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FUNCTIONS WITH MEASURABLE DIFFERENCES

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1. Let $\mathbf{R}$ denote the set of real numbers and let F be a class of real valued functions defined on $\mathbf{R}$. We say that F has the *difference property*, provided that every function $f: \mathbf{R}\to\mathbf{R}$ for which $f(x+h)-f(x)\in F$ holds for every h, can be written in the form $f=g+H$ where $g\in F$ and H is additive, that is H satisfies the functional equation $H(x+y)=H(x)+H(y)$. The notion of difference property was introduced by N. G. de Bruijn who proved that a series of important classes have the difference property (e.g. the classes of continuous, differentiable, analytic, absolute continuous, Riemann-integrable functions, respectively; see [1] and [2]). The results of de Bruijn have been extended and generalized in various ways, see [3], [4], [5], [6], [11], [13].

However, the following example given by Erdős shows that the class of Lebesgue measurable functions does not have the difference property if we assume the continuum hypothesis. Indeed, the continuum hypothesis implies the existence of a bounded and non-measurable function $S: \mathbf{R}\to\mathbf{R}$ such that for every $h\in\mathbf{R}$, $S(x+h)-S(x)=0$ holds for all but countably many values of x (see [16], p. 27). Now S is not of the form $g+H$ where g is measurable and H is additive because otherwise $H=S-g$ would be bounded on a set of positive measure. By a theorem of Ostrowski, this implies that H is linear (see [15] or [12]) and thus S is measurable, a contradiction.

It was conjectured by Erdős that every function $f: \mathbf{R}\to\mathbf{R}$ for which $f(x+h)-f(x)$ is measurable for every h, is of the form $f=g+H+S$, where g is measurable, H is additive and S has the property that, for every h, $S(x+h)-S(x)=0$ for almost every x.

We say that a class F has the *weak difference property* if every function $f: \mathbf{R}\to\mathbf{R}$ for which $f(x+h)-f(x)\in F$ holds for every h admits a decomposition $f=g+H+S$ with $g\in F$, H additive, and S satisfying the condition that for every h, $S(x+h)-S(x)=0$ holds for a.e. x (see [6]). Let L denote the class of Lebesgue measurable functions defined on $\mathbf{R}$. Then Erdős' conjecture can be formulated as follows: the class L has the weak difference property. The main purpose of this paper is to prove this conjecture (Theorem 3).

We remark that the weak difference property has been established for the classes $L_p(0, 1)$ if $p\geqq 1$ (see [4] and for a generalization, [13]). F. W. Carroll raised the question whether this is true for $0<p<1$. We give an affirmative answer in Theorem 4.

We prove Theorems 3 and 4 in Section 2, making use of the preparatory results of Lemmas 1, 2 and Theorem 2. In Section 3 we give some applications of Theorem 3. These applications will be based on Theorem 5 which states that the classes of measurable functions of one and two variables have a "double difference property" in the following sense.

Let F_1 be a class of real functions defined on $\mathbf{R}$ and let F_2 be a class of real functions defined on $\mathbf{R}^2$. We say that the pair (F_1, F_2) has the *double difference property* if whenever $f(x+y)-f(x)-f(y)\in F_2$ holds for a function $f\colon \mathbf{R}\to\mathbf{R}$ then f is of the form $f=g+H$, where $g\in F_1$ and H is additive.

For example, the pair of classes of bounded functions of one and two variables, respectively has the double difference property (see [1], Theorem 1.2, p. 196). First we prove that the same is true for the classes of functions whose limit equals zero at the origin. More precisely we prove

THEOREM 1. *If f is defined on $\mathbf{R}$ and*

$$\lim_{\substack{x\to 0\\ y\to 0}} \{f(x+y)-f(x)-f(y)\} = 0$$

holds then $f=g+H$ where H is additive and $\lim\limits_{x\to 0} g(x)=g(0)=0$.

PROOF. Let $\delta>0$ be such that

$$(1)\qquad |f(x+y)-f(x)-f(y)| \leqq \max\big(1, |f(0)|\big) \stackrel{\text{def}}{=} A \quad \text{for every} \quad |x|, |y| \leqq \delta.$$

We put $f^*(x)\stackrel{\text{def}}{=}f(x-k\delta)$ if $k\delta\leqq x<(k+1)\delta$ $(k=0, \pm 1, \dots)$. We show that $F(x,y)\stackrel{\text{def}}{=}|f^*(x+y)-f^*(x)-f^*(y)|$ is bounded on $\mathbf{R}^2$. Let $x, y\in\mathbf{R}$ be arbitrary and let $k=\left[\frac{x}{\delta}\right]$, $n=\left[\frac{y}{\delta}\right]$. If $\left[\frac{x+y}{\delta}\right]=k+n$ then we have

$$F(x,y) = \big|f\big(x+y-(k+n)\delta\big)-f(x-k\delta)-f(y-n\delta)\big| \leqq A$$

by (1). If $\left[\frac{x+y}{\delta}\right]=k+n+1$ then we have

$$F(x,y) = \big|f\big(x+y-(k+n+1)\delta\big)-f(x-k\delta)-f(y-n\delta)\big| \leqq$$

$$\leqq \big|f\big(x-k\delta+y-(n+1)\delta\big)-f(x-k\delta)-f\big(y-(n+1)\delta\big)\big|+$$

$$+\big|f\big(y-(n+1)\delta\big)-f(y-n\delta)-f(-\delta)\big|+|f(-\delta)| \leqq 2A+|f(-\delta)|$$

using (1) again. Hence $F(x,y)\leqq 2A+|f(-\delta)|$ for every $x, y\in R$. Thus, by the above mentioned theorem ([1], Theorem 1.2, p. 196), there exists an additive function H such that f^*-H is bounded. Hence the function $g(x)\stackrel{\text{def}}{=}f(x)-H(x)$ is bounded in $[0,\delta)$. If $x\in(-\delta, 0)$ then we have

$$|g(x)| = |f(x)-H(x)| \leqq |f(x)+f(-x)-f(0)|+|f(0)|+|H(-x)-f(-x)| \leqq$$

$$\leqq A+|f(0)|+|g(-x)|$$

and hence g is bounded in $(-\delta, \delta)$.

We show $\lim_{x\to+0} g(x)=0$ (the proof of $\lim_{x\to-0} g(x)=0$ is similar). We put

$$M_n = \sup\left\{g(x);\ \frac{\delta}{2^n} \leqq x < \frac{\delta}{2^{n-1}}\right\} \quad (n = 1, 2, \ldots).$$

Let $\varepsilon>0$ be arbitrary and let N be such that

$$|g(x+y)-g(x)-g(y)| = |f(x+y)-f(x)-f(y)| < \varepsilon$$

holds for every $0<|x|, |y|<\frac{\delta}{2^N}$. Then we have

$$M_{n+1} \leqq \frac{1}{2} M_n+\frac{1}{2}\varepsilon \quad (n \geqq N). \tag{2}$$

Indeed, for every $x\in\left[\frac{\delta}{2^{n+1}}, \frac{\delta}{2^n}\right)$, $n\geqq N$, we have $|g(2x)-2g(x)|<\varepsilon$ from which

$$g(x) < \frac{1}{2} g(2x)+\frac{1}{2}\varepsilon \leqq \frac{1}{2} M_n+\frac{1}{2}\varepsilon$$

and hence $M_{n+1} \leqq \frac{1}{2} M_n+\frac{1}{2}\varepsilon$.

