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BESICOVITCH TYPE MAXIMAL OPERATORS AND APPLICATIONS TO FOURIER ANALYSIS

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Introduction and Summary

Averaging over tubes of unit length and width $\delta < 1$ leads to two maximal functions $f_\delta^*, f_\delta^{**}$. The first is defined on the sphere S_{d-1} in $\mathbf{R}^d$ and for a fixed direction one takes the maximum over all translates of the tube (the Kakeya maximal function). The second is defined on $\mathbf{R}^d$ and at a given point x one considers all δ-tubes centered at x and varying direction (the Nikodym maximal function). It is a natural conjecture that the L^p-bound on $f_\delta^*, f_\delta^{**}$ is essentially given by the formula

$$\left(\frac{1}{\delta}\right)^{\frac{d}{p}-1+\varepsilon} \tag{0.1}$$

for $1 \leq p \leq d$. This fact is verified for $d = 2$ but open in higher dimensions. In particular, it seems unknown whether a set in $\mathbf{R}^3$ containing a line in every direction needs to have full Hausdorff dimension (in the paper, one gets a lower bound $\frac{7}{3}$). The estimate (0.1) is verified here if $p < p(d)$, where

$$\frac{d+1}{2} < p(d) < \frac{d}{2} + 1 \tag{0.2}$$

is some exponent given by a recursive formula; $p(3) = \frac{7}{3}$. This fact enables one to obtain some new results on Radon transforms, for instance:

The function $f^*(L)$ defined on $G_2(\mathbf{R}^4)$, obtained by taking the maximum over all averages over translates of L, is bounded on L^p for $p > 2$ (we restrict ourselves to functions with a support contained in the unit ball). In particular there are no so-called Besicovitch (4,2)-sets (cf. [Fa1]). In general, it is shown that there are no Besicovitch (d,k)-sets provided the relation

$$d \leq 2^{k-1} + k \tag{0.3}$$

holds.

Relations between these geometric problems and Fourier analysis (restriction and multiplier problems) appear for instance in the work of C. Fefferman [Fe2]. His argument implies that the boundedness of the multipliers m_λ defined by

$$\begin{cases} (1-|\xi|^2)^\lambda & |\xi| \leq 1 \\ 0 & |\xi| > 1 \end{cases} \tag{0.4}$$

on $L^p(\mathbf{R}^d)$ for $\lambda > 0$, $\frac{2d}{d+1+2\lambda} < p < \frac{2d}{d-1-2\lambda}$ (i.e. the spherical multiplier conjecture) has as formal consequence the dimension property

$$\dim A = d \ , \tag{0.5}$$

whenever A is a Nikodym set in $\mathbf{R}^d$.

Similarly, a restriction theorem such as

$$\|\hat{f}|_S\|_{L^1(S_{d-1})} \leq C\|f\|_{L^p(\mathbf{R}^d)} \quad \text{for} \quad 1 \leq p < \frac{2d}{d+1} \tag{0.6}$$

would imply (0.5) if A is a Besicovitch $(d,1)$ set. Our aim is to try to reverse these considerations and derive certain facts in Fourier Analysis from our partial knowledge of the (0.1)- estimate. It is shown that (0.6) holds for certain

$$\frac{2d+2}{d+3} < p < \frac{2d}{d+1} \tag{0.7}$$

The exponent $\frac{2d+2}{d+3}$ is the (sharp) exponent for which an L^2-restriction holds. In particular, for $d = 3$, one gets (0.6) for $p < \frac{31}{23}$ $(> \frac{4}{3})$.

The last section of the paper deals with applications to the spherical multipliers (0.4).Given $\lambda > 0$, the interval $\frac{2d}{d+1+2\lambda} < p < \frac{2d}{d-1-2\lambda}$ is the optimal range where m_λ may be bounded on $L^p(\mathbf{R}^d)$ ([He]). If $\lambda \geq \lambda(d) = \frac{d-1}{2(d+1)}$, it is known that the spherical multiplier conjecture is valid, as a consequence of the L^2-restriction theory (see [Fe1]). We will verify here this conjecture for certain $\lambda < \lambda(d)$. For instance, if $d = 3$, m_λ is bounded if

$\lambda > 1 - \frac{3}{p}$ and $\frac{127}{32} \le p \vee p' \le 4$. This statement is not covered by the L^2-restriction theorem. The technique is related to A. Cordoba's paper [Co].

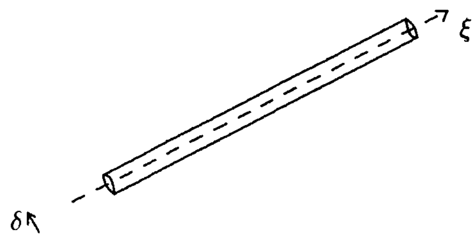
This work is rather the beginning of a certain investigation than something final. In particular, several arguments in section 6 and 7 may be improved at the cost of additional work which would lead to better numerical statements.

The author wishes to thank M. Christ and T. Wolff for discussions on the subject.

The letter C will stand for various constants.

1. Estimates on the 3-dimensional Kakeya Maximal Function

Let $\xi \in S_2$ be a unit vector and $\delta > 0$. By a (ξ, δ)-tube in $\mathbf{R}^3$, we mean a cylinder τ of unit length in direction ξ and of thickness δ.



For f a locally integrable function of $\mathbf{R}^3$, define

$$f_\delta^*(\xi) = \sup_\tau \frac{1}{|\tau|} \int_\tau f(x)dx \tag{1.1}$$

where the supremum refers to all (ξ, δ)-tubes τ and $|\tau|$ stands for the measure of τ, i.e. $|\tau| \sim \delta^2$. By Kakeya maximal function, we mean a function of the form (1.1) (although other related definitions are possible). In this section, we will obtain certain a priori bounds on the f_δ^*, of the form

$$\|f_\delta^*\|_{L^p(S_2)} \le K_3(p, \delta)\|f\|_{L^p(\mathbf{R}^3)} \, . \tag{1.2}$$

Remarks:

(1.3) Inequality (1.2) is a local problem in the sense that there is no restriction in assuming that f is supported by the unit ball in $\mathbf{R}^3$. The reader will easily verify this fact.

(1.4) The case $p = 2$ is well-understood (in any dimension). One has

$$K_2(2,\delta) \sim \left(\log \frac{1}{\delta}\right)^{1/2} \quad \text{(in dimension 2)} \tag{1.5}$$

$$K_d(2,\delta) \sim \left(\frac{1}{\delta}\right)^{\frac{d-2}{2}} \quad \text{(in dimension } d\text{)} \tag{1.6}$$

These bounds may be obtained by using Fourier Analysis only, without further geometric considerations (see [Fa1], section 7 and related references).

(1.7) It is a conjecture that

$$K_d(p,\delta) = O(\delta^{-\varepsilon}) \quad \text{for} \quad p \geq d\ . \tag{1.8}$$

This would imply that a set in $\mathbf{R}^d$ containing a line segment in every direction has Hausdorff dimension d. This is an open problem, already for $d = 3$. It is directly related to certain conjectures on exponential sums (Montgomery's conjectures) and problems in Fourier analysis (behavior of the restriction of Fourier transforms to surfaces). See [Bo1] and [Fe1] for these matters. The relevance to Fourier Analysis will also be discussed in later sections of this paper.

Presently (as will be shown in this section) I know to show a lower estimate $\frac{7}{3}$ for the Hausdorff dimension of a set in $\mathbf{R}^3$ containing a line segment in every direction. Also the right bound on $K_3(p,\delta)$ for $2 \leq p \leq \frac{7}{3}$ will be obtained (given by interpolation between $p = 2$, i.e. (1.6) and $p = 3$, i.e. (1.8)). Thus

$$K_3(p,\delta) = O\left(\left(\frac{1}{\delta}\right)^{\frac{3}{p}-1+\varepsilon}\right) \quad \text{for} \quad 2 \leq p \leq \frac{7}{3}\ . \tag{1.9}$$

(1.10) This work is mainly meant as progress on well-known problems. Several results are indeed not final and only improve on what is given by interpolation.

Inequality (1.5) will be used and we include a proof. (An analogous argument works for (1.6) as well.)

Proof of (1.5): Denote

$$\widehat{f}(\lambda) = \int f(x) e^{-2\pi i \langle x, \lambda \rangle} dx \tag{1.11}$$

the Fourier transform of f. Consider a function $\varphi \in \mathcal{S}(\mathbf{R})$ satisfying

$$\left.\begin{array}{c} 0 \le \varphi \le 2 \\ \varphi \ge 1 \quad \text{on} \quad [0,1] \\ \widehat{\varphi} \ge 0 \ , \quad \widehat{\varphi} \ \ \text{supported by} \quad [-10, 10] \end{array}\right\} \tag{1.12}$$

Clearly the indicator function of the $[0,1] \times [0,\delta]$- rectangle in $\mathbf{R}^2$ is bounded by the function

$$\delta \cdot \psi(x) \equiv \varphi(x_1)\varphi(\delta^{-1} x_2) \tag{1.13}$$

Hence, for $\xi = Oe_2$, $O \in \mathrm{SO}(2)$, we have

$$\begin{aligned} f_\delta^*(\xi) &\le \sup_x \Big| \int f(y)\psi(x - O^{-1}y) dy \Big| \\ &= \sup_x \Big| \int \widehat{f}(\lambda)\widehat{\psi}(O^{-1}\lambda) e^{2\pi i \langle x, O^{-1}\lambda \rangle} d\lambda \Big| \\ &\le \int |\widehat{f}(\lambda)| |\widehat{\psi}(O^{-1}\lambda)| d\lambda \ . \end{aligned} \tag{1.14}$$

Observe that by (1.12), (1.13), $\widehat{\psi}(\lambda) = 0$ unless $|\lambda| < c \cdot \delta^{-1}$. Hence, (1.14) is bounded by $\int_0^{c\cdot\delta^{-1}} \int_0^{2\pi} |\widehat{f}(r \cdot \xi)| \widehat{\varphi}\big(r \cdot \cos(\psi - \theta)\big) r \, dr \, d\psi$, where

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.15}$$

and

$$\xi = (\cos\psi \ , \ \sin\psi) \tag{1.16}$$

By the preceding and Hölder's inequality

$$\|f_\delta^*\|_{L^2(S_1)}^2 \le \left\{ \int_0^{c\cdot\delta^{-1}} \int_0^{2\pi} \int_0^{2\pi} \big|\widehat{f}(r \cdot \xi)\big|^2 \widehat{\varphi}\big(r \cdot \cos(\psi - \theta)\big) r^2 dr \, d\theta \, d\psi \right\}$$

$$\left\{ \int_0^{c\cdot\delta^{-1}} \int_0^{2\pi} \widehat{\varphi}(r \cdot \cos\psi) dr \, d\psi \right\} \tag{1.17}$$

$$\leq c\left\{\int_0^{c\cdot\delta^{-1}} \frac{1}{1+r}dr\right\}\left\{\int_0^{c\cdot\delta^{-1}}\int_0^{2\pi} \left|\widehat{f}(r\cdot\xi)\right|^2 r\,dr\,d\psi\right\} \tag{1.18}$$

$$\leq c\cdot\left(\log\frac{1}{\delta}\right)\|\widehat{f}\|_2^2 \tag{1.19}$$

Hence

$$\|f_\delta^*\|_2 \leq c\cdot\left(\log\frac{1}{\delta}\right)^{1/2}\|f\|_2 \tag{1.20}$$

Using the general results from [St], (1.9) will follow from a distributional inequality (corresponding to the case $p=\frac{7}{3}$ in (1.9))

$$|A| \geq c\cdot\delta^{\frac{4}{3}+\varepsilon}\sigma^{\frac{7}{3}}\left|\left\{(\chi_A)_\delta^* > \sigma\right\}\right| \tag{1.21}$$

for A a subset of $B(0,1)$ and $0\leq\sigma\leq 1$. Considering the action of the orthogonal group, (1.21) is also equivalent to the following statement:

$$|A| > c\cdot\delta^{\frac{4}{3}+\varepsilon}\sigma^{\frac{7}{3}} \quad \text{provided} \quad \left\{(\chi_A)_\delta^* > \sigma\right\}\subset S_2 \quad \text{has measure} > \tfrac{1}{2} \tag{1.22}$$

(see again [St] for details).

