_{i+1})$, then the integral can be decreased. Thus, in a minimal situation, the q_i lie in arithmetic progression. We may assume that $q_1 = -q_n$ or else the integral may be reduced by translating the q_i. Subject to the above conditions, the integral equals $(1 - q_n)^2 + q_n^2/(n-1)$. This function has a minimal value $1/n$, attained for $q_n = 1 - \frac{1}{n}$.

In the proof of the right inequality of (1) we use Haar integrals over the special orthogonal group SO(3) acting on the 2-sphere. We assume that the measure is normalized. The integrals will be evaluated by means of the following lemma which we state without proof.

LEMMA 2.3. *Let f be a real integrable function on the 2-sphere, and p_0 be a distinguished point on the sphere. Let F be defined on* SO(3) *by $F(\tau) = f(\tau p_0)$. Then*

$$\overline{\int} F(\tau)d\tau = \frac{1}{4\pi} \int f(p)d\sigma(p). \tag{5}$$

As a notational aid, a barred integral will be a Haar integral. Otherwise, the integral will be a surface integral over the sphere.

We give a useful application of the lemma. Let q_1 and q_2 be points in E^3. Then

$$|q_1 - q_2| = \frac{1}{2\pi} \int |(q_1 - q_2) \cdot p| \, d\sigma(p) = 2 \overline{\int} |(q_1 - q_2) \cdot \tau(p_0)| \, d\tau. \tag{6}$$

Let $p_1, p_2, \ldots, p_n$ be points on the 2-sphere such that $\sum_{i<j} |p_i - p_j| = S(n)$. We show that

$$\overline{\int} \sum_{i,j} |p_i - \tau_1(p_j)| \, d\tau_1 = \frac{4}{3} n^2. \tag{7}$$

Note that the integral equals $n^2 \overline{\int} | p_0 - \tau_1(p_0) | \, d\tau_1$; applying Lemma 2.3 we obtain $\frac{n^2}{4\pi} \int | p_0 - p | \, d\sigma(p)$. The surface integral is $\frac{16}{3}\pi$.

We apply (6) to each summand in (7) and the integral (7) becomes

$$2 \overline{\int}\,\overline{\int} \Big(\sum_{i,j} | (p_i - \tau_1(p_j)) \cdot \tau_2(p_0) | \Big) d\tau_2 d\tau_1 . \tag{8}$$

The sum in (8) may be interpreted as the sum of the distances between two sets of n colinear points. These sets may be realized by orthogonally projecting the points $p_1, p_2, \ldots, p_n$ and $\tau_1(p_1), \tau_1(p_2), \ldots, \tau_1(p_n)$ onto a line having the direction of $\tau_2(p_0)$. We apply Lemma 2.1 to assert that (8) equals

$$2 \overline{\int}\,\overline{\int} \Big\{ \sum_{i<j} | (p_i - p_j) \cdot \tau_2(p_0) | + \sum_{i<j} | \big(\tau_1(p_i) - \tau_1(p_j)\big) \cdot \tau_2(p_0) | + E(\tau_1, \tau_2) \Big\} d\tau_2 d\tau_1 \tag{9}$$

where $E(\tau_1, \tau_2) \geqq 0$. The (double) integral of each of the sums is $\frac{1}{2} S(n)$. We will estimate $2 \overline{\int}\,\overline{\int} E d\tau_1 d\tau_2$, the order of integration being reversed. Let $\rho_i(\tau_1, \tau_2) = \inf_j | \big(p_j - \tau_1(p_i)\big) \cdot \tau_2(p_0) |$. In terms of the geometric realization, $\rho_i(\tau_1, \tau_2)$ is the same function that occurs in Corollary 2.1. Therefore,

$$\overline{\int}\,\overline{\int} E d\tau_1 d\tau_2 \geqq \overline{\int}\,\overline{\int} \sum_{i=1}^{n} \rho_i d\tau_1 d\tau_2 = \overline{\int} \Big\{ \sum_{i=1}^{n} \overline{\int} \rho_i d\tau_1 \Big\} d\tau_2 . \tag{10}$$