If $M_n\leqq\varepsilon$ holds for at least one $n\geqq N$ then by (2) we have $M_{n+1}\leqq\frac{1}{2}\varepsilon+ +\frac{1}{2}\varepsilon=\varepsilon$ and by induction $M_k\leqq\varepsilon$ for every $k\geqq n$. Thus in this case $\limsup_{n\to\infty} M_n\leqq\varepsilon$ holds.

If $M_n>\varepsilon$ holds for every $n\geqq N$ then $M_{n+1}<\frac{1}{2} M_n+\frac{1}{2} M_n=M_n$ for every $n\geqq N$ i.e. the sequence $\{M_n\}_{n=N}^{\infty}$ is decreasing. Let $M\overset{\text{def}}{=}\lim_{n\to\infty} M_n$. Then (2) implies $M\leqq\frac{1}{2} M+\frac{1}{2}\varepsilon$, $M\leqq\varepsilon$ and thus we have $\limsup_{n\to\infty} M_n\leqq\varepsilon$ again.

Since $\varepsilon>0$ was arbitrary we proved $\limsup_{x\to+0} g(x)\leqq \limsup_{n\to\infty} M_n\leqq 0$. A similar argument shows that $\liminf_{x\to+0} g(x)\geqq 0$ and hence $\lim_{x\to+0} g(x)=0$ as we stated.

Finally $\lim_{x\to 0} |g(0)-g(x)-g(-x)|=0$ gives $g(0)=0$, which completes the proof.

2. Let S denote the class of all functions defined on **R** which are Lebesgue measurable and periodic mod 1. For $f\in S$ we denote

$$I\{f \lesseqqgtr c\} = \{x\in[0, 1];\ f(x) \lesseqqgtr c\}$$

and

$$\|f\| = \inf\left\{a+\lambda(I\{|f| \geqq a\});\ a > 0\right\}$$

where λ denotes the Lebesgue measure.[1] The following properties of the "pseudonorm" $\|\cdot\|$ are well-known (and can be easily verified).

(3) $0 \leqq \|f\| \leqq 1 \quad (f \in S)$,

(4) $\|f\| = 0$ iff $f(x) = 0$ for a.e. $x \in \mathbf{R}$,

(5) $\|f+g\| \leqq \|f\| + \|g\| \quad (f, g \in S)$,

(6) $\|f(x+h)\| = \|f(x)\| \quad (f \in S,\ h \in \mathbf{R})$,

(7) If $f \in S$ and $\|f\| < a$ then $\lambda(I\{|f| \geqq a\}) < a$,

(8) $\lim\limits_{h \to 0} \|f(x+h) - f(x)\| = 0 \quad (f \in S)$,

(9) $\lim\limits_{n \to \infty} \|f_n - f\| = 0$ iff the sequence f_n converges to f in measure $(f_n, f \in S)$.

LEMMA 1. *Let* $s(f) \overset{\text{def}}{=} \inf\{\|f(x)-c\|;\ c \in \mathbf{R}\}$. *Then for every* $f \in S$,
a) *there exists* $c_0 \in \mathbf{R}$ *such that* $\|f(x) - c_0\| = s(f)$,

b) *if* $s(f) = d$ *then there exists* $c_1 \in \mathbf{R}$ *such that* $\lambda(I\{f \leqq c_1\}) \geqq \frac{d}{3}$ *and* $\lambda\left(I\left\{f \geqq c_1 + \frac{d}{3}\right\}\right) \geqq \frac{d}{3}$.

PROOF. a) Let $f \in S$ be given. First we show that $c_n \in \mathbf{R}$, $|c_n| \to \infty$ implies $\|f - c_n\| \to 1$. For every $\varepsilon > 0$ and $n = 1, 2, \ldots$ there exists $a_n > 0$ such that

$$a_n + \lambda(I\{|f - c_n| \geqq a_n\}) < \|f - c_n\| + \varepsilon. \tag{10}$$

Since $|f(x) - c_n| \to \infty$ for every $x \in \mathbf{R}$ hence $\lambda(I\{|f - c_n| \geqq 1\}) \to 1$, thus there exists $N > 0$ such that $\lambda(I\{|f - c_n| \geqq 1\}) > 1 - \varepsilon$ for $n \geqq N$. If $a_n > 1$ then $\|f - c_n\| > 1 - \varepsilon$ by (10). If $a_n \leqq 1$ and $n \geqq N$ then

$$\|f - c_n\| > \lambda(I\{|f - c_n| \geqq a_n\}) - \varepsilon \geqq \lambda(I\{|f - c_n| \geqq 1\}) - \varepsilon > 1 - 2\varepsilon.$$

That is, $\|f - c_n\| > 1 - 2\varepsilon$ for $n \geqq N$ and hence $\|f - c_n\| \to 1$.

Now let $\|f - c_n\| \to s(f)$. If the sequence $\{c_n\}$ is not bounded, then for a suitable subsequence we have $|c_{n_k}| \to \infty$, consequently $s(f) = \lim\limits_{k \to \infty} \|f - c_{n_k}\| = 1$. Then $1 = s(f) \leqq \|f - 0\| \leqq 1$ and we put $c_0 = 0$.

If $\{c_n\}$ is bounded then it has a convergent subsequence $c_{n_k} \to c_0$. We have $\|f - c_0\| = \lim\limits_{k \to \infty} \|f - c_{n_k}\| = s(f)$ which proves a).

b) We can suppose $d > 0$. Let $C = \left\{c \in \mathbf{R};\ \lambda(I\{f \leqq c\}) \leqq \frac{d}{3}\right\}$. C is non-empty and bounded from above. Indeed, $\bigcap\limits_{n=1}^{\infty} I\{f \leqq -n\} = \emptyset$ and hence $\lambda(I\{f \leqq -n\}) \to 0$; $\bigcup\limits_{n=1}^{\infty} I\{f \leqq n\} = [0, 1]$ from which $\lambda(I\{f \leqq n\}) \to 1 > \frac{d}{3}$. Consequently $-n \in C$ and

[1] This norm was introduced by M. Fréchet. See his book *Les Espaces Abstraits*, Gauthier-Villars (Paris, 1928), p. 92.

$C\subset(-\infty, n)$ if n is large enough. We put $c_1=\sup C$. Then $c_1+\frac{1}{n}\notin C$ from which $\lambda(I\{f\leqq c_1\})=\lim\limits_{n\to\infty}\lambda\left(I\left\{f\leqq c_1+\frac{1}{n}\right\}\right)\geqq\frac{d}{3}$. On the other hand we have

$$d=s(f)\leqq\|f-c_1\|\leqq\frac{d}{3}+\lambda\left(I\left\{|f-c_1|\geqq\frac{d}{3}\right\}\right)=$$

$$=\frac{d}{3}+\lambda\left(I\left\{f\leqq c_1-\frac{d}{3}\right\}\right)+\lambda\left(I\left\{f\geqq c_1+\frac{d}{3}\right\}\right)\leqq\frac{2d}{3}+\lambda\left(I\left\{f\geqq c_1+\frac{d}{3}\right\}\right)$$

since $c_1-\frac{d}{3}\in C$. Hence $\lambda\left(I\left\{f\geqq c_1+\frac{d}{3}\right\}\right)\geqq\frac{d}{3}$ and b) is proved.