Denote $\mathbf{E}$ the averaging operator on cubes of size δ. Clearly

$$(\chi_A)_\delta^* \leq c\big(\mathbf{E}[\chi_A]\big)_\delta^* . \tag{1.23}$$

For $0\leq t\leq 1$, define the set

$$A_t = \left\{x\in\mathbf{R}^3 \;\middle|\; t\leq \mathbf{E}[\chi_A]\right\} . \tag{1.24}$$

These sets are unions of δ-cubes and

$$t|A_t| \leq |A| . \tag{1.25}$$

Write

$$\mathbf{E}[\chi_A] \leq \int_{c\sigma}^1 \chi_{A_t}dt + c\sigma$$

for an appropriate constant c,

$$\big(\mathbf{E}[\chi_A]\big)_\delta^* \leq \int_{c\sigma}^1 (\chi_{A_t})_\delta^* dt + c\sigma . \tag{1.26}$$

Hence, by (1.23) and an appropriate choice of constants

$$\{(\chi_A)^*_\delta > \sigma\} \subset \bigcup_{c\sigma < t < 1} \left\{(\chi_{A_t})^*_\delta > c\frac{\sigma}{\sqrt{t}}\right\} . \tag{1.27}$$

Assume inequality (1.21) holds for sets which are unions of δ- cubes. Letting t range over dyadic values, it then follows from (1.25)

$$\left|\{(\chi_A)^*_\delta > \sigma\}\right| \le c\left(\log\frac{1}{\sigma}\right)\delta^{-\frac{2}{3}-\varepsilon}\left(\frac{\sigma}{\sqrt{t}}\right)^{-\frac{7}{3}}\frac{1}{t}|A| \tag{1.28}$$

for some $t < 1$. Hence clearly

$$|A| \ge c\delta^{\frac{2}{3}+\varepsilon}\sigma^{\frac{7}{3}}\left|\{(\chi_A)^*_\delta > \sigma\}\right| . \tag{1.21}$$

Therefore it suffices to prove (1.21) or (1.22) assuming A a union of δ-cubes. The argument is of combinatorial nature.

Proof of (1.22) for A a union of δ-cubes: We start by constructing inductively a sequence of "bushes", i.e. collections of δ-tubes having a common point. Let $A \subset B(0,1)$ such that

$$\mathcal{D} \equiv \{\xi \in S^2 \mid (\chi_A)^*_\delta > \sigma\} \quad \text{satisfies} \quad |\mathcal{D}| > \tfrac{1}{2} \tag{1.29}$$

(we may clearly assume $\sigma > \delta$).

Let $\mathcal{E}$ be a $\frac{10\delta}{\sigma}$-separated set in $\mathcal{D}$ with

$$\#\mathcal{E} \sim \left(\frac{\sigma}{\delta}\right)^2 . \tag{1.30}$$

For $\xi \in \mathcal{E}$, there is a (ξ, δ)-tube τ_ξ such that

$$|A \cap \tau_\xi| > \sigma\delta^2 . \tag{1.31}$$

Hence

$$\int_A \left(\sum_{\xi\in\mathcal{E}} \chi_{\tau_\xi}\right) > \sigma^3 \tag{1.32}$$

and therefore there is a subset $\mathcal{F}_0$ of $\mathcal{E}$ and a point $x_0 \in A$ such that

$$x_0 \in \tau_\xi \quad \text{for} \quad \xi \in \mathcal{F}_0 \tag{1.33}$$

$$\#\mathcal{F}_0 > \frac{\sigma^3}{|A|} \,. \tag{1.34}$$

Define

$$B_0 = \bigcup_{\xi\in\mathcal{F}_0} \tau_\xi \,. \tag{1.35}$$

Observe that

$$\left|A \cap \left(\tau_\xi \backslash B\left(x_0, \frac{\sigma}{3}\right)\right)\right| > \frac{\sigma}{3}\delta^2 \tag{1.36}$$

for $\xi \in \mathcal{F}_0$, where the $\left(\tau_\xi\backslash B(x_0, \frac{\sigma}{3})\right)_{\xi\in\mathcal{F}_0}$ are mutually disjoint, by definition of $\mathcal{E}$. Consequently

$$|B_0| \le \frac{3}{\sigma}|A\cap B_0| \,. \tag{1.37}$$

Define

$$A' = A\backslash B_0 \tag{1.38}$$

and

$$\mathcal{D}_1 = \left\{\xi \in S^2 \,\middle|\, (\chi_{A'})^*_\delta > \frac{\sigma}{10}\right\} . \tag{1.39}$$

If $|\mathcal{D}_1| < \frac{1}{10}$, this ends the construction. Otherwise, one may repeat previous construction to get a point x_1, a set $\mathcal{F}_1$ of directions such that

$$x_1 \in \tau_\xi \quad \text{for} \quad \xi \in \mathcal{F}_1 \tag{1.40}$$

$$\#\mathcal{F}_1 \gtrsim \frac{\sigma^3}{|A_1|} > \frac{\sigma^3}{|A|} \tag{1.41}$$

and

$$|B_1| \lesssim \frac{1}{\sigma}|A'\cap B_1| \tag{1.42}$$

where

$$B_1 = \bigcup_{\xi\in\mathcal{F}_1} \tau_\xi \,. \tag{1.43}$$

Define then

$$A_1' = A\backslash(B_0\cup B_1) \tag{1.44}$$

and start again.

The construction stops after s steps, where (1.34), (1.35), (1.37), (1.41), (1.42), (1.43) etc. yields the bound

$$\frac{|A|}{\sigma} \gtrsim \sum_{t\le s} |B_t| \gtrsim s\cdot\frac{\sigma^3}{|A|}\delta^2 \,. \tag{1.45}$$

i.e.

$$s \lesssim \frac{|A|^2}{\sigma^4}\delta^{-2} . \tag{1.46}$$

Also, by construction, the set

$$\overline{A} = \bigcup_{t=0}^{s}(A \cap B_t) \tag{1.47}$$

satisfies

$$(\chi_{\overline{A}})^*_\delta(\xi) > \frac{\sigma}{2} \tag{1.48}$$

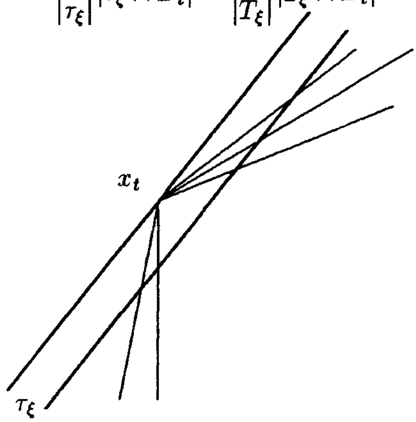
for $\xi \in \overline{\mathcal{D}}$, where $\overline{\mathcal{D}} \subset S_2$ has measure $> \frac{1}{4}$.

It follows in particular that for each $\xi \in \overline{\mathcal{D}}$ one may find a (ξ, δ)-tube τ_ξ such that

$$\sum_{t=0}^{s} |\tau_\xi \cap B_t| > \frac{\sigma}{2}\delta^2 . \tag{1.49}$$

Fix $t = 0, \dots, s$. Denote T_ξ the parallellepipedum generated by τ_ξ and its translate centered at x_t. Because of the geometry of B_t, one has that

$$\frac{1}{|\tau_\xi|}|\tau_\xi \cap B_t| \sim \frac{1}{|T_\xi|}|T_\xi \cap B_t| \tag{1.50}$$



It is natural at this point to consider the maximal function

$$\mathcal{M}_\delta f(\xi) = \sup_T \frac{1}{|T|}\int_T f(x)dx \tag{1.51}$$

where the supremum ranges over parallellepipeds T obtained as the δ-neighborhood of some rectangle axed along ξ of dimensions $1, \rho$, $\delta \leq \rho \leq 1$. Here $\xi \in S_2$.

We will use the following estimate, the proof of which is postponed.

LEMMA 1.52. *$\|\mathcal{M}_\delta f\|_2 \lesssim r^{-1/2}\delta^{-\varepsilon}\|f\|_2$ if $\operatorname{supp} f \subset B(0,2r)\backslash B(0,r)$ $(0 < r < 1)$.*

It follows from (1.49),(1.50) that for $\xi \in \overline{\mathcal{D}}$

$$\sigma \lesssim \sum_{t\leq s} \sum_{\substack{\delta<r<1 \\ r \text{ dyadic}}} \mathcal{M}_\delta(\chi_{B_t^r}) \tag{1.53}$$

where

$$B_t^r = (B_t - x_t) \cap \left[B(0,2r)\backslash B(0,r)\right] . \tag{1.54}$$

Hence

$$\sigma^2 \lesssim s \log\frac{1}{\delta}\sum_t\sum_r \mathcal{M}_\delta(\chi_{B_t^r})^2 \tag{1.55}$$

and integrating (1.56) on $\overline{\mathcal{D}}$ yields by (1.52)

$$\sigma^2 \lesssim s\delta^{-\varepsilon}\sum_t\sum_r \frac{1}{r}|B_t^r| \,. \tag{1.56}$$

Observe that again by the geometry of B_t

$$|B_t^r| \lesssim r\cdot|B_t| \,. \tag{1.57}$$

Thus, by (1.45),(1.46)

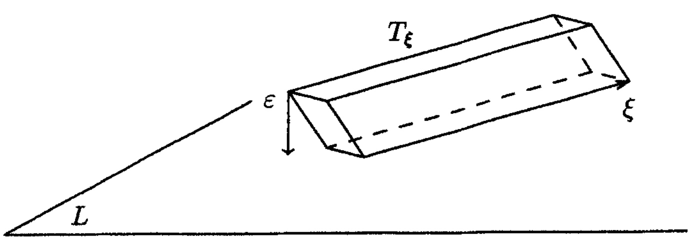
$$\sigma^2 \lesssim s\delta^{-\varepsilon}\frac{|A|}{\sigma} \lesssim \delta^{-2-\varepsilon}\frac{|A|^3}{\sigma^5} \tag{1.58}$$

which is inequality (1.22).

Proof of Lemma 1.52: We will use the Kakeya maximal function estimate in dimension 2. Fix a plane L through 0 with normal vector e_L. If ξ is in the unit sphere $S_L \sim S_1$ of L and T_ξ the associated parallellepiped, one has

$$\frac{1}{|T_\xi|}\int_{T_\xi} f \lesssim \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} (f|_{L+te_L})_\delta^*(\xi)\,dt \tag{1.59}$$

where ε depends on the position of T_ξ with respect to L.



Hence, using the bound on g_δ^* in a 2-plane

$$|\mathcal{M}f(\xi)|^2 \lesssim \log\frac{1}{\delta} \sum_{\substack{\varepsilon>\delta \\ \varepsilon \text{ dyadic}}} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \left[(f|_{L+te_L})_\delta^*(\xi)\right]^2 dt \tag{1.60}$$

$$\int_{S_L} |\mathcal{M}f|^2 \lesssim \left(\log\frac{1}{\delta}\right)^2 \sum_{\varepsilon} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \left(\int |f(x)|^2 dL_t(x)\right) dt \tag{1.61}$$

denoting $L_t = L + te_L$.