We will show that each integral in the inner sum is at least $\frac{1}{2n}$. We may assume without loss of generality that $\tau_2(p_0) = (1, 0, 0)$ so that the points may be viewed as being projected onto the interval $[-1, 1]$ as real numbers $p_1', p_2', \ldots, p_n'$. Using Lemmas 2.3 and 2.2, we may write

$$\overline{\int} \rho_i d\tau_1 = \frac{1}{4\pi} \int \rho_i(p) d\sigma(p) = \frac{1}{4\pi} \int_{-1}^{1} 2\pi \rho(x) dx \geqq \frac{1}{2n} . \tag{11}$$

Here $\rho_i(p) = \rho_i\big(\tau_2, \tau_1(p)\big)$ and $\rho(x) = \min_i | x - p_i' |$. From (10) and (11) it follows that

$$2 \overline{\int}\,\overline{\int} E(\tau_1, \tau_2) d\tau_1 d\tau_2 > 2 \overline{\int} \frac{1}{2} d\tau_2 = 1 . \tag{12}$$

It is clear that the integral in (11) exceeds $\frac{1}{2n}$ for almost all τ_2. This justifies the strict inequality in (12). Consideration of (7), (9) and (12) gives

$$\frac{4}{3} n^2 > 2S(n) + 1 . \tag{13}$$

The right inequality of Proposition 1.1 follows at once.

Next we turn our attention to the left inequality in (1). Our method is surprisingly simple, but gives much better results than we had been able to obtain previously.

LEMMA 2.4. *Let K be a compact subset of a Euclidean space, and let μ be a Borel measure for which $\mu(K) = n$. Let $A_1, \ldots, A_n$ be Borel subsets of K such that $\mu(A_i \cap A_j) = \delta_{ij}$. Then*

$$2S(n, K) \geqq \int_K \int_K | p - q | \, d\mu(p) d\mu(q) - \sum_{i=1}^{n} D_i \tag{14}$$

where D_i is the diameter of A_i.

PROOF. Define f on $A_1 \times \ldots \times A_n$ by $f(p_1, \ldots, p_n) = \sum_{i \neq j} | p_i - p_j |$. Clearly $f \leqq 2S(n)$. Since $\mu(A_i) = 1$ for all i,

$$2S(n) \geqq \int_{A_1} \ldots \int_{A_n} f d\mu(p_1) \ldots d\mu(p_n) = \sum_{i \neq j} \int_{A_i} \int_{A_j} | p_i - p_j | \, d\mu(p_i) d\mu(p_j) = \tag{15}$$

$$= \int_K \int_K | p - q | \, d\mu(p) d\mu(q) - \sum_{i=1}^{n} \int_{A_i} \int_{A_i} | p - q | \, d\mu(p) d\mu(q).$$

The result follows immediately.

If one knows which μ maximizes the energy integral (as we do for spheres), we are faced with a dissection problem. For the circle we choose the A_i in the obvious manner to get $D_i < \frac{2\pi}{n}$. This gives the estimate $S(n) > \frac{2}{\pi} n^2 - \pi$, which is not bad. For the 2-sphere we claim that we can choose the A_i so that $D_i = O\left(n^{-\frac{1}{2}}\right)$, although we have no elegant method for doing this. For simplicity suppose $n = 6m^2$. We begin with a spherical cube, and consider one of its faces. Using $m - 1$ great circles from the pencil determined by two opposite edges we can cut the face into m slices of equal area. Each slice can be cut into m quadrilaterals of equal area using great circles in the pencil in the other pair of opposite edges of the face. The diameters of the quadralaterals are of the right magnitude. We suppress the details.

§3. Final remarks

How accurate is the right inequality in (1)? If the points are close to being uniformly distributed, then the distribution of the orthogonal projection of them onto a line does not vary much the direction of the line. In Lemma 2.1 if the p_i and q_i are distributed in much the same manner, then (2) is close to equality. It is difficult to judge what is lost when $| p_i - q_i |$ is replaced by $\rho(p_i)$. We can check this for the circle, and it is not too much. We conjecture that for a Euclidean sphere $\alpha n^2 - S(n) = O(1)$. If it were known that $S(n)$ is "analytic in n", as it is for the circle, then the left inequality of (1) settles the issue. The conjecture is seen to be false for the Hilbert sphere.

We remark that in the case of the 2-sphere $S(8)$ is not attained by the inscribed cube. If we rotate the four points of one face through an angle of 45^0 in the plane of that face, the distance sum increases.

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