LEMMA 2. *Let A and B be measurable subsets of the interval* $[0, a]$ *with* $\lambda(A)\geqq c$, $\lambda(B)\geqq c$ $(c\geqq 0)$. *Then there exists* $|h|\leqq a$ *such that*

$$\lambda\big((A+h)\cap B\big)\geqq\frac{c^2}{2a}$$

where $A+h$ denotes the set $\{x+h;\ x\in A\}$.

PROOF. We put $D=\{(x, y);\ -a\leqq x\leqq a, y-x\in A\}\cap(\mathbf{R}\times B)$; it is easy to see that D is measurable. For every $y\in B$ we have

$$D^y\overset{\text{def}}{=}\{x;\ (x, y)\in D\}=(-A)+y$$

and hence $\lambda(D^y)=\lambda(A)$. By Fubini's theorem $\lambda_2(D)=\int\limits_B\lambda(D^y)\,dy=\lambda(B)\cdot\lambda(A)$. On the other hand $D_x\overset{\text{def}}{=}\{y;\ (x, y)\in D\}=(A+x)\cap B$ for every $x\in[-a, a]$ and thus we have

$$\int\limits_{-a}^{a}\lambda\big((A+x)\cap B\big)\,dx=\int\limits_{-a}^{a}\lambda(D_x)\,dx=\lambda(B)\lambda(A)\geqq c^2.$$

Hence $\lambda\big((A+x)\cap B\big)\geqq\frac{c^2}{2a}$ holds for at least one $|x|\leqq a$, q.e.d.

Our next theorem is a generalization of the simple fact that a function $f\in S$ is constant a.e. (that is $s(f)=0$) if and only if $\|f(x+h)-f(x)\|=0$ for every h.

THEOREM 2. *Let $\{f_n\}$ be an arbitrary sequence of functions belonging to S. Then* $\lim\limits_{n\to\infty}s(f_n)=0$ *if and only if* $\lim\limits_{n\to\infty}\|f_n(x+h)-f_n(x)\|=0$ *holds for every* $h\in\mathbf{R}$.

PROOF.[2] Suppose first $\lim\limits_{n\to\infty}s(f_n)=0$. By Lemma 1 a), there exists a sequence $\{c_n\}$ such that $s(f_n)=\|f_n(x)-c_n\|$. Then, for every $h\in\mathbf{R}$ we have

$$\|f_n(x+h)-f_n(x)\|\leqq\|f_n(x+h)-c_n\|+\|c_n-f_n(x)\|=2s(f_n)\to 0.$$

[2] A simpler proof can be found in A. J. E. M. Janssen, Note on a paper by M. Laczkovich on functions with measurable differences (Erdős' conjecture) (to appear).

Now suppose indirectly that $\lim_{n\to\infty} \|f_n(x+h)-f_n(x)\|=0$ holds for every h but $s(f_n)\not\to 0$. Then, after selecting a suitable subsequence, we may assume that $s(f_n)\geqq 3d>0$ for $n=1, 2, \ldots$. We prove that for every non-degenerate interval $[a, b]$ and for every $N\geqq 0$ there exist $h\in(a, b)$ and $n>N$ such that

$$\|f_n(x+h)-f_n(x)\| \geqq \frac{d^2}{4}. \tag{11}$$

Let $\frac{p}{q}$ be a rational number such that $a<\frac{p-1}{p}<\frac{p+1}{p}<b$. By our assumption $\lim_{n\to\infty}\left\|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right\|=0$; hence there exists $n>N$ such that $\left\|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right\|<\eta$ where $\eta=\frac{d^2}{8q^2}$. Thus by (7) we have

$$\lambda\left(I\left\{\left|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right| \geqq \eta\right\}\right) < \eta. \tag{12}$$

This easily implies

$$\lambda\left(I\left\{\left|f_n\left(x+\frac{k}{q}\right)-f_n(x)\right| \geqq q\eta\right\}\right) < q\eta \quad \text{for every} \quad k=1, 2, \ldots, q. \tag{13}$$

Indeed, $\left|f_n\left(x+\frac{k}{q}\right)-f_n(x)\right|\geqq q\eta$ implies

$$\left|f_n\left(x+\frac{i+1}{q}\right)-f_n\left(x+\frac{i}{q}\right)\right| \geqq \eta$$

for at least one of the values $i=0, 1, \ldots, k-1$. Thus

$$I\left\{\left|f_n\left(x+\frac{k}{q}\right)-f_n(x)\right| \geqq q\eta\right\} \subset \bigcup_{i=0}^{k-1}\left[I\left\{\left|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right| \geqq \eta\right\}-\frac{i}{q}\right]; \tag{14}$$

and (13) follows from (12) and (14).

Since $s(f_n)\geqq 3d$, by Lemma 1 b), there is a $c\in\mathbf{R}$ such that for the level sets $A=I\{f_n\leqq c\}$, $B=I\{f_n\geqq c+d\}$ we have $\lambda(A)\geqq d$, $\lambda(B)\geqq d$. Then there are indices $1\leqq i\leqq q$, $1\leqq j\leqq q$ such that

$$\lambda\left(A\cap\left[\frac{i-1}{q}, \frac{i}{q}\right]\right) \geqq \frac{d}{q}, \quad \lambda\left(B\cap\left[\frac{j-1}{q}, \frac{j}{q}\right]\right) \geqq \frac{d}{q}.$$

Applying Lemma 2 for $a=\frac{1}{q}$ and for the sets

$$A'=\left(A\cap\left[\frac{i-1}{q}, \frac{i}{q}\right]\right)-\frac{i-1}{q}, \quad B'=\left(B\cap\left[\frac{j-1}{q}, \frac{j}{q}\right]\right)-\frac{j-1}{q}$$

we get $|h_1| \leqq \frac{1}{q}$ such that $\lambda((A'+h_1)\cap B') \geqq \frac{d^2}{2q}$. We put $h = \frac{p}{q} - h_1$. Obviously $h \in (a, b)$; we are going to show (11). Let $1 \leqq k \leqq q$ be arbitrary and let

$$D = I\left\{\left|f_n\left(x+\frac{p}{q}-h_1\right)-f_n\left(x-\frac{k-i}{q}-h_1\right)\right| \geqq q\eta\right\}$$

and

$$E = I\left\{\left|f_n(x)-f_n\left(x-\frac{k-j}{q}\right)\right| \geqq q\eta\right\}.$$

It follows from (13) (and from the periodicity of f_n) that

$$\lambda(D) < q\eta, \quad \lambda(E) < q\eta. \tag{15}$$

Let $F = [(A'+h_1)\cap B'] + \frac{k-1}{q}$, then

$$F \subset \left[\frac{k-1}{q}, \frac{k}{q}\right] \quad \text{and} \quad \lambda(F) \geqq \frac{d^2}{2q}. \tag{16}$$

If $x \in F \setminus (D \cup E)$ then $x - \frac{k-i}{q} - h_1 \in A$ and $x - \frac{k-j}{q} \in B$. Therefore, by the definition of A and B we have $f_n\left(x - \frac{k-i}{q} - h_1\right) \leqq c$, $f_n\left(x - \frac{k-j}{q}\right) \geqq c+d$ from which

$$\left|f_n\left(x-\frac{k-i}{q}-h_1\right)-f_n\left(x-\frac{k-j}{q}\right)\right| \geqq d.$$

On the other hand $x \notin D \cup E$ and hence we get

$$\left|f_n\left(x+\frac{p}{q}-h_1\right)-f_n(x)\right| \geqq d-2q\eta.$$

Consequently

$$\lambda\left(\left\{x \in \left[\frac{k-1}{q}, \frac{k}{q}\right];\ |f_n(x+h)-f_n(x)| \geqq d-2q\eta\right\}\right) \geqq$$

$$\geqq \lambda(F \setminus (D \cup E)) \geqq \frac{d^2}{2q} - 2q\eta = \frac{d^2}{4q}$$

by (15) and (16). This inequality holds for every $k = 1, 2, \ldots, q$, therefore

$$\lambda(I\{|f_n(x+h)-f_n(x)| \geqq d-2q\eta\}) \geqq \frac{d^2}{4}.$$

Since $d - 2q\eta > \frac{d^2}{4}$ this implies

$$\lambda\left(I\left\{|f_n(x+h)-f_n(x)| \geqq \frac{d^2}{4}\right\}\right) \geqq \frac{d^2}{4}$$

and hence we have (11) by (7).