Integration (1.61) over the Grassmannian $G(3,2)$ in L easily yields

$$\|\mathcal{M}f\|_2^2 \lesssim \left(\log\frac{1}{\delta}\right)^2 \sum_{\varepsilon} \frac{1}{\varepsilon}\int_G \left(\int_{D_{L,\varepsilon}} |f(x)|^2 dx\right) dL \tag{1.62}$$

$$\lesssim \left(\log\frac{1}{\delta}\right)^3 \int |f(x)|^2 \frac{dx}{|x|} \tag{1.63}$$

$$\sim \left(\log\frac{1}{\delta}\right)^3 r^{-1}\|f\|_2^2 \,. \tag{1.64}$$

Here $D_{L,\varepsilon}$ denotes and ε-neighborhood of the annulus $\left[x \in L \mid r < |x| < 2r\right]$. This proves Lemma 1.52.

2. Estimates on the Kakeya Maximal Function in Dimension > 3

It is an easy exercise to see that a set in $\mathbf{R}^d$ containing a line segment in every direction has Hausdorff dimension $\geq \frac{d+1}{2}$. We may improve this lower bound by using the same technique as for $d = 3$ (previous section). In fact, we will prove the "correct" estimate (assuming $p \leq d$)

$$K_d(p,\delta) \lesssim \left(\frac{1}{\delta}\right)^{\frac{d}{p}-1+\varepsilon} \tag{2.1}$$

provided p is in the range

$$2 \leq p \leq p(d) \tag{2.2}$$

where $\frac{d+1}{2} < p(d) < \frac{d+2}{2}$ will be specified in what follows.

The argument is based on induction. We essentially repeat the construction of the previous section, replacing (1.5) by the bound on some $K_{d-1}(p,\delta)$.

The analogue of (1.22) is now

$$|A| > c \cdot \delta^{d-p+\varepsilon}\sigma^p \tag{2.3}$$

where $A \subset B(0,1)$ is a union of δ-cubes such that $\{(\chi_A)^*_\delta > \sigma\} \subset S_{d-1}$ has measure $> \frac{1}{2}$. We then repeat the construction leading to the proof of (1.22). Here

$$\#\mathcal{E} \sim \left(\frac{\sigma}{\delta}\right)^{d-1} \tag{2.4}$$

$$\int_A \Big(\sum_{\xi\in\mathcal{E}} \chi_{\tau_\xi}\Big) > \sigma^d \tag{2.5}$$

$$\#\mathcal{F}_t > \frac{\sigma^d}{|A|} \, . \tag{2.6}$$

Inequality (1.45) becomes now

$$\frac{|A|}{\sigma} \gtrsim \sum_{t=0}^{s} |B_t| \gtrsim s\frac{\sigma^d}{|A|}\delta^{d-1} \, . \tag{2.7}$$

The T_ξ are defined as in the 3-dimensional case and (1.50) holds. Lemma 1.52 gets replaced by the inequality

$$\|\mathcal{M}_\delta f\|_q \lesssim r^{-1/q}\left(\frac{1}{\delta}\right)^{\frac{d-1}{q}-1+\varepsilon} \tag{2.8}$$

provided

$$\operatorname{supp} f \subset B(0,2r)\backslash B(0,r) \, .$$

The proof of (2.8) is analogous to (1.52) up to the L^q-bound of g^*_δ in dimension $d-1$, where one invokes the induction hypothesis.

Inequality (1.56) becomes

$$\sigma^q \lesssim s^{q-1}\left(\frac{1}{\delta}\right)^{d-1-q+\varepsilon} \sum_{t\leq s} \sum_{\substack{\delta<r<1 \\ r \text{ dyadic}}} \frac{1}{r}|B^r_t| \, . \tag{2.9}$$

Hence, by (2.7),

$$s^{q-1}\left(\frac{1}{\delta}\right)^{d-1-q+\varepsilon} \frac{|A|}{\sigma} \gtrsim \sigma^q \tag{2.10}$$

and substituting s

$$|A|^{2q-1}\left(\frac{1}{\delta}\right)^{(d-1)(q-1)+d-1-q+\varepsilon}\sigma^{-(d+1)(q-1)-1-q}\gtrsim 1\ . \tag{2.11}$$

This yields us

$$|A|\gtrsim \delta^{\frac{qd-2q}{2q-1}+\varepsilon}\sigma^{\frac{qd+2q-d}{2q-1}} \tag{2.12}$$

hence

$$p(d)=\frac{p(d-1)(d+2)-d}{2p(d-1)-1}\ . \tag{2.13}$$

For p satisfying (2.2), one has (2.1).

Thus one gets in particular

$$\begin{aligned} p(2)&=2\\ p(3)&=\frac{7}{3}\\ p(4)&=\frac{30}{11}\\ p(5)&=\frac{155}{49}\end{aligned} \tag{2.14}$$

Inequality (2.1) implies a lower estimate on the Hausdorff dimension as shown next.

LEMMA 2.15. *Suppose that for a given p, (2.1) holds for all $\delta > 0$. Then a set A in $\mathbb{R}^d$ containing a line segment in every direction has dimension $\geq p$.*

Proof: Let $D \subset S_{d-1}$ have positive measure and assume A contains a unit segment in every direction $\xi \in D$. Assume

$$A\subset\bigcup_{j\geq j_0}B_j \tag{2.16}$$

where B_j is a union of 2^{-j}-cubes. We have to show that for some $j \geq j_0$ the number of these cubes is at least

$$2^{j(p-\varepsilon)} \tag{2.17}$$

(letting j_0 be large enough). Take any sequence $(\eta_j)_{j\geq j_0}$ satisfying

$$\sum_{j\geq j_0} \eta_j < \frac{1}{100} \,. \tag{2.18}$$

It is clear from the definition of the B_j that for $\xi \in D$

$$\sum_{j\geq j_0} (\chi_{B_j})^*_{2^{-j+1}} > \frac{1}{10} \,. \tag{2.19}$$

Hence, for some $j \geq j_0$

$$\int_D (\chi_{B_j})^*_{2^{-j+1}} > \eta_j |D| \,. \tag{2.20}$$

The left member of (2.32) may then be evaluated as

$$\left\|(\chi_{B_j})^*_{2^{-j+1}}\right\|_p \leq K_d(p, 2^{-j+1})|B_j|^{1/p} \lesssim (2^j)^{\frac{d}{p}-1+\varepsilon}|B_j|^{1/p} \tag{2.21}$$

by (2.1). Hence

$$|B_j| \gtrsim |D|^p \eta_j^p (2^{-j})^{d-p+\varepsilon} \,. \tag{2.22}$$

Choosing η_j appropriately, (2.17) immediately follows.

The numbers (2.14) consequently yield lower bounds for the dimension of "Kakeya-sets" in various dimensions.

3. Averages over Planes in $\mathbf{R}^4$

We will show in this section that a measurable set in $\mathbf{R}^4$ containing a translate of any 2-plane (or simply intersecting the translate in a set of positive 2-dimensional Lebesgue measure) has positive measure. Other results of a similar nature will be obtained later on in the paper. In fact, one obtains also L^p-bounds on the maximal function associated to the corresponding Radon transform.

If L is a k-plane in $\mathbf{R}^d$ and f a bounded measurable function with bounded support, define

$$F(L) = \int_L f(x)dx \tag{3.1}$$

where dx stands for the k-dimensional Lebesgue measure on L. Consider the maximal function

$$f^*(L) = \sup_{x\in\mathbf{R}^d} F(x+L) \tag{3.2}$$

defined on the Grassmannian $G(d,k)$. The statement made at the beginning of this section may then be derived from the following analytical fact

PROPOSITION 3.3. *There is an a priori inequality*

$$\|f^*\|_{L^p_{G(4,2)}} \leq c(p)\|f\|_p \tag{3.4}$$

for $p > 2$ and assuming f supported by the unit ball in $\mathbf{R}^4$.

Remarks:

(3.5) The result of this section may be rephrased in the language of [Fa1] by saying that there are no $(4,2)$ "Besicovitch sets". The case $(3,2)$ was known and proved by various authors, using either geometric or Fourier Analysis techniques. The reader is referred to [Fa1] for a survey. In [Fa2] it is claimed that $(d,2)$ Besicovitch sets do not exist in any dimension d. Unfortunately, the proof is incorrect and although the statement is most likely true, the problem is open in general (the case $d \geq 5$ is open, to the best of the author's knowledge).

(3.6) In proving Proposition 3.3, we will rely on the $K_3(p,\delta)$ estimate (1.9) and standard Fourier transform techniques appearing in the study of Radon transforms. The method is thus based on both geometric considerations and Fourier Analysis. This idea is applicable to various similar problems (see later sections in the paper).

Proof of Proposition 3.3: Take $2 < p < \frac{7}{3}$ and f a bounded measurable function supported by the unit ball $B(0,1)$ of $\mathbf{R}^4$. Write

$$f = \sum_{j \geq 0} f_j \tag{3.7}$$

where $f_j = (f * \varphi_j) - (f * \varphi_{j-1})$, $\varphi_j(x) = 2^j \varphi(2^j x)$ and $\varphi \in \mathcal{S}(\mathbf{R})$ satisfies

$$\widehat{\varphi} = 1 \quad \text{on} \quad B(0,1)\ , \quad \operatorname{supp} \widehat{\varphi} \subset B(0,2)\ . \tag{3.8}$$

Fix $g = f_j$ and let $\delta = 2^{-j}$. For a unit vector ξ, define a Radon transform

$$\overline{g}_\xi(x) = \int_0^1 g(x + t\xi)dt \quad \text{for} \quad x \in [\xi]^\perp\ . \tag{3.9}$$

Let ξ, ξ' be perpendicular unit vectors in $\mathbf{R}^4$. It follows from the definition of g that

$$\left| \int_0^1 \int_0^1 g(x + t\xi + t'\xi')dt\, dt' \right| = \left| \int_0^1 \overline{g}_\xi(x + t'\xi')dt' \right| \lesssim \left(|\overline{g}_\xi|\right)^*_\delta(\xi')\ . \tag{3.10}$$

Consequently, also

$$g^*([\xi,\xi']) \lesssim (|\overline{g}_\xi|)^*_\delta(\xi') . \tag{3.11}$$

Fixing $\xi \in S_3$, denote $S_\xi \simeq S_2$ the 2-sphere $S_3 \cap [\xi]^\perp$. One has

$$\|g^*\|_{L^p(G_{4,2})} = \left(\int_{S_3}\int_{S_\xi} |g^*([\xi,\xi'])|^p \sigma_2(d\xi')\sigma_3(d\xi)\right)^{1/p} \tag{3.12}$$

where, by (1.9), applied in $[\xi]^\perp$, and (3.11)

$$\int_{S_\xi} |g^*([\xi,\xi'])|^p \sigma_2(d\xi') \lesssim \left(\frac{1}{\delta}\right)^{3-p+\varepsilon} \int_{[\xi]^\perp} |\overline{g}_\xi|^p dx . \tag{3.13}$$

Further

$$\int_{S_3}\int_{[\xi]^\perp} |\overline{g}_\xi|^p dx\, d\xi \lesssim \|g\|_\infty^{p-2} \cdot \int_{S_3}\int_{[\xi]^\perp} |\overline{g}_\xi|^2 dx\, d\xi \tag{3.14}$$

where, taking into account the support of $\widehat{g} = \widehat{f_j}$

$$\int_{S_3}\int_{[\xi]^\perp} |\overline{g}_\xi|^2 dx\, d\xi \lesssim \delta \|g\|_2^2 \tag{3.15}$$

(cf. the proof of (1.5) for this fact).