Let $n_0=0$ and $[a_0, b_0]=[0, 1]$. Suppose that $k>0$, and the index n_{k-1} and the non-degenerate interval $[a_{k-1}, b_{k-1}]$ have been defined. Then, applying the foregoing argument with $[a, b]=[a_{k-1}, b_{k-1}]$ and $N=n_{k-1}$, we get an index $n_k>n_{k-1}$ and $h_k\in(a_{k-1}, b_{k-1})$ such that

$$\|f_{n_k}(x+h_k)-f_{n_k}(x)\| \geqq \frac{d^2}{4}.$$

It follows easily from (8) that there exists $\delta>0$ such that

$$[a_k, b_k] \overset{\text{def}}{=} [h_k-\delta, h_k+\delta] \subset [a_{k-1}, b_{k-1}]$$

and

$$\|f_{n_k}(x+h)-f_{n_k}(x)\| > \frac{d^2}{8} \quad \text{holds for every} \quad h\in[a_k, b_k]. \tag{17}$$

Thus by induction we define the sequence $n_1, n_2, \ldots$ and the nested sequence of intervals $[a_k, b_k]$ such that (17) holds for every k. Let $h_0\in \bigcap_{k=1}^{\infty} [a_k, b_k]$. Then by (17) we have

$$\|f_{n_k}(x+h_0)-f_{n_k}(x)\| > \frac{d^2}{8} \quad (k=1, 2, \ldots).$$

This obviously contradicts our assumption $\|f_n(x+h_0)-f_n(x)\|\to 0$ and this contradiction proves Theorem 2.

Now we turn to prove our main result.

THEOREM 3. *The class L has the weak difference property.*

PROOF. Suppose that, for a function f: $\mathbf{R}\to\mathbf{R}$, $f(x+h)-f(x)\in L$ holds for every h. We have to prove that f can be written in the form $f=g+H+S$, where $g\in L$, H is additive and, for every h, $S(x+h)-S(x)=0$ holds for a.e. x. We may suppose that f is periodic mod 1. Indeed, let the periodic function f^* be defined by

$$f^*(x)=f(x) \quad (0\leqq x<1) \quad \text{and} \quad f^*(x+1)=f^*(x) \quad (x\in\mathbf{R}).$$

Then $f-f^*$ is measurable since for $n\leqq x<n+1$ we have $f^*(x)-f(x)=f(x-n)-f(x)$. On the other hand

$$f^*(x+h)-f^*(x) = [f^*(x+h)-f(x+h)]+[f(x+h)-f(x)]+[f(x)-f^*(x)]$$

is measurable for every h. Hence, if $f^*=g^*+H+S$ where g^*, H and S have the desired properties then we have $f=g+H+S$ where $g=(f-f^*)+g^*$ is measurable. (This argument is due to DE BRUIJN [1], § 1.)

Now suppose that f is periodic mod 1, then $f(x+h)-f(x)\in S$ for every h. By Lemma 1 a), for every h there exists a constant $c(h)$ such that

$$s(f(x+h)-f(x)) = \|f(x+h)-f(x)-c(h)\|.$$

We may suppose that $c(0)=0$ and the function $c(x)$ is periodic mod 1. We show that

$$\lim_{\substack{h\to 0\\ k\to 0}} (c(h+k)-c(h)-c(k)) = 0. \tag{18}$$

First we prove that

$$\lim_{h\to 0} s\big(f(x+h)-f(x)\big) = 0. \tag{19}$$

Indeed, let h_n be an arbitrary sequence tending to zero and let $f_n(x)=f(x+h_n)-f(x)$. Then for every fixed $k\in R$ we have

$$\|f_n(x+k)-f_n(x)\| = \|f(x+h_n+k)-f(x+k)-f(x+h_n)+f(x)\| = \|F(x+h_n)-F(x)\|$$

where $F(x)=f(x+k)-f(x)\in S$. It follows from (8) that $\|F(x+h_n)-F(x)\|\to 0$, therefore $\|f_n(x+k)-f_n(x)\|\to 0$ for every $k\in\mathbf{R}$. Applying Theorem 2 we have $s(f_n)=s\big(f(x+h_n)-f(x)\big)\to 0$ which gives (19).

Now let $h_n\to 0$ and $k_n\to 0$ be arbitrary, then we have

$$s\big(f(x+h_n+k_n)-f(x)\big) = \|f(x+h_n+k_n)-f(x)-c(h_n+k_n)\| \to 0,$$

$$s\big(f(x+h_n)-f(x)\big) = \|f(x+h_n)-f(x)-c(h_n)\| = \|f(x+h_n+k_n)-f(x+k_n)-c(h_n)\| \to 0$$

and

$$s\big(f(x+k_n)-f(x)\big) = \|f(x+k_n)-f(x)-c(k_n)\| \to 0.$$

Hence

$$\|c(h_n+k_n)-c(h_n)-c(k_n)\| \leqq \|f(x+h_n+k_n)-f(x)-c(h_n+k_n)\| +$$

$$+\|c(h_n)-f(x+h_n+k_n)+f(x+k_n)\|+\|c(k_n)+f(x)-f(x+k_n)\| \to 0.$$

Since $\|c\|=\min(1,|c|)$ holds for every constant function c, therefore

$$|c(h_n+k_n)-c(h_n)-c(k_n)| \to 0$$

which proves (18).

Now we can apply Theorem 1 for the function $c(x)$ and get the functions $H(x)$ and $u(x)$ such that $c(x)=H(x)+u(x)$, H is additive and

$$\lim_{x\to 0} u(x) = u(0) = 0. \tag{20}$$

We may suppose $H(1)=0$ since otherwise we put

$$H_1(x) = H(x)-x\cdot H(1), \quad u_1(x) = u(x)+x\cdot H(1).$$

Then $H(x)$ (and thus $u(x)$ as well) is periodic mod 1. We put

$$K(x, y) \overset{\text{def}}{=} f(x+y)-f(x)-H(y).$$

Obviously, for every fixed y, $K(x, y)$ (as a function of x) belongs to S. We show that

$$\|K(x, y_n)-K(x, y_0)\| \to 0 \quad \text{whenever} \quad y_n \to y_0. \tag{21}$$

(Here and in the sequel the "norm" $\|\cdot\|$ of a function of x and y denotes the norm of that function as a function of x; the variable y is always fixed.)