Hence, (3.14) may be estimated by

$$\delta \cdot \|f\|_\infty^{p-2} \|f\|_2^2 \tag{3.16}$$

and (3.12) is therefore bounded by

$$\delta^{1-\frac{2}{p}-\varepsilon} \|f\|_\infty^{1-\frac{2}{p}} \|f\|_2^{\frac{2}{p}} \tag{3.17}$$

which is summable over $\delta = 2^{-j}$, since $p > 2$. This yields an estimate

$$\|f^*\|_{L^p(G_{4,2})} \leq \sum_{j\geq 0} \|f_j^*\|_{L^p(G_{4,2})} \leq C_p \|f\|_\infty^{1-\frac{2}{p}} \|f\|_2^{\frac{2}{p}} . \tag{3.18}$$

Hence, for $2 < p < \frac{7}{3}$

$$\|f^*\|_{L^p(G_{4,2})} \leq C_p \|f\|_{p,1} . \tag{3.19}$$

Inequality (3.4) then follows from the results of [St] and interpolation.

A direct adaptation of the previous argument yields that there are no $(7,3)$ Besicovitch sets. More precisely, PROPOSITION 3.20. *There is an a priori inequality*

$$\|f^*\|_{L^p_{G(7,3)}} \leq C(p)\|f\|_p \tag{3.21}$$

for $p > 3$ and assuming f supported by the unit ball of $\mathbf{R}^7$.

Here we consider first $3 < p < \frac{155}{49} = p(5)$ (cf. (226)) and apply in 5-dimensional subspaces inequality (2.1) on the 2-dimensional Radon transform $\overline{g}_{[\xi,\xi']} = \int_0^1 \int_0^1 g(x + t\xi + t'\xi')dt\, dt'$ where $x \in [\xi,\xi']^\perp$. Moving the frame ξ, ξ', a gain of δ^2 is obtained, i.e.

$$\int_{G(7,2)} \int_{[\xi,\xi']^\perp} |\overline{g}_{[\xi,\xi']}|^2 dx\, d\xi\, d\xi' \lesssim \delta^2 \|g\|_2^2 \ . \tag{3.22}$$

The bound (2.1) now gives the estimate $\delta^{1-\frac{5}{p}+\frac{2}{p}-\varepsilon}$ on $\|f_j^*\|_p$ and hence $p > 3$ yields a convergent series.

Remarks:

(3.23) The condition $p > 2$ in Proposition 3.3 is optimal, as the standard example shows.

(3.24) When studying the (d,k)-properties for larger k, it is more efficient to apply geometric dimension estimates to sets containing translates of k'-cubes, $k' > 2$, rather than using a Kakeya-maximal function estimate on a $(k-1)$-Radon transform in dimension $d - k + 1$. Relevant geometrical estimates will be obtained later in the paper and the applications to (d,k)-properties discussed subsequently.

4. Further Results on the Besicovitch Property

By Besicovitch (d,k)-property, we mean the non-existence of (d,k) Besicovitch sets, i.e. the fact that sets in $\mathbf{R}^d$ containing a translate of any k-plane are of positive measure. The so-called Kakeya set in the plane disproves this $(2,1)$-property. The (d,k)-property for $k > \frac{d}{2}$ has been obtained by several authors and may be derived from the L^2-boundedness of the corresponding maximal function. In the previous section, we proved the $(4,2)$ and the $(7,3)$ property. We will obtain here further results of the same nature

PROPOSITION 4.1. *The (d,k) property holds for $d \leq 2^{k-1} + k$.*

The argument is again based on entropy estimates and Fourier Analysis and we follow the same scheme as before. We first develop the geometrical inequalities.

If A is a subset of a finite dimensional metric space X and $\varepsilon > 0$, denote

$$h_\varepsilon(A) = \frac{\log N(A,\varepsilon)}{\log \frac{1}{\varepsilon}} \tag{4.2}$$

where $N(A,\varepsilon)$ stands for the metrical entropy numbers of A, i.e. the minimum number of ε-balls needed to cover A. The following observation is elementary and left as an exercise to the reader. Let $f : X \to Y$ be a Lipschitz map between metric spaces and assume

$$h_\varepsilon\big(f^{-1}(y)\big) \geq \rho \tag{4.3}$$

for any $y \in Y$. Then

$$h_\varepsilon(X) \geq h_\varepsilon(Y) + \rho \,. \tag{4.4}$$

For $\sigma > 0$, define $\rho_\varepsilon(d,k,\sigma)$ as the lower bound on $h_\varepsilon(A)$ where A ranges over all subsets of $B(0,1) =$ the unit ball in $\mathbf{R}^d$ having the following property:

$$\sup_x \big|A \cap (x+L)\big| > \sigma \tag{4.5}$$

for all $L \in G(d,k)$ taken in a a subset of $G(d,k)$ of measure $> \frac{1}{2}$. Our purpose is to get a recursive estimate on the $\rho_\varepsilon(d,k,\sigma)$. Let A have the property considered above and $G \subset G(d,k)$ the set of measure $> \frac{1}{2}$ of subspaces L satisfying (4.5). Let $H = [e_1,\dots,e_{d-1}]$. Let $\xi \in S_{d-1}$, $\text{dist}(\xi,H) > c_d$ and $M \in G(H,k-1)$ such that $L = [M,\xi] \in G$. By (4.5), there is a translate M' of $M, M' \subset H$, such that

$$\big|A \cap [M',\xi]\big| > \sigma \,. \tag{4.6}$$

Thus, defining

$$\Gamma(\xi,M) = \big\{t \in [-1,1] \;\big|\; \text{mes}\,\big[y \in M' \;\big|\; \big(y+[\xi]\big) \cap (H+te_d) \in A\big] > c\sigma\big\}$$

(4.6) yields by a Fubini argument

$$\big|\Gamma(\xi,M)\big| > c\sigma \tag{4.8}$$

(we denote again by c various constants, which here may depend on d). The set $\mathcal{H} \subset S_{d-1} \times G(H,k-1)$ defined by

$$\mathcal{H} = \big\{\,\text{dist}(\xi,H) > c_d \quad \text{and} \quad [M,\xi] \in G\big\} \tag{4.9}$$

may be assumed for measure $> \frac{1}{3}$. Again by Fubini applied in $\mathcal{H} \times [-1,1]$, one gets from (4.8) a set $\Gamma \subset [-1,1]$ such that

$$|\Gamma| > c\sigma \tag{4.10}$$

and for $t \in \Gamma$

$$\big|\{(\xi, M) \in \mathcal{H} \mid t \in \Gamma(\xi, M)\}\big| > c\sigma \ . \tag{4.11}$$

One easily deduces from (4.10),(4.11) the existence of a sequence $(t_i)_{i \leq K}$ in Γ and a subset $\mathcal{K}$ of $\mathcal{H}$ such that the following holds

$$|t_i - t_j| > c\sigma K^{-1} \quad \text{for} \quad i \neq j \tag{4.12}$$

$$\#\{i \leq K \mid t_i \in \Gamma(\xi, M)\} > c\sigma K \quad \text{for all} \quad (\xi, M) \in \mathcal{K} \tag{4.13}$$

$$|\mathcal{K}| > c\sigma \ . \tag{4.14}$$

The number $K = K(\sigma)$ will be specified later.

Define $H_i = H + t_i e_d$ $(i \leq K)$ and consider the Lipschitz map

$$f : \bigcup_{i \neq j} \left[(H_i \cap A) \times (H_j \cap A)\right] \to S_{d-1} : (x, y) \mapsto \frac{x - y}{|x - y|} \ . \tag{4.15}$$

Define $\Omega \subset S_{d-1}$ as follows.

$$\Omega = \big\{\xi \in S_{d-1} \mid (\xi, M) \in \mathcal{K} \quad \text{for} \quad M \in \mathcal{K}_\xi \subset G(H, k-1) \quad \text{where} \quad |\mathcal{K}_\xi| > c\sigma\big\} \ . \tag{4.16}$$

Thus, by (4.14)

$$|\Omega| > c\sigma \ . \tag{4.17}$$

Fix $\xi \in \Omega$. Then, by construction, for $M \in \mathcal{K}_\xi$, there is a translate M' of M, $M' \subset H$ such that $\text{mes}\,\big[y \in M' \mid (y + [\xi]) \cap H_i \in A\big] > c\sigma$ for $c\sigma K$ i's. Hence, taking $K \sim \sigma^{-2}$, it follows that

$$|M' \cap B_\xi| > c\sigma^2 \tag{4.18}$$

where one has defined

$$B_\xi = \{y \in H \mid (y + [\xi]) \cap H_i \in A \quad \text{for at least 2 distinct } 1 \leq i \leq K\} \ . \tag{4.19}$$

Replacing B_ξ by a suitable union $\overline{B}_\xi$ of $\sim \sigma^{-1}$ images of B_ξ under orthogonal transformation in H, one gets a set satisfying

$$h_\varepsilon(\overline{B}_\xi) < h_\varepsilon(B_\xi) + c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \tag{4.20}$$

and

$$\sup_{x\in H} \left|\overline{B}_\xi \cap (x+M)\right| > c\sigma^2 \tag{4.21}$$

for M taken in a half-measure subset of $G(H, k-1)$. Consequently

$$h_\varepsilon(\overline{B}_\xi) \geq \rho_\varepsilon(d-1, k-1, c\sigma^2) \ . \tag{4.22}$$

Coming back to the map f given by (4.15), it follows from the definition of B_ξ for $\xi \in \Omega$, that $h_\varepsilon\big(f^{-1}(\xi)\big) \geq h_\varepsilon(B_\xi)$. Hence, by (4.20),(4.22)

$$h_\varepsilon\big(f^{-1}(\xi)\big) \geq \rho_\varepsilon(d-1, k-1, c\sigma^2) - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \ . \tag{4.23}$$

Also, by (4.17)

$$h_\varepsilon(\Omega) > d - 1 - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \ . \tag{4.24}$$

Applying (4.4), one obtains from (4.23),(4.24)

$$\begin{aligned} 2 \max_{1\leq i\leq K} h_\varepsilon(A\cap H_i) &\geq h_\varepsilon\big(f^{-1}(\Omega)\big) > \\ &> \rho_\varepsilon(d-1, k-1, c\sigma^2) + d - 1 - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \ . \end{aligned} \tag{4.25}$$

Thus $h_\varepsilon(H_i \cap A) > \frac{1}{2}\rho_\varepsilon(d-1, k-1, c\sigma^2) + \frac{d-1}{2} - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}}$ for some $i \leq K$. Taking into account the construction of the H_i, the reader will easily verify that

$$h_\varepsilon\big(A \cap (H + te_d)\big) > \tfrac{1}{2}\rho_\varepsilon(d-1, k-1, c\sigma^2) + \frac{d-1}{2} - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \tag{4.26}$$

for t ranging in a set $\Gamma' \subset \Gamma$ satisfying also

$$h_\varepsilon(\Gamma') > 1 - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \ . \tag{4.27}$$

Hence one obtains finally

$$h_\varepsilon(A) > \tfrac{1}{2}\rho_\varepsilon(d-1, k-1, c\sigma^2) + \frac{d+1}{2} - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}}$$

and therefore

$$\rho_\varepsilon(d,k,\sigma) > \tfrac{1}{2}\rho_\varepsilon(d-1, k-1, c\sigma^2) + \frac{d+1}{2} - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \tag{4.28}$$

which is the desired recursive inequality.

It follows also from section 2 that

$$\rho_\varepsilon(d,1,\sigma) > p(d) - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \tag{4.29}$$

for some

$$\frac{d+1}{2} < p(d) \equiv \frac{d+1+\tau}{2} < \frac{d}{2} + 1\ . \tag{4.30}$$

It is now straightforward by iterating (4.28) and inserting (4.29),(4.30) at the end to show that

$$\rho_\varepsilon(d,k,\sigma) > d - \frac{d-k-\tau'}{2^k} - c\frac{\log \sigma^{-1}}{\log \varepsilon^{-1}} \tag{4.31}$$

for some $\tau' = \tau'(d,k) > 0$.