Let $y_n\to 0$, then

$$\|K(x, y_n)\| = \|f(x+y_n)-f(x)-H(y_n)\| \leqq \|f(x+y_n)-f(x)-c(y_n)\|+\|c(y_n)-H(y_n)\| =$$

$$= s\big(f(x+y_n)-f(x)\big)+\min\big(1, |u(y_n)|\big) \to 0$$

by (19) and (20). If $y_n \to y_0$ then we have

$$\|K(x, y_n) - K(x, y_0)\| = \|f(x+y_n) - f(x+y_0) - H(y_n - y_0)\| =$$
$$= \|f(x+y_n-y_0) - f(x) - H(y_n - y_0)\| = \|K(x, y_n - y_0)\| \to 0$$

and hence (21) is proved.

The next step of our proof is the construction of a measurable function $G(x, y)$ satisfying the following condition:

(22) For every $y \in R$, $G(x, y) = K(x, y)$ for a.e. $x \in \mathbf{R}$.

Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $\|K(x, y) - K(x, y')\| < \varepsilon$ whenever $|y - y'| < \delta$. Indeed, otherwise we could find two sequences y_n and y'_n such that $y_n - y'_n \to 0$ and $\|K(x, y_n) - K(x, y'_n)\| \geqq \varepsilon$. Since $K(x, y)$ is periodic in y, we may suppose $y_n, y'_n \in [0, 3]$ $(n = 1, 2, \ldots)$. Then, for a suitable subsequence n_k we have $y_{n_k} \to y_0$, $y'_{n_k} \to y_0$. By (21) we have $\|K(x, y_{n_k}) - K(x, y'_{n_k})\| \to 0$ which is a contradiction.

Now let $\delta_n > 0$ be such that $|y - y'| < \delta_n$ implies $\|K(x, y) - K(x, y')\| < \frac{1}{2^n}$ $(n = 1, 2, \ldots)$ and put

$$G_n(x, y) \stackrel{\text{def}}{=} K(x, i\delta_n) \quad \text{if} \quad i\delta_n \leqq y < (i+1)\delta_n \quad (i = 0, \pm 1, \pm 2, \ldots;\ n = 1, 2, \ldots).$$

Then G_n is measurable for every n and

$$\|G_n(x, y) - K(x, y)\| = \|K(x, i\delta_n) - K(x, y)\| < \frac{1}{2^n} \tag{23}$$

holds for every $y \in \mathbf{R}$.

We define

$$G(x, y) = \begin{cases} \lim_{n\to\infty} G_n(x, y), & \text{if the finite limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

G is measurable and satisfies (22). Indeed, let y be fixed. By (7) and (23) we have

$$\lambda\left(I\left\{|G_n(x, y) - K(x, y)| \geqq \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}$$

and hence, by the Borel—Cantelli lemma we have

$$G(x, y) = \lim_{n\to\infty} G_n(x, y) = K(x, y) \quad \text{for a.e.} \quad x \in [0, 1].$$

Thus the periodicity of the functions G_n and K proves $G(x, y) = K(x, y)$ for a.e. $x \in \mathbf{R}$.

Let

$$S_1(x, y) \stackrel{\text{def}}{=} K(x, y) - G(x, y) = f(x+y) - f(x) - H(y) - G(x, y). \tag{24}$$

According to (22), for every fixed y we have

$$S_1(x, y) = 0 \quad \text{for a.e.} \quad x \in \mathbf{R}. \tag{25}$$

We shall prove that there exists a point $x_0 \in \mathbf{R}$ such that

(26) for every fixed h, $S_1(x_0, x+h) - S_1(x_0, x) = 0$ for a.e. $x \in \mathbf{R}$.

We have:

$$S_1(x, y+z) = f(x+y+z) - f(x) - H(y+z) - G(x, y+z);$$

$$-S_1(x+y, z) = -f(x+y+z) + f(x+y) + H(z) + G(x+y, z);$$

$$-S_1(x, y) = -f(x+y) + f(x) + H(y) + G(x, y).$$

By adding we get

(27) $$S_1(x, y+z) - S_1(x+y, z) - S_1(x, y) = -G(x, y+z) + $$

$$+ G(x+y, z) + G(x, y) \overset{\text{def}}{=} L(x, y, z).$$

The measurability of G implies that L is measurable, too. On the other hand it follows from (25) and (27) that for every fixed y and z, $L(x, y, z) = 0$ for a.e. x. Therefore $L(x, y, z) = 0$ for almost every $(x, y, z) \in \mathbf{R}^3$. Hence there exists a point x_0 such that $L(x_0, y, z) = 0$ for almost every pair $(y, z) \in \mathbf{R}^2$. Thus there exists a subset $Z \subset \mathbf{R}$ such that $\lambda\ (\mathbf{R} \setminus Z) = 0$ and for every $z \in Z$ we have $L(x_0, y, z) = S_1(x_0, y+z) - S_1(x_0 + y, z) - S_1(x_0, y) = 0$ for a.e. y. However $S_1(x_0 + y, z) = 0$ for a.e. y by (25) hence

(28) $$S_1(x_0, y+z) - S_1(x_0, y) = 0 \quad \text{holds for a.e. } y.$$

Now let $h \in \mathbf{R}$ be arbitrary. Then there are $z_1, z_2 \in Z$ such that $h = z_1 + z_2$, since $Z \cap (h - Z) \neq \emptyset$. Therefore

$$S_1(x_0, x+h) - S_1(x_0, x) =$$

$$= [S_1(x_0, x+z_1+z_2) - S_1(x_0, x+z_2)] + [S_1(x_0, x+z_2) - S_1(x_0, x)] = 0$$

holds for a.e. x by (28) and hence (26) is proved.

Now we apply (24) by replacing x by x_0 and y by $x - x_0$:

$$S_1(x_0, x - x_0) = f(x) - f(x_0) - H(x) + H(x_0) - G(x_0, x - x_0)$$

from which

$$f(x) = [G(x_0, x - x_0) + f(x_0) - H(x_0)] + H(x) + S_1(x_0, x - x_0) \overset{\text{def}}{=} g(x) + H(x) + S(x).$$

It is easy to see from the construction of $G(x, y)$ that $G(x_0, x)$ is measurable for every fixed x_0. Hence $g(x) = G(x_0, x - x_0) + f(x_0) - H(x_0)$ is measurable. Furthermore, for every fixed h we have

$$S(x+h) - S(x) = S_1(x_0, x+h-x_0) - S_1(x_0, x - x_0) = 0$$

for a.e. $x \in \mathbf{R}$ by (26), q.e.d.

In our next theorem $L_p(0, 1)$ denotes the class of those functions $f \in L$ which are periodic mod 1 and for which $\|f\|_p = \left(\int_0^1 |f(x)|^p\, dx\right)^{1/p} < \infty$.

THEOREM 4. *The classes $L_p(0, 1)$ have the weak difference property for every $p>0$.*

PROOF. Suppose that for a function $f\colon \mathbf{R}\to\mathbf{R}$, $f(x+h)-f(x)\in L_p(0, 1)$ holds for every h. Then, by our preceding theorem, $f=g+H+S$, where $g\in L$, H is additive and, for every h, $S(x+h)-S(x)=0$ holds for a.e. x. We may assume that g, H and S are periodic mod 1 since otherwise we consider the functions $g_1(x)=g(x-[x])+(x-[x])\cdot H(1)$, $H_1(x)=H(x)-H(1)\cdot x$ and $S_1(x)=S(x-[x])$ instead of g, H and S.

For every fixed h we have

$$g(x+h)-g(x)=[f(x+h)-f(x)]-H(h)-[S(x+h)-S(x)]=[f(x+h)-f(x)]-H(h)$$

for a.e. x and thus

$$N(h)\overset{\text{def}}{=}\|g(x+h)-g(x)\|_p^p<\infty$$

holds for every h.

We prove that the function N is bounded on $[0, 1]$. First observe that

$$N(-h)=\|g(x-h)-g(x)\|_p^p=\|g(x)-g(x+h)\|_p^p=N(h)$$

holds for each $h\in\mathbf{R}$.

Furthermore, for every $f_1, f_2\in L_p(0, 1)$ we have

$$\|f_1+f_2\|_p^p=\int_0^1 |f_1+f_2|^p\,dx\leqq\int_0^1 [2\max(|f_1|, |f_2|)]^p\,dx\leqq$$

$$\leqq 2^p\int_0^1 (|f_1|^p+|f_2|^p)\,dx=2^p(\|f_1\|_p^p+\|f_2\|_p^p)$$

and thus

$$N(y_1+y_2)=\|g(x+y_1+y_2)-g(x)\|_p^p=$$

$$=\|[g(x+y_1+y_2)-g(x+y_2)]+[g(x+y_2)-g(x)]\|_p^p\leqq$$

$$\leqq 2^p\big(\|g(x+y_1+y_2)-g(x+y_2)\|_p^p+\|g(x+y_2)-g(x)\|_p^p\big)=$$

$$=2^p\big(\|g(x+y_1)-g(x)\|_p^p+N(y_2)\big)=2^p\big(N(y_1)+N(y_2)\big)$$

holds for every $y_1, y_2\in\mathbf{R}$. The function $G(x, y)\overset{\text{def}}{=}|g(x+y)-g(x)|^p$ is measurable on $[0, 1]\times[0, 1]$ hence $N(y)=\int_0^1 G(x, y)\,dx$ is measurable on $[0, 1]$. Thus there exists $K>0$ such that the set $A=\{y\in[0, 1];\ N(y)<K\}$ is of positive measure. By a theorem of Steinhaus (see [12], p. 145), there is $\delta>0$ such that if $|y|<\delta$ then $y=y_1-y_2$ for suitable $y_1, y_2\in A$ and so,

$$0\leqq N(y)\leqq 2^p\big(N(y_1)+N(-y_2)\big)=2^p\big(N(y_1)+N(y_2)\big)\leqq 2^{p+1}K.$$

Hence, if $1/2^n<\delta$ then for every $y\in[0, 1]$ we have

$$0\leqq N(y)\leqq 2^{p+1}N\left(\frac{y}{2}\right)\leqq 2^{2(p+1)}\cdot N\left(\frac{y}{4}\right)\leqq\ldots\leqq 2^{n(p+1)}\cdot N\left(\frac{y}{2^n}\right)\leqq 2^{(n+1)(p+1)}\cdot K.$$

It follows by Fubini's theorem that

$$\int_0^1 \Big(\int_0^1 G(x, y)\, dy\Big)\, dx = \int_0^1 \Big(\int_0^1 G(x, y)\, dx\Big)\, dy = \int_0^1 N(y)\, dy < \infty$$

and hence, for at least one value of x we have

$$\int_0^1 G(x, y)\, dy = \int_0^1 |g(x+y)-g(x)|^p\, dy < \infty.$$

By the periodicity of g, this obviously implies $g\in L_p(0, 1)$, q.e.d.

3. Our next theorem states that the pair $(L, L^{(2)})$ has the double difference property, where $L^{(2)}$ denotes the class of Lebesgue measurable functions defined on $\mathbf{R}^2$.

THEOREM 5. *If a function $f\colon \mathbf{R}\to\mathbf{R}$ is such that $f(x+y)-f(x)-f(y)$ is Lebesgue measurable (as a function of two variables), then f is of the form $g+H$ where $g\in L$ and $H\colon \mathbf{R}\to\mathbf{R}$ is additive.*

PROOF. There exists a subset $Y\subset\mathbf{R}$ such that $\lambda(\mathbf{R}\setminus Y)=0$ and, for every $y\in Y$, $f(x+y)-f(x)-f(y)$ is measurable, as a function of x. Let $h\in\mathbf{R}$ be arbitrary. Then there are $y_1, y_2\in Y$ such that $h=y_1+y_2$ since $Y\cap[(-Y)+h]\neq\emptyset$. Since

$$f(x+h)-f(x) = [f(x+y_1+y_2)-f(x+y_2)-f(y_1)]+$$
$$+[f(x+y_2)-f(x)-f(y_2)]+f(y_1)+f(y_2),$$

hence $f(x+h)-f(x)$ is measurable for every h. According to Theorem 3, $f=g+H+S$ where g is measurable, H is additive and, for every h, $S(x+h)-S(x)=0$ for a.e. x.

Let $F(x, y)\overset{\text{def}}{=}S(x+y)-S(x)-S(y)$, then

$$F(x, y) = [f(x+y)-f(x)-f(y)]-g(x+y)+g(x)+g(y)$$

and thus $F(x, y)$ is measurable. For every fixed x we have

$$-F(x, y) = S(x)-[S(x+y)-S(y)] = S(x) \quad \text{for a.e. } y.$$

Consequently $S(x)=-\int_0^1 F(x, y)\, dy$ holds for every x which proves that S is measurable, too. Hence $f=[g+S]+H$ is a sum of a measurable and an additive function, q.e.d.

For the analogous theorem concerning Borel measurable functions we need the following simple

LEMMA 3. *Let $f(x, y)\colon \mathbf{R}\times[0, 1]\to\mathbf{R}$ be a bounded function of Baire class α. Then $F(x)=\int_0^1 f(x, y)\, dy$ is of Baire class α, too.*

PROOF. We prove by transfinite induction. If $\alpha=0$, that is if f is continuous, then the continuity of the function F is well-known.

Let $\alpha>0$ and suppose the assertion is true for $\beta<\alpha$. Let $|f(x,y)|\leqq M$ and let $f_n(x,y)$ be a sequence of functions of Baire class $\alpha_n<\alpha$ converging to f. We may suppose $|f_n(x,y)|\leqq M$ because otherwise we take the functions $\min(M, \max(f_n, -M))$ instead of f_n. Then $F_n(x)=\int_0^1 f_n(x,y)\,dy$ is of Baire class α_n by the induction hypothesis. Furthermore Lebesgue's theorem implies

$$F(x)=\int_0^1 f(x,y)\,dy=\lim_{n\to\infty}\int_0^1 f_n(x,y)\,dy=\lim_{n\to\infty}F_n(x)$$

which proves that F is of class α.

THEOREM 6. *If a function $f\colon \mathbf{R}\to\mathbf{R}$ is Lebesgue measurable and such that $f(x+y)-f(x)-f(y)$ is of Baire class α (as a function of two variables), then f is of Baire class α.*

PROOF. In the case of $\alpha=0$ the assertion is a simple consequence of de Bruijn's theorem on the difference property of the class of continuous functions. For, if $f(x+y)-f(x)-f(y)$ is continuous, then $f(x+h)-f(x)$ is continuous for every h and hence $f=g+H$ where g is continuous and H is additive. By our assumptions $H=f-g$ is measurable and thus H must be linear.