Proof of Proposition 4.1: We use the same method as in section 3. Let f be the indicator function of a subset A of the unit ball $B(0,1)$ and assume

$$f^*(L) > \sigma > 0 \quad \text{for} \quad L \in G \subset G(d,k) \tag{4.32}$$

where $|G| > \frac{1}{2}$. Let

$$f = \sum_{j\geq 0} f_j \tag{4.33}$$

be a Littlewood Paley decomposition as in (3.7) and estimate

$$f^*(L) \leq \sum (f_j^*)(L)\ . \tag{4.34}$$

For $\xi \in S_{d-1}$, let $\overline{g}_\xi$ be defined as in (3.9) and observe again that for $M \in G\big([\xi]^\perp, k-1\big)$

$$g^*\big([\xi, M]\big) \sim (\overline{g}_\xi)^*(M)\ . \tag{4.35}$$

Also, cf. (3.15)

$$\int_{S_{d-1}} \int_{[\xi]^\perp} \Big\{ \sum_j 2^j \big|(\overline{f}_j)_\xi\big|^2 \Big\} \leq C\|f\|_2^2 \tag{4.36}$$

and hence, there is $\xi \in S_{d-1}$ satisfying

$$|G_\xi| > \frac{1}{10} \quad \text{where} \quad G_\xi = \big\{M \in G\big([\xi]^\perp, k-1\big) \mid [\xi, M] \in G\big\} \tag{4.37}$$

and

$$\int_{[\xi]^\perp} \sum_j 2^j \big|(\overline{f}_j)_\xi\big|^2 < C|A| \,. \tag{4.38}$$

Let $(\eta_j)_{j \geq 0}$ be a sequence of positive numbers to be specified later. It follows from (4.38) that

$$A_j \equiv \big\{x \in [\xi]^\perp \mid \big|(\overline{f}_j)_\xi\big|(x) > \eta_j\big\} \tag{4.39}$$

satisfies

$$|A_j| < c\eta_j^{-2} 2^{-j} |A| \,. \tag{4.40}$$

Since, roughly speaking, the f_j may be considered essentially constant on 2^{-j}-cubes, this means that

$$h_{2^{-j}}(A_j) < d - 2 + c\frac{\log \eta_j^{-1}}{j} - c\frac{\log |A|^{-1}}{j} \,. \tag{4.41}$$

Also

$$\big[(\overline{f}_j)_\xi\big]^*(M) < (\chi_{A_j})^*(M) + c\eta_j \,. \tag{4.42}$$

Thus, if

$$\sum \eta_j < c\sigma \tag{4.43}$$

it follows from (4.34),(4.35),(4.42) that

$$\sum_{j \geq 0} (\chi_{A_j})^*(M) > \frac{\sigma}{2} \tag{4.44}$$

for $M \in G_\xi$. Clearly, by (4.37),(4.43), there is some $j \geq 0$ satisfying

$$(\chi_{A_j})^*(M) > c\sigma\eta_j \tag{4.45}$$

for $M \in \mathcal{H} \subset G_\xi$ where

$$|\mathcal{H}| > c\sigma\eta_j \ . \tag{4.46}$$

Consequently, by (4.31)

$$\begin{aligned} h_{2^{-j}}(A_j) \geq \rho_{2^{-j}}(d-1, k-1, c\sigma\eta_j) - c\frac{\log(\sigma\eta_j)^{-1}}{j} > \\ > d - 1 - \frac{d-k-\tau'}{2^{k-1}} - c\frac{\log(\sigma\eta_j)^{-1}}{j} \ , \end{aligned} \tag{4.47}$$

and thus, invoking (4.41) and the hypothesis $d \leq k + 2^{k-1}$

$$\log |A|^{-1} < \log(\sigma\eta_j)^{-1} - j\frac{\tau'}{2^{k-1}} \ . \tag{4.48}$$

Take $\eta_j \sim \sigma j^{-2}$. Then the right member of (4.48) is bounded by a function of σ, implying a lower bound on $|A|$. This completes the proof of Proposition 4.1.

Remark 4.49: The preceding also permits deriving a maximal inequality on the Radon transform

$$\|f^*\|_{L^p_{G(d,k)}} \geq C\|f\|_p \tag{4.50}$$

for some $p < \infty$, provided $d \leq k + 2^{k-1}$. We again assume f supported by a bounded region.

5. The Nikodym Maximal Function

For f a locally integrable function on $\mathbf{R}^d$, define for given $0 < \delta < 1$

$$f_\delta^{**}(x) = \sup_\tau \frac{1}{|\tau|} \int_\tau f(y)dy \tag{5.1}$$

where the supremum is taken over all δ-tubes of unit length centered at x. Let us call f_δ^{**} the Nikodym maximal function of eccentricity δ.

Remarks:

(5.2) The Nikodym set in $\mathbf{R}^2$ is a measure zero set A such that for every point $x \in \mathbf{R}^2$ there is a line segment L centered at x so that $L - x \subset A$.

(5.3) One may again conjecture that a Nikodym set in $\mathbf{R}^d$ needs to have Hausdorff dimension d. This fact is correct for $d = 2$ and open for $d \geq 3$. In this section we will use the same techniques as in section 1,2 of the paper to obtain the same estimates on the Nikodym maximal operators.

(5.4) C. Fefferman's remarkable paper on ball-multipliers (see [Fe2]) implies that the dimension conjecture stated in (5.3) is a corollary of the following fact (unknown for $d \geq 3$) on radial multipliers:

$$(M_\lambda f)^\wedge(\xi) = \left(1 - |\xi|^2\right)_+^\lambda \cdot \widehat{f}(\xi) \tag{5.5}$$

defines a bounded multiplier on $L^p(\mathbf{R}^d)$, provided $\lambda > 0$, $\frac{d-1}{2d} \leq p \leq \frac{d+1}{2d}$, (see [St2] for a discussion).

The interest of the results obtained in this paper to multiplier and restriction problems in Harmonic analysis will be discussed later. Our attention will mostly be directed to dimension 3.

PROPOSITION 5.6.

(a) *In $\mathbf{R}^3$, one has the inequality*

$$\|f_j^{**}\|_{L^p(\mathbf{R}^3)} \leq \left(\frac{1}{\delta}\right)^{\frac{3}{p}-1+\varepsilon} \|f\|_{L^p(\mathbf{R}^3)} \tag{5.7}$$

for $2 \leq p \leq \frac{7}{3}$.

(b) *In $\mathbf{R}^d$, there is the inequality*

$$\|f_j^{**}\|_p \leq \left(\frac{1}{\delta}\right)^{\frac{d}{p}-1+\varepsilon} \|f\|_p \tag{5.8}$$

for $2 \leq p \leq p(d)$, where $p(d)$ are the numbers obtained in section 2.

Taking for f the indicator function of a ball $B(0, \delta)$, one verifies again the essential optimality of these estimates.

For $d = 2$, inequality (5.8), i.e.

$$\|f_\delta^{**}\|_p \leq \delta^{-\varepsilon}\|f\|_p \tag{5.9}$$

holds for $p \geq 2$. This fact may be verified using Fourier transform, in the same way as for the Kakeya maximal function f_δ^*.

Our purpose is to indicate the modifications in the proof of the Kakeya estimates obtained in sections 1,2. Only the case $d = 3$ will be of relevance. The problem is reduced to showing the analogue of (1.4), i.e.

$$|A| \geq c\delta^{\frac{2}{3}+\epsilon}\sigma^{\frac{7}{3}}\left|\left\{(\chi_A)_\delta^{**} > \sigma\right\}\right| \tag{5.10}$$

assuming A a union of δ-cubes.

Considering unions of suitable translates of A, (5.10) follows from the statement

$$|A| > c\delta^{\frac{2}{3}+\epsilon}\sigma^{\frac{7}{3}} \tag{5.11}$$

whenever $A \subset B(0,1)$, A a union of δ-cubes and

$$\left|\left\{(\chi_A)_\delta^{**} > \sigma\right\}\right| > \tfrac{1}{2}\ . \tag{5.12}$$

We now consider the construction in the proof of (1.22). Letting

$$\mathcal{D} = \left\{(\chi_A)_\delta^{**} > \sigma\right\} \tag{5.13}$$

we get for $x \in \mathcal{D}$ a δ-tube τ_x such that

$$|A \cap \tau_x| > \sigma\delta^2\ . \tag{5.14}$$

Let $\mathcal{E}$ be a δ-separated net in $\mathcal{D}$. One has by (5.12)

$$\#\mathcal{E} \sim \delta^{-3} \tag{5.16}$$

and

$$\int_A \sum_{x\in\mathcal{E}} \chi_{\tau_x} \gtrsim \sigma\delta^{-1}\ . \tag{5.17}$$

Thus there is a point x_0 such that

$$\#\{x \in \mathcal{E} \mid x_0 \in \tau_x\} \gtrsim \frac{\sigma}{\delta|A|} \tag{5.18}$$

and we may therefore extract a subset $\mathcal{F}_0$ of $\mathcal{E}$ such that

$$\#\mathcal{F}_0 \gtrsim \frac{\sigma^3}{|A|} \tag{5.19}$$

and the tubes $(\tau_x)_{x\in\mathcal{F}_0}$ containing x_0 point in $\frac{\delta}{\sigma}$-separated directions. Letting

$$B_0 = \bigcup_{x\in\mathcal{F}_0} \tau_x \tag{5.20}$$

one then has again, by (5.14),

$$|B_0| \lesssim \frac{1}{\sigma}|A \cap B_0| \,. \tag{5.21}$$

Define $A' = A\backslash B_0$, $\mathcal{D}_1 = \left\{(\chi_{A_1})^{**}_\delta > \frac{\sigma}{10}\right\}$. If $|\mathcal{D}_1| > \frac{1}{10}$, one gets from the preceding construction a subset $\mathcal{F}_1 \subset \mathcal{D}_1$ so that

$$\#\mathcal{F}_1 \gtrsim \frac{\sigma^3}{|A_1|} > \frac{\sigma^3}{|A|} \tag{5.22}$$

$(\tau_x)_{x\in\mathcal{F}_1}$ have a common point x_1 and

$$B_1 = \bigcup_{x\in\mathcal{F}_{x_1}} \tau_x \tag{5.23}$$

satisfies

$$|B_1| \lesssim \frac{1}{\sigma}|A_1 \cap B_0| \,. \tag{5.24}$$

If the construction stops after s steps, one gets the bound (1.46), i.e.

$$s \lesssim \frac{|A|^2}{\sigma^4}\delta^{-2} \tag{5.25}$$

and

$$\overline{A} = \bigcup_{t=0}^{s}(A \cap B_t) \tag{5.26}$$

satisfies

$$(\chi_{\overline{A}})^{**}_\delta > \frac{\sigma}{2} \tag{5.27}$$

on a set $\overline{\mathcal{D}} \subset \mathcal{D}$ of measure $> \frac{1}{4}$.