Now we suppose $\alpha\geqq 2$. It easily follows from Luzin's theorem that every measurable function equals almost everywhere to a Baire 2 function. Hence there exists a Baire 2 function $p(x)$ such that

$$q(x)\overset{\text{def}}{=}f(x)-p(x)=0 \quad \text{for a.e. } x. \tag{29}$$

$$q(x+y)-q(x)-q(y)=[f(x+y)-f(x)-f(y)]-p(x+y)+p(x)+p(y)$$

is a Baire α function since $f(x+y)-f(x)-f(y)$ is Baire α by our assumption, p is Baire 2 and $\alpha\geqq 2$. Hence

$$F(x,y)\overset{\text{def}}{=}-\operatorname{arctg}[q(x+y)-q(x)-q(y)]$$

is a bounded Baire α function. For every fixed x we have $F(x,y)=\operatorname{arctg}[q(x)]$ for a.e. y by (29). Hence by Lemma 3, $\operatorname{arctg}[q(x)]=\int_0^1 F(x,y)\,dy$ is Baire α and thus so is the function $f(x)=p(x)+q(x)$.

Finally suppose $\alpha=1$. By Luzin's theorem, for every natural number n there exists a closed subset $F_n\subset\mathbf{R}$ such that $\lambda(\mathbf{R}\setminus F_n)<\frac{1}{n}$ and the restriction $f|_{F_n}$ is continuous. Let

$$p_n(x)=\begin{cases} f(x) & \text{if } x\in F_n\\ 0 & \text{if } x\notin F_n,\end{cases} \qquad q_n(x)=\begin{cases} 0 & \text{if } x\in F_n\\ f(x) & \text{if } x\notin F_n.\end{cases}$$

Then $p_n(x)$ is Baire 1, $q_n(x)$ is measurable,

$$f(x) = p_n(x)+q_n(x) \quad (x\in\mathbf{R}), \tag{30}$$

$$\lambda(\{x;\ q_n(x) \neq 0\}) < \frac{1}{n} \tag{31}$$

and

$$|q_n(x)| \leqq |f(x)| \quad \text{for every} \quad x\in\mathbf{R}. \tag{32}$$

$$q_n(x+y)-q_n(x)-q_n(y) = [f(x+y)-f(x)-f(y)]-p_n(x+y)+p_n(x)+p_n(y)$$

is Baire 1 by our assumption. Hence

$$F_n(x, y) \overset{\text{def}}{=} -\operatorname{arctg}[q_n(x+y)-q_n(x)-q_n(y)]$$

is a bounded Baire 1 function and thus by Lemma 3, $G_n(x) \overset{\text{def}}{=} \int_0^1 F_n(x,y)\,dy$ is Baier 1, too. It follows from (31) that

$$\lambda(\{y;\ F_n(x, y) \neq \operatorname{arctg}[q_n(x)]\}) < \frac{2}{n}$$

for every $x\in\mathbf{R}$. Hence

$$|G_n(x)-\operatorname{arctg}[q_n(x)]| < \frac{2\pi}{n} \tag{33}$$

that is

$$G_n(x)-\frac{2\pi}{n} < \operatorname{arctg}[q_n(x)] < G_n(x)+\frac{2\pi}{n} \quad \text{for every } x. \tag{34}$$

Let

$$U_n = \left\{x;\ -\frac{\pi}{2} < G_n(x)-\frac{2\pi}{n} < G_n(x)+\frac{2\pi}{n} < \frac{\pi}{2}\right\}.$$

U_n is an F_σ set since G_n is Baire 1.

Let

$$a_n(x) \overset{\text{def}}{=} p_n(x)+\operatorname{tg}\left[G_n(x)-\frac{2\pi}{n}\right], \quad b_n(x) \overset{\text{def}}{=} p_n(x)+\operatorname{tg}\left[G_n(x)+\frac{2\pi}{n}\right] \quad (x\in U_n),$$

then by (30) and (34) we have

$$a_n(x) < f(x) < b_n(x) \quad (x\in U_n). \tag{35}$$

Since p_n and G_n are Baire 1 functions, hence a_n and b_n are Baire 1 functions, too (on the F_σ set U_n) and thus the level sets $\{x\in U_n;\ a_n(x)>c\}$ and $\{x\in U_n, b_n(x)<c\}$ are F_σ sets for every $c\in\mathbf{R}$ and $n=1, 2, \ldots$.

We prove that

$$\{x; f(x) < c\} = \bigcup_{n=1}^{\infty} \{x\in U_n;\ b_n(x) < c\} \tag{36}$$

for every $c\in\mathbf{R}$. The inclusion

$$\{x; f(x)<c\}\supset\bigcup_{n=1}^{\infty}\{x\in U_n;\ b_n(x)<c\}$$

is obvious by (35). Suppose $f(x)<c$ and let $|f(x)|=A$. It follows from (32) that

$$|\operatorname{arctg}[q_n(x)]|\leqq \operatorname{arctg} A \overset{\text{def}}{=} \frac{\pi}{2}-\varepsilon$$

for every n. Hence by (33) we have

$$-\frac{\pi}{2}+\frac{\varepsilon}{2}<G_n(x)-\frac{2\pi}{n}<G_n(x)+\frac{2\pi}{n}<\frac{\pi}{2}-\frac{\varepsilon}{2} \tag{37}$$

if $n\geqq\frac{8\pi}{\varepsilon}$ and thus $x\in U_n$ for $n\geqq\frac{8\pi}{\varepsilon}$. (37) implies

$$\lim_{n\to\infty}(b_n(x)-a_n(x))=\lim_{n\to\infty}\left(\operatorname{tg}\left[G_n(x)+\frac{2\pi}{n}\right]-\operatorname{tg}\left[G_n(x)-\frac{2\pi}{n}\right]\right)=0$$

since $\operatorname{tg}(x)$ is uniformly continuous on the interval $\left[-\frac{\pi}{2}+\frac{\varepsilon}{2},\frac{\pi}{2}-\frac{\varepsilon}{2}\right]$. Hence $\lim\limits_{n\to\infty} a_n(x)=\lim\limits_{n\to\infty} b_n(x)=f(x)$ by (35), consequently $b_n(x)<c$ if n is large enough. Thus we have

$$\{x; f(x)<c\}\subset\bigcup_{n=1}^{\infty}\{x\in U_n;\ b_n(x)<c\}$$

and (36) is proved.

Hence $\{x; f(x)<c\}$ is an F_σ set for every c. The same argument shows that $\{x; f(x)>c\}$ is F_σ, too which proves that f is Baire 1, q.e.d.

Now Theorems 5 and 6 immediately imply

THEOREM 7. *Suppose that $f(x+y)-f(x)-f(y)$ is Baire α (as a function of two variables). Then there are a Baire α function $g\colon \mathbf{R}\to\mathbf{R}$ and an additive function H such that $f(x)=g(x)+H(x)$.*

For, by Theorem 5, f is of the form $g+H$ where g is measurable and H is additive. Since

$$g(x+y)-g(x)-g(y)=f(x+y)-f(x)-f(y)$$

is Baire α, Theorem 6 gives that g is Baire α, too.

We remark that de Bruijn's theorem on the class of continuous functions can be deduced from Theorem 7. Indeed, suppose that $f(x+h)-f(x)$ is continuous for every h. Then the function $f(x+y)-f(x)-f(y)$ is separately continuous in both variables. Then, by Baire's theorem, it is a Baire 1 function (as a function of two variables) and thus, by Theorem 7, there are a Baire 1 function g and an additive function H such that $f=g+H$. Let x_0 be a point of continuity of the function g. Then the continuity of $g(x+h)-g(x)=f(x+h)-f(x)-H(h)$ implies that g is continuous at x_0+h. Since h is arbitrary, g is continuous everywhere. Obviously, this proof is much more complicated than de Bruijn's. Yet, the same argument applies for the following theorems.