Thus for $x \in \overline{\mathcal{D}}$, there is a tube τ_x centered at x such that

$$\sum_0^s |\tau_x \cap B_t| > \sigma\delta^2 \,. \tag{5.28}$$

Denote again

$$B_t^r = B_t \cap \left[B(x_t, 2r)\backslash B(x_t, r)\right] \quad \text{for} \quad \delta < r < 1 \ . \tag{5.29}$$

Hence

$$|B_t^r| \lesssim r|B_t| \tag{5.30}$$

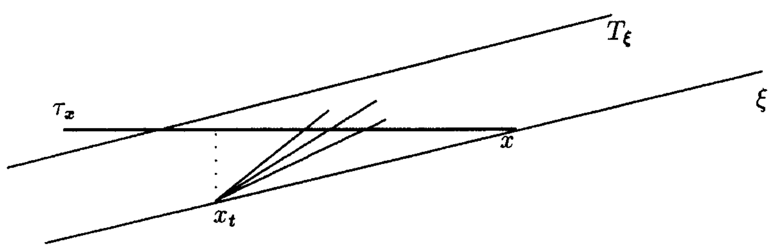
and form (5.28)

$$\sum_{\substack{\delta < r < 1 \\ r \text{ dyadic}}} \sum_{t=0}^{s} |\tau_x \cap B_t^r| > \sigma\delta^2 \ . \tag{5.31}$$

Fix t, r. If $|x - x_t| > r$, one has

$$\delta^{-2}|\tau_x \cap B_t^r| \lesssim \frac{1}{|T_\xi|}\int_{T_\xi} \chi_{B_t^r} \tag{5.32}$$

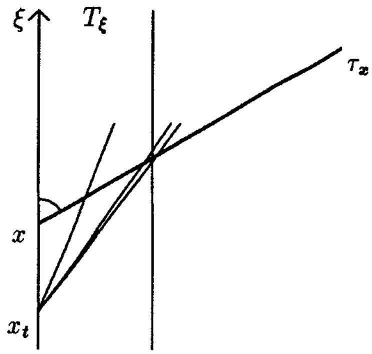
where $\xi = \frac{x - x_t}{|x - x_t|}$ and T_ξ has dimensions ~ 1, dist $\left(x_t, [\tau_x]\right)$, δ



If $|x - x_t| < r$, one gets

$$\delta^{-2}|\tau_x \cap B_t^r| \leq \frac{r}{|x - x_t|}\frac{1}{|T_\xi|}\int_{T_\xi} \chi_{B_t^r} \tag{5.33}$$

with $\xi = \frac{x-x_t}{|x-x_t|}$ and T_ξ with dimensions $1, r \cdot \text{Angle}(x - x_t, \tau_x), \delta$.



Consider the maximal operator $\mathcal{M}_\delta$ defined in (1.51). It follows from (5.31),(5.32),(5.33) that for $x \in \overline{\mathcal{D}}$

$$\sum_{\substack{\delta<r<1 \\ r \text{ dyadic}}} \sum_{t=0}^{s} \left[\mathcal{M}_\delta(B_t^r - x_t)\left(\frac{x - x_t}{|x - x_t|}\right)\right]^2 \frac{1}{|x - x_t|^2} > \frac{\sigma^2}{s \log \frac{1}{\delta}} \; . \tag{5.34}$$

Integrating (5.34) on $\overline{\mathcal{D}}$ and invoking Lemma 1.52 yields

$$\sum_{\delta<r<1} \sum_{t\leq s} \int_{S_2} [\mathcal{M}_\delta(B_t^r - x_t)(\xi)]^2 d\xi \gtrsim \frac{\sigma^2}{s \log \frac{1}{\delta}} \tag{5.35}$$

$$\sum_{\delta<r<1} \sum_{t\leq s} \frac{1}{r}|B_t^r| > \frac{\sigma^2 \delta^\epsilon}{s} \; . \tag{5.36}$$

Consequently, from (5.30),(5.21),(5.24), etc.

$$|A| > \frac{\sigma^3 \delta^\epsilon}{s} \; . \tag{5.37}$$

Combine with (5.25), (5.11) follows.

This proves Proposition (5.6),(a).

The case $d > 3$ is analogous and we omit details.

6. Restriction of Fourier Transforms to Spheres

A result of P. Tomas [T] states that if $f \in L^p(\mathbb{R}^d)$, $1 \le p < \frac{2d+2}{d+3}$, the Fourier transform $\hat{f}$ is defined a.e. on the sphere S_{d-1} and the restriction yields and $L^2(d\sigma)$ functions. There is the inequality

$$\left(\int_{S_{d-1}} |\hat{f}|^2 d\sigma\right)^{1/2} \le C_p \|f\|_p \,. \tag{6.1}$$

The result was extended to the end point $p = \frac{2d+2}{d+3}$ by E. Stein (see [St3], p. 326). In this section, we will consider non-L^2 restriction theorems and prove that the restriction $\hat{f}\big|_S$ makes sense for $f \in L^p(\mathbb{R}^d)$ and certain $\frac{2d+2}{d+3} < p < \frac{2d}{d+1}$. These facts are intimately related to the behavior of Kakeya maximal operators and our argument is based in part on results from the first two sections.

We first present an alternative proof of (6.1) (in a dual form), i.e. we look for a distributional inequality for the function

$$\widehat{\varphi\sigma}(\xi) \equiv \int_{S_{d-1}} \varphi(x) e^{-2\pi i \langle x,\xi\rangle} \sigma(dx) \tag{6.2}$$

assuming

$$\int_{S_{d-1}} |\varphi|^2 d\sigma \le 1 \,. \tag{6.3}$$

This inequality will give some additional information on the level sets

$$\left[|\widehat{\varphi\sigma}| > \lambda\right] \,. \tag{6.4}$$

LEMMA 6.5. *With previous notations and* $0 < R < \lambda^{-\frac{4}{d-1}}$, *one has*

$$\begin{aligned} &\lambda^2 \operatorname{mes}\left[|\widehat{\varphi\sigma}| > \lambda\right] \lesssim \\ &\quad \lesssim R + R^{-\frac{d-1}{2}} \sup_{z \in \mathbb{R}^d} \operatorname{mes}\left[x \in B(z, \lambda^{-\frac{4}{d-1}}) \,\big|\, |\widehat{\varphi\sigma}(x)| > \lambda\right] \,. \end{aligned} \tag{6.6}$$

Proof: Denote

$$A = [\operatorname{Re} \widehat{\varphi\sigma} > \lambda] \tag{6.7}$$

$\{A_j\}$ a partition of A in cubes of size $\lambda^{-\frac{4}{d-1}}$ and $\{A_{j,j'}\}$ a subpartition of A_j in cubes of size R. Denote $\{\chi_j\}$, $\{\chi_{j,j'}\}$ the corresponding indicator functions. At the cost of replacing A by a set of proportional measure, one may assume

$$\operatorname{dist}(A_{j_1}, A_{j_2}) > c\lambda^{-\frac{4}{d-1}} \quad \text{for} \quad j_1 \neq j_2 \tag{6.8}$$

and

$$\operatorname{dist}(A_{j,j_1'}, A_{j,j_2'}) > R \quad \text{for} \quad j_1' \neq j_2' \,. \tag{6.9}$$

Since

$$A = \bigcup_j A_j = \bigcup_{jj'} A_{j,j'} \tag{6.10}$$

one has

$$\lambda|A| < \big|\langle \widehat{\varphi\sigma}, \chi_A\rangle\big| = \left|\int \varphi \widehat{\chi}_A d\sigma\right| \leq \left(\int \Big|\sum_j \widehat{\chi}_j\Big|^2 d\sigma\right)^{1/2} . \tag{6.11}$$

Expressing the square as single and mixed products yields

$$\begin{aligned} \lambda^2|A|^2 &< \sum_j \int |\widehat{\chi}_j|^2 d\sigma + 2\sum_{j\neq k} \langle \widehat{\chi}_j, \widehat{\chi}_k\rangle_\sigma \\ &\leq \sum_j \int |\widehat{\chi}_j|^2 d\sigma + 2\sum_{j\neq k} \big|\langle \chi_j, \chi_k * \check{\sigma}\rangle\big| \,. \end{aligned} \tag{6.12}$$

Repeating this for $\chi_j = \sum_{j'} \chi_{j,j'}$ gives thus

$$\begin{aligned} \lambda^2|A|^2 \lesssim \sum_{j,j'} \int |\widehat{\chi}_{j,j'}|^2 d\sigma + \sum_j \sum_{j'\neq k'} \big|\langle \chi_{j,j'}, \chi_{j,k'} * \check{\sigma}\rangle\big| + \\ + \sum_{j\neq k} \big|\langle \chi_j, \chi_k * \check{\sigma}\rangle\big| \,. \end{aligned} \tag{6.13}$$

To estimate the first terms in (6.13), observe that $|\widehat{\chi}_{j,j'}|^2$ has a Fourier transform contained in a ball of radius CR, hence

$$\int |\widehat{\chi}_{j,j'}|^2 d\sigma = \int |\widehat{\chi}_{j,j'}|^2 d\sigma_{\frac{1}{R}} \tag{6.14}$$

where $\sigma_{1/R} = \sigma * \varphi_{1/R}$ and $\varphi \in \mathcal{S}$ satisfies $\widehat{\varphi} = 1$ on $B(0, C)$. Since $\|\sigma_{1/R}\|_\infty \lesssim R$, one gets the estimate $C \cdot R \|\chi_{j,j'}\|_2^2$ on (6.14). Hence,

$$\sum_{j,j'} \int |\widehat{\chi}_{j,j'}|^2 d\sigma \lesssim R \sum_{j,j'} |A_{j,j'}| = R|A| \ . \tag{6.15}$$

To bound the second and third terms in (6.13), observe that if

$$\operatorname{dist}(A, A') > \rho \tag{6.16}$$

then by the decay property of $\widehat{\sigma}$

$$\big| \langle \chi_A, \chi_{A'} * \check{\sigma} \rangle \big| \le \|\chi_A\|_1 \|\chi_{A'}\|_1 \|\check{\sigma}\big|_{B(0,\rho)^c}\|_\infty \lesssim \rho^{-\frac{d-1}{2}} |A||A'| \ . \tag{6.17}$$

Inserting this bound in (6.13) give the following estimate on the mixed terms,

$$\sum_j \sum_{j' \neq k'} R^{-\frac{d-1}{2}} |A_{j,j'}||A_{j,k'}| + c \sum_{j \neq k} \lambda^2 |A_j||A_k| \le$$

$$\le R^{-\frac{d-1}{2}} \sum_j |A_j|^2 + c\lambda^2 |A|^2 \tag{6.18}$$

relying on (6.8),(6.9).

Since the last term in (6.18) is $o(\lambda^2 |A|^2)$, one concludes from (6.13), (6.15), (6.18) that

$$\lambda^2 |A|^2 \lesssim R|A| + R^{-\frac{d-1}{2}} \sum_j |A_j|^2 \ . \tag{6.19}$$

Estimating

$$\sum |A_j|^2 \lesssim |A| \max_z \operatorname{mes} \left[x \in B(z, \lambda^{-\frac{4}{d-1}}) \ \big| \ |\widehat{\varphi\sigma}| > \lambda \right] \ . \tag{6.20}$$

(6.6) easily follows. This proves the lemma.

Remark 6.21: If one takes in (6.6) $R \sim \lambda^{-\frac{4}{d-1}}$, it clearly follows that

$$\text{mes}\left[|\widehat{\varphi\sigma}| > \lambda\right] \lesssim \lambda^{-2-\frac{4}{d-1}} = \lambda^{-2\frac{d+1}{d-1}} . \tag{6.22}$$

In a dual formulation, it means that $\widehat{f}\big|_S \in L^2(d\sigma)$, provided $f \in L^{\frac{2d+2}{d+3},1}(\mathbf{R}^d)$.

Our next purpose it to develop an estimate on the quantity

$$\text{mes}\left[x \in B(0,\rho) \mid |\widehat{\varphi\sigma}| > \lambda\right] \tag{6.23}$$

assuming $|\varphi| \leq 1$ and $\lambda < 1$, $\rho > 1$. This estimate will result from a majoration of

$$\| \widehat{\varphi\sigma} \mid_{B(0,\rho)} \|_q \tag{6.24}$$

where $q < 2\frac{d+1}{d-1}$ will be taken such that

$$\frac{d+1}{2} < \left(\frac{q}{2}\right)' \leq p(d) \tag{6.25}$$

and $p(d)$ is the exponent appearing in (2.2).

LEMMA 6.26. *If $\varphi \in L^\infty(S_{d-1})$, $|\varphi| \leq 1$, one has for q satisfying (6.25)*

$$\| \widehat{\varphi\sigma} \mid_{B(0,\rho)} \|_q < C_\varepsilon \rho^{\frac{1}{2}\left(\frac{d+1}{q} - \frac{d-1}{2}\right)+\varepsilon} . \tag{6.27}$$

Part of the considerations in the proof are related to the geometric construction appearing for instance in [Fe1]. This is the decomposition of a (2-dimensional) annulus $1 - \delta < |x| < 1$ in $(\delta \times \sqrt{\delta})$-size rectangles. The reader may find it more convenient to read what follows in dimension 3. Define

$$q_0 = q_0(d) = 2\frac{d+1}{d-1} . \tag{6.28}$$

LEMMA 6.29. *Let $\{x_\alpha\}$ be a $\frac{1}{R}$-separated set of points on S_{d-1} and $2 \leq q \leq q_0$. Then*

$$\Big\| \sum_\alpha a_\alpha e^{i\langle x_\alpha, \xi\rangle} \Big\|_{L^q(B(0,R))} \lesssim R^{\frac{1}{2}\left(\frac{d-1}{2} + \frac{d+1}{q}\right)} \quad \Big(\sum |a_\alpha|^2\Big)^{1/2} \tag{6.30}$$

where the $\{a_\alpha\}$ are arbitrary scalars.

Proof of (6.29): The result is gotten from interpolation between 2 and q_0. The case $q = 2$ yields a bound $R^{\frac{d}{2}}$, simply by the almost orthogonality of the restrictions $\{e^{i\langle x_\alpha,\xi\rangle}|_{B(0,R)}\}$. For $q = q_0$, the left member of (6.30) may be evaluated by $\left\| \int_S \varphi(x)e^{i\langle x,\xi\rangle}\sigma(dx)\right\|_{q_0} \cdot R^{d-1}$, where φ is constant on cells of size R^{-1} on S and $\left(\int |\varphi|^2 d\sigma\right)^{1/2} \sim R^{-\frac{d-1}{2}}\left(\sum |a_\alpha|^2\right)^{1/2}$. Hence, by the estimate $\|\widehat{\varphi\sigma}\|_{q_0} \leq C\|\varphi\|_{L^2(S)}$, there is the bound $R^{\frac{d-1}{2}}$ if $q = q_0$. The result follows.

Coming back to $\widehat{\varphi\sigma}$, partition the unit cube in cells of size $\sim \frac{1}{\sqrt{\rho}}$ and let $S = \bigcup S_\alpha$ be the induced partition of S_{d-1}.

Write for $x_\alpha \in S_\alpha$

$$\begin{aligned}\int_S \varphi(x)e^{-2\pi i\langle x,\xi\rangle}\sigma(dx) &= \\ &= \sum_\alpha e^{-2\pi i\langle x_\alpha,\xi\rangle}\left(\int_{S_\alpha} \varphi(x)e^{-2\pi i\langle x-x_\alpha,\xi\rangle}\sigma(dx)\right) .\end{aligned} \tag{6.31}$$

To evaluate its L^q-norm on $B(0,\rho)$, partition the domain into cubes Q of size $\sqrt{\rho}$. Since $|x-x_\alpha| < c\rho^{-1/2}$ for $x \in S_\alpha$, one may see the $\int_{S_\alpha}\varphi(x)e^{-2\pi i\langle x-x_\alpha,\xi\rangle}d\sigma$-factors as constants for $\xi \in Q$. This may be formalized the usual way, making a change of variable $\xi' = \xi + \eta$ where $\eta \in B(0,\sqrt{\rho})$ is a new variable. We skip the details. Applying (6.29) with $R = \sqrt{\rho}$, one gets on a Q-cube

$$\begin{aligned}&\|\widehat{\varphi\sigma}\|_{L^q(Q)} \lesssim \\ &\lesssim \rho^{\frac{1}{4}\left(\frac{d-1}{2}+\frac{d+1}{q}\right)}|Q|^{-1/q}\left\|\left(\sum_\alpha\left|\int_{S_\alpha} \varphi(x)e^{-2\pi i\langle x-x_\alpha,\xi\rangle}d\sigma\right|^2\right)^{1/2}\right\|_{L^q(Q)} .\end{aligned} \tag{6.32}$$

Summing over these subcubes of $B(0,\rho)$ yields

$$\begin{aligned}&\|\widehat{\varphi\sigma}\|_{L^q(B(0,\rho))} \lesssim \\ &\lesssim \rho^{\frac{1}{4}\left(\frac{d-1}{2}+\frac{d+1}{q}\right)-\frac{d}{2q}}\left\|\left(\sum_\alpha\left|\int_{S_\alpha} \varphi(x)e^{-2\pi i\langle x,\xi\rangle}d\sigma\right|^2\right)^{1/2}\right\|_{L^q(B(0,\rho))} .\end{aligned} \tag{6.33}$$

Our next purpose is to estimate the square function expression.

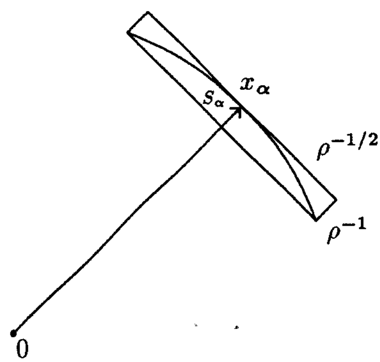
Write

$$\left\| \left(\sum_\alpha \left| \int_{S_\alpha} \varphi(x) e^{-2\pi i \langle x,\xi \rangle} d\sigma \right|^2 \right)^{1/2} \right\|_{L^q(B(0,\rho))} = \\ = \left\| \sum_\alpha \left| \int_{S_\alpha} \varphi(x) e^{-2\pi i \langle x,\xi \rangle} d\sigma \right|^2 \right\|^{1/2}_{L^{q/2}(B(0,\rho))} . \tag{6.34}$$

Fix α. Let x_α be the center of S_α and consider a function K_α such that

$$\widehat{K}_\alpha = 1 \quad \text{on} \quad (S_\alpha - S_\alpha) + B(0, \rho^{-1}) \tag{6.35}$$

where K_α has the shape of a tube in direction x_α of length ρ and width $\sqrt{\rho}$



This tube appears as the polar of the $(\underbrace{\rho^{-1/2} \times \ldots \times \rho^{-1/2}}_{d-1} \times \rho^{-1})$-box indicated above. Its volume-measure is normalized.

Write for $\xi \in B(0,\rho)$

$$\left| \int_{S_\alpha} \varphi(x) e^{-2\pi i \langle x,\xi \rangle} d\sigma \right|^2 (\xi) = \left(| \widehat{\varphi|_{S_\alpha} \cdot \sigma} |^2 \psi(\rho^{-1} \cdot) * K_\alpha \right)(\xi) = \\ = \int K_\alpha(\xi - \eta) | \widehat{\varphi|_{S_\alpha} \cdot \sigma} |^2(\eta) \varphi(\rho^{-1}\eta) d\eta . \tag{6.36}$$

Here $\psi \in \mathcal{S}$ is the usual bump function and we assume

$$\operatorname{supp} \widehat{\psi} \subset B(0,1) . \tag{6.37}$$

Interpret (6.36) as an average of translates of K_α, where the averaging measure

$$| \widehat{\varphi|_{S_\alpha} \cdot \sigma} |^2 \psi(\rho^{-1} \cdot) \tag{6.38}$$

has weight

$$\begin{aligned} \int | \widehat{\varphi|_{S_\alpha} \cdot \sigma} |^2(\eta) \psi(\rho^{-1}\eta) d\eta \leq \\ \leq \rho^d \iint_{S_\alpha \times S_\alpha} \widehat{\psi}\big(\rho(x_1 - x_2)\big) \sigma(dx_1) \sigma(dx_2) \gtrsim \rho^{d - \frac{3}{2}(d-1)} \end{aligned} \tag{6.39}$$

as the reader will easily verify.

By convexity, one may find points $\{\eta_\alpha\}$ for which the right member of (6.34) is bounded by

$$\rho^{-\frac{d}{4}+\frac{3}{4}} \Big\| \sum_\alpha K_\alpha(\cdot - \eta_\alpha) \Big\|^{1/2}_{L^{q/2}(B(0,\rho))} . \tag{6.40}$$

Making a change of variable $\eta = \rho x$, $x \in B(0,1)$, (6.40) becomes

$$\rho^{\frac{1}{2}+\frac{d}{q}-\frac{d}{2}} \Big\| \sum_\alpha \rho^{-\frac{d-1}{2}} \widetilde{K}_\alpha(x - z_\alpha) \Big\|^{1/2}_{L^{q/2}(B(0,1))} \tag{6.41}$$

where $z_\alpha = \rho^{-1}\eta_\alpha$ and

$$\widetilde{K}_\alpha(x) = \rho^d K_\alpha(\rho x) . \tag{6.42}$$

Observe that $\widetilde{K}_\alpha$ corresponds to a tube of unit length in direction x_α and thickness $\frac{1}{\sqrt{\rho}}$. Estimating the $\| \quad \|_{L^{q/2}}$-norm in (6.41) is done by duality and pairing with a function $f \in L^{(\frac{q}{2})'}\big(B(0,1)\big)$ of norm 1. The norm then gets essentially evaluated by

$$\sum_\alpha \rho^{-\frac{d-1}{2}} f^*_{\rho^{-\frac{1}{2}}}(x_\alpha) \sim \int_{S_{d-1}} f^*_{\rho^{-\frac{1}{2}}} \leq \|f^*_{\rho^{-\frac{1}{2}}}\|_{L^{(q/2)'}} , \tag{6.43}$$

since the $\{x_\alpha\}$ form a $\rho^{-1/2}$-net in S.

In view of (2.1), (6.25), there is a bound $(\rho^{1/2})^{\frac{d}{(\frac{q}{2})'}-1+\varepsilon} = \rho^{-\frac{1}{2}+\frac{d}{2}-\frac{d}{q}+\varepsilon}$. Collecting estimates (6.33), (6.34), (6.41) and the previous one, leads finally to the bound

$$\|\widehat{\varphi\sigma}\|_{L^q(B(0,\rho))} \lesssim \rho^{\frac{1}{4}\left(\frac{d-1}{2}+\frac{d+1}{q}\right)-\frac{d}{2q}+\frac{1}{2}+\frac{d}{q}-\frac{d}{2}-\frac{1}{4}+\frac{d}{4}-\frac{d}{2q}+\varepsilon} \tag{6.44}$$

which is (6.27). This proves lemma 6.26.

Coming back to inequality (6.6), application of (6.27) with $\rho = \lambda^{-\frac{4}{d-1}}$ yields the additional inequality

$$\text{mes}\left[x \in B(z, \lambda^{-\frac{4}{d-1}}) \;\middle|\; |\widehat{\varphi\sigma}(x)| > \lambda\right] \lesssim \lambda^{-\frac{q}{2}-1-\frac{2}{d-1}} \tag{6.45}$$

for q satisfying (6.25).

Inserting (6.45) in (6.6) and optimizing in R leads to

$$\text{mes}\left[|\widehat{\varphi\sigma}| > \lambda\right] \lesssim \lambda^{-\left(\frac{q}{d+1}+\frac{2d}{d-1}\right)-\varepsilon} . \tag{6.46}$$

Consequently

PROPOSITION 6.47. *The map $\varphi \mapsto \int \varphi(x) e^{i\langle x,\xi\rangle} \sigma(dx)$ is bounded from $L^\infty(S_{d-1})$ to $L^p(\mathbf{R}^d)$, provided*

$$p > 2\left\{\frac{p(d)'}{d+1} + \frac{d}{d-1}\right\} . \tag{6.48}$$

Since $p(d) > \frac{d+1}{2}$, this exponent is less than $\frac{2(d+1)}{d-1}$, corresponding to the case $\varphi \in L^2(S_{d-1})$. This leads to a Fourier transform restriction theorem for functions $f \in L^{p'}(\mathbf{R}^d)$ where $p' > \frac{2(d+1)}{d+3}$.