THEOREM 8. *The class of approximately continuous functions has the difference property.*

PROOF. Suppose that $f(x+h)-f(x)$ is approximately continuous for every h. Then $f(x+y)-f(x)-f(y)$ is separately approximately continuous and hence, by a theorem of R. O. DAVIES [7], it is measurable. Applying Theorem 5 we get $f=g+H$ where g is measurable and H is additive. It is well-known that every measurable function is approximately continuous almost everywhere (see [10], Theorem 5.9, p. 118). Let x_0 be a point at which g is approximately continuous. Then $g(x+h)= =g(x)+[f(x+h)-f(x)]-H(h)$ implies that g is approximately continuous at x_0+h. Since h is arbitrary, g is approximately continuous everywhere.

THEOREM 9. *Let f: $\mathbf{R}\to\mathbf{R}$ be a bounded function for which $f(x+h)-f(x)$ is a derivative for every h. Then f is a derivative.*

PROOF. The function $F(x, y)=f(x+y)-f(x)-f(y)$ is bounded and for every fixed x_0, y_0, the functions $F(x_0, y)$ and $F(x, y_0)$ are derivatives. Then, according to a theorem of Z. GRANDE [9], F is measurable. Applying Theorem 5 we get $f=g+H$ where g is measurable and H is additive. Since $H=f-g$ is bounded on a set of positive measure, H is linear (see [12]) and thus f is measurable. Now the assertion of Theorem 9 immediately follows from the following

THEOREM 10. *Suppose that $f(x+h)-f(x)$ is a derivative for every h. If there exists an interval on which f is measurable and summable, then f is a derivative.*

PROOF. Suppose that f is summable on $[a, b]$ and let $F(x)=\int_a^x f(t)\,dt$ $(x\in[a, b])$. We can choose a point $x_0\in(a, b)$ such that $F'(x_0)=f(x_0)$ (since $F'(x)= =f(x)$ holds a.e. in (a, b)). Let $0<h<b-x_0$ be fixed and let $G(x)$ be a primitive of $f(x+h)-f(x)$. Then $G'(x)$ is summable on the interval $[a, b-h]$ and hence $G(x)-G(a)=\int_a^x G'(t)\,dt$ for every $x\in[a, b-h]$ (see [10], Theorem 6.6, p. 143). That is, for $x\in[a, b-h]$ we have

$$G(x)-G(a)=\int_a^x [f(t+h)-f(t)]\,dt=\int_a^{x+h} f(t)\,dt-$$

$$-\int_a^x f(t)\,dt-\int_a^{a+h} f(t)\,dt=F(x+h)-F(x)-F(a+h).$$

This implies

$$F'(x_0+h)=G'(x_0)+F'(x_0)=[f(x_0+h)-f(x_0)]+f(x_0)=f(x_0+h).$$

Since $h\in(0, b-x_0)$ was arbitrary, we have $F'(x)=f(x)$ $(x\in(x_0, b))$. Hence $f(x+h)$ has a primitive on $(x_0-h, b-h)$ and the same holds for

$$f(x)=f(x+h)-[f(x+h)-f(x)].$$

That is, f has a primitive on every open interval of length $b-x_0$. This easily implies that f has a primitive everywhere, q.e.d.

REMARKS. Let B_α denote the class of Baire α functions defined on $\mathbf{R}$. Since every measurable function equals a Baire 2 function a.e., it follows from Theorem 3 that the classes B_α have the weak difference property for $\alpha \geqq 2$. On the other hand, assuming the continuum hypothesis, the classes B_α do not have the difference property for $\alpha \geqq 2$. This can be shown by the same example as in the case of measurable functions; for if the set $\{x;\ S(x+h) \neq S(x)\}$ is countable for every h then $S(x+h)-S(x) \in B_2$ for every h (see Section 1). This raises the following

PROBLEM 1. *Has the class B_1 the difference property?*[3]

We remark here that *if B_1 has the difference property, then so is the class of derivatives.* Indeed, suppose that $f(x+h)-f(x)$ is a derivative for every h. Then, by assumption $f=g+H$ where $g \in B_1$ and H is additive. Then $g(x+h)-g(x)= =[f(x+h)-f(x)]-H(h)$ is a derivative for every h. On the other hand, g has a point of continuity and in a sufficiently small neighbourhood of this point g is measurable and bounded. Hence, by Theorem 10, g is a derivative.

PROBLEM 2. Suppose that $f(x+h)-f(x)$ is Borel measurable for every h. Is it true that the functions $f(x+h)-f(x)$ belong to the *same* Baire class of order $\alpha<\omega_1$? (In the example above, $\alpha=2$.)

Now we prove that (assuming the continuum hypothesis) there exists a Lebesgue measurable function $S(x)$ such that $S(x+h)-S(x) \in B_2$ for every h and S is not Borel measurable. This means that Theorem 6 fails to remain valid if we replace the condition "$f(x+y)-f(x)-f(y)$ is Baire α" by "$f(x+h)-f(x)$ is Baire α for every h".

Let $\{a_\alpha\}_{\alpha<\omega_1}$ be a well-ordering of $\mathbf{R}$. Let U be an everywhere dense G_δ set of measure zero and let $\{P_\alpha\}_{\alpha<\omega_1}$ be a well-ordering of the family of perfect subsets of U. Let G_α denote the additive group generated by the set $\{a_\beta;\ \beta<\alpha\}$. Then G_α is countable for every $\alpha<\omega_1$ and $G_0=\{0\}$. Let $p_0 \in P_0$ and $x_0 \in U \setminus \{p_0\}$ be arbitrary and put $H_0=\{x_0\}$.

Suppose that $\alpha>0$ and the points p_β and the countable sets H_β have been defined for every $\beta<\alpha$. Then the set

$$A = \bigcup_{\beta<\alpha} H_\beta \cup \{p_\beta;\ \beta<\alpha\}$$

is countable. Let $p_\alpha \in P_\alpha \setminus A$. $V=U \setminus (A \cup \{p_\alpha\})$ is an everywhere dense G_δ set and hence so is $V'=\bigcap_{h \in G_\alpha} (V+h)$. Let $x_\alpha \in V'$ and define $H_\alpha=G_\alpha+x_\alpha$. Hence the points $p_\alpha \in P_\alpha$ and the countable sets H_α are defined for every $\alpha<\omega_1$. Let $X \overset{\text{def}}{=} \bigcup_{\alpha<\omega_1} H_\alpha$. It is easy to see that

(38) $X \subset U$ and hence $\lambda(X)=0$,
(39) $p_\alpha \notin X$ $(\alpha<\omega_1)$ and hence X does not contain any perfect set,
(40) X has the cardinality $\aleph_1$ and
(41) $(X+h) \setminus X$ is countable for every $h \in R$.

[3] Meanwhile I succeeded in giving an affirmative answer.

(We remark that a set with the properties (38), (40) and (41) was constructed in [16], Theorème I.) Then X is not a Borel set by (40) and (39) (see [14], p. 355). Hence the function

$$S(x)=\begin{cases}1, & x\in X\\ 0, & x\notin X\end{cases}$$

is not Borel measurable. On the other hand, S is Lebesgue measurable by (38) and $\{x;\ S(x+h)\neq S(x)\}$ is countable for every h by (41).

PROBLEM 3. *Let f be Borel measurable and suppose that $f(x+h)-f(x)$ is of class α for every h. Does it follow that f is of class α, too?*

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