From Proposition 6.47, the general Nikishin-Maurey-Pisier factorization theory [Pi] and the invariance of the problem under the orthogonal group, it follows formally that the map $\varphi \mapsto \int \varphi(x) e^{i\langle x,\xi\rangle} \sigma(dx)$ is also bounded from $L^{p,1}(S)$ to L^p, for p satisfying (6.48). Interpolating this result and the $L^2(S) - L^{\frac{2(d+1)}{d-1}}$ bound yields then further improvement. In particular, in the case $d = 3$, one has $p(d) = \frac{7}{3}$ and following restriction theorem

THEOREM 6.49. *The restriction map $f \to \hat{f}|_S$ is bounded from $L^r(\mathbf{R}^3)$ to $L^{\bar{r}}(S_2)$ for $\frac{4}{3} \le r \le \frac{31}{23}$ and $\bar{r} = \frac{r}{23r-30}$.*

Remarks:

(6.50) Based on the technique described above, it is possible in fact to get slightly better results. In particular, Lemma 6.26 may be improved. We did not want to complicate the exposition however.

(6.51) The method employed here applies equally well if the sphere is replaced by a compact smooth surface of non-vanishing curvature (cf. also Stein's exposition in [St3]).

7. Applications to Bochner-Riesz Summability

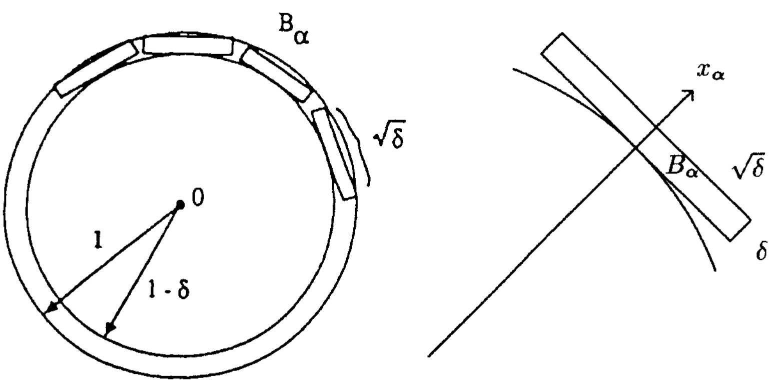
Consider the multiplier

$$m_\lambda(\xi) = \left(1 - |\xi|^2\right)^\lambda \quad \text{if} \quad |\xi| < 1 \\ = 0 \quad \text{if} \quad |\xi| \geq 1 \tag{7.1}$$

It is a conjecture that m_λ acts boundedly on $L^p(\mathbb{R}^d)$ for $p_0(\lambda) < p < p_0(\lambda)'$, $p_0(\lambda) = \frac{2d}{d+1+2\lambda}$, $\lambda > 0$ (cf. [Fe1], [Ho]). The conjecture is verified for $d = 2$ and in arbitrary dimension d, for $\lambda > \frac{d-1}{2(d+1)}$. The reader is referred to the expository paper [St3] and its bibliography for a discussion of the $d = 2$ case. If $\lambda > \frac{d-1}{2(d+1)}$, the result follows from the L^2-restriction theorem, along the lines of [Fe1], p. 51. An alternative proof of this fact will be given here. The method used follows a known pattern. It may be summarized as follows. One considers for $0 < \delta < 1$ the contribution of a substitute for

$$\left(\hat{f}\,\big|_{\left[1-\delta<|\xi|<1-\frac{\delta}{2}\right]}\right)^\vee \tag{7.2}$$

and decomposes the spherical shell $A_\delta = \left[1 - \delta < |\xi| < 1 - \frac{\delta}{2}\right]$ in boxes of size $\underbrace{\sqrt{\delta} \times \ldots \times \sqrt{\delta}}_{d-1} \times \delta$, say B_α.



One may indeed build a multiplier with spherical symmetry from $\sum_\alpha \chi_{B_\alpha}$ by averaging over the orthogonal groups.

Let $p \leq \frac{2(d+1)}{d-1}$.

An estimate on a (7.2)-related multiplier in L^p will thus follow from a bound on $\|\sum_\alpha (\hat f|_{B_\alpha})^\vee\|_p$. Assume $f \in \mathcal{S}$, $p > 2$ (the multiplier problem is self dual). The first step consists of an estimate by a square function. For $d = 2$, $p = 4$ one has equivalence. For $d \geq 3$, we don't know the accurate estimate unless $p \geq 2\frac{d+1}{d-1}$. In any case, it follows from Lemma 6.29 as in the proof of Lemma 6.26 that for $p \leq 2\frac{d+1}{d-1}$

$$\Big\|\sum_\alpha (\hat f|_{B_\alpha})^\vee\Big\|_p \lesssim \delta^{-\frac{1}{4}(\frac{d-1}{2}+\frac{d+1}{p})+\frac{d}{2p}} \big\|\big(\sum |(\hat f|_{B_\alpha})^\vee|^2\big)^{1/2}\big\|_p\,. \tag{7.3}$$

If Q_α denotes a $\sqrt{\delta}$-cube containing B_α (they are essentially disjoint), one also has that

$$\Big\|\big(\sum |(\hat f|_{Q_\alpha})^\vee|^2\big)^{1/2}\Big\|_p \lesssim \|f\|_p \tag{7.4}$$

since $p \geq 2$. Denoting $f_\alpha = (\hat f|_{Q_\alpha})^\vee$, one may write

$$(\hat f|_{B_\alpha})^\vee = f_\alpha * (\chi_{B_\alpha})^\vee\,. \tag{7.5}$$

Roughly speaking $(\chi_{B_\alpha})^\vee$ has the shape of a tube of length $\frac{1}{\delta}$ and width $\frac{1}{\sqrt{\delta}}$, oriented perpendicularly on B_α. The left member of (7.4) may therefore be controlled by expressions of the form

$$\Big\|\big(\sum (|f_\alpha| * K_\alpha)^2\big)^{1/2}\Big\|_p \leq \Big\|\sum |f_\alpha|^2 * K_\alpha\Big\|_{p/2}^{1/2} \tag{7.6}$$

where K_α is the indicator function of a tube in direction x_α, of length ρ and width $\rho^{1/2}$, $\rho \sim \delta^{-1}$. Proceeding as in the previous section, estimate $\|\sum (|f_\alpha|^2 * K_\alpha)\|_{p/2}$ by duality and hence, consider $g \in L^{(p/2)'}$ of norm 1 and write

$$\Big\langle \sum |f_\alpha|^2 * K_\alpha, g\Big\rangle \leq \big\|\sum |f_\alpha|^2\big\|_{p/2} \big\|\max_\alpha |g * K_\alpha|\big\|_{(p/2)'}\,. \tag{7.7}$$

The second factor will lead to a Nikodym maximal function after rescaling.

Putting $x = \rho x'$, $g'(x') = g(\rho x)$, one has indeed

$$(g * K_\alpha)(x) = \big(g' * (K_\alpha)_{\rho^{-1}}\big)(x') \tag{7.8}$$

hence

$$|g * K_\alpha|(x) \leq \big[(g')^{**}_{\rho^{-1/2}}\big](x')\,. \tag{7.9}$$

consequently, assuming

$$\left(\frac{p}{2}\right)' \leq p(d) \tag{7.10}$$

it follows from Proposition 5.6 that

$$\left\| \max_{\alpha} |g * K_\alpha| \right\|_{(p/2)'} \lesssim \rho^{\frac{1}{2}\left[\frac{d}{\left(\frac{p}{2}\right)'} - 1\right] + \varepsilon} . \tag{7.11}$$

Collecting estimates (7.3),(7.4),(7.7),(7.11), one obtains the following bound on the L^p-norm of (7.2)

$$\delta^{-\frac{1}{4}\left(\frac{d-1}{2} + \frac{d+1}{p}\right) + \frac{d}{2p}} \delta^{-\frac{1}{4}\left[\frac{d}{\left(\frac{p}{2}\right)'} - 1\right] - \varepsilon} = \delta^{-\frac{1}{4}\left(\frac{3d}{2} - \frac{3d}{p} + \frac{1}{p} - \frac{3}{2}\right) - \varepsilon} . \tag{7.12}$$

Therefore

$$\lambda > \frac{1}{4}\left(\frac{3d}{2} - \frac{3d}{p} + \frac{1}{p} - \frac{3}{2}\right) \tag{7.13}$$

implies boundedness of the multiplier m_λ given by (7.1) on L^r, $p' \leq r \leq p$. Our assumption on p is that

$$2p(d)' < p \leq \frac{2(d+1)}{d-1} . \tag{7.14}$$

Recall that $p(d) > \frac{d+1}{2}$, so $p(d)' < \frac{d+1}{d-1}$ and hence (7.14) defines a non-empty range.

In particular, for $p = \frac{2(d+1)}{d-1}$, (7.13) becomes $\lambda > \frac{d-1}{2(d+1)}$ which is the sharp result in this case. In fact, one may obtain the sharp result also beyond the range of the L^2-restriction theory. For instance, if $d = 3$.

PROPOSITION 7.15. *m_λ is bounded for $\lambda > 1 - \frac{3}{p}$ and $p > \frac{127}{32}$.*

Take $p_0 = \frac{31}{8}$. By discretization of Proposition 6.47 and application of Pisier's factorization theorem (see [Pi]), one gets the following analogue to inequality (6.30)

$$\left\| \sum_\alpha a_\alpha e^{i\langle x_\alpha, \xi\rangle} \right\|_{L^{p_0}(B(0,R))} \lesssim R^{\frac{2}{p_0}+\varepsilon} \left(\sum |a_\alpha|^{p_0}\right)^{1/p_0} \tag{7.16}$$

where the $\{a_\alpha\}$ are $\frac{1}{R}$-separated on S_2.

Let $p_0 < p < 4$

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{4}\,. \tag{7.20}$$

Interpolating (7.16) and the inequality

$$\Big\| \sum_\alpha a_\alpha e^{i\langle x_\alpha, \xi\rangle} \Big\|_{L^4(B(0,R))} \lesssim R\Big(\sum |a_\alpha|^2\Big)^{1/2} \tag{7.21}$$

yields

$$\Big\| \sum_\alpha e^{i\langle x_\alpha, \xi\rangle} \Big\|_{L^p(B(0,R))} \leq R^{\frac{2}{p_0'}\theta + (1-\theta)+\varepsilon}\Big(\sum |a_\alpha|^q\Big)^{1/q} \tag{7.22}$$

where q is given by

$$\frac{1}{q} = \frac{\theta}{p_0} + \frac{1-\theta}{2}\,. \tag{7.23}$$

One may then replace inequality (7.3) by

$$\Big\| \sum_\alpha (\widehat{f}|_{B_\alpha})^\vee \Big\|_p \leq \delta^{-\frac{\theta}{p_0'} - \frac{1-\theta}{2} + \frac{3}{2p} - \varepsilon} \Big\| \Big(\sum |(\widehat{f}|_{B_\alpha})^\vee|^q\Big)^{1/q} \Big\|_p\,. \tag{7.24}$$

If q satisfies

$$\left(\frac{p}{q}\right)' < \frac{7}{3} \tag{7.25}$$

the previous reasoning involving the Nikodym maximal function leads now to the estimate

$$\delta^{-\frac{\theta}{p_0'} - \frac{1-\theta}{2} + \frac{3}{2p} - \frac{\lambda}{2q}\left\{\frac{3}{\left(\frac{p}{q}\right)'} - 1\right\} - \varepsilon} = \delta^{-1+\frac{3}{p}-\varepsilon}\,. \tag{7.26}$$

This correspond to the condition $\lambda > 1 - \frac{3}{p}$. Analyzing (7.25) yields the restriction $p > \frac{127}{32}$. This number in particular may be improved.

Added in proof: The reader may find some refinements of the techniques and results of the present paper in [Bo2]. Using similar ideas the author obtains in [Bo3] results on the L^p behaviour of oscillatory integrals in $\mathbf{R}^3$, making progress on L. Hörmander's problem described in [Ho].

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