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PERIODIC SOLUTIONS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH AN IMPULSE EFFECT

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Abstract

The paper applies a numerical-analytical method for finding periodic solutions of the system of integro-differential equations

$$\dot{x} = f(t, x, \int_0^t \varphi(t, s, x(s))ds), \quad t \neq t_i(x),$$

$$\Delta x|_{t=t_i(x)} = I_i(x).$$

Two theorems for existence of periodic solutions are proved for the cases when $t = t_i$ and $t = t_i(x)$.

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In the present paper a numerical-analytical method is applied (see [2], [3], [5]) for finding periodic solutions of a system of integro-differential equations of the following form:

$$\dot{x} = f(t, x, \int_0^t \varphi(t, s, x(s))\,ds), \qquad t \neq t_i(x) \tag{1}$$

$$\Delta x|_{t=t_i(x)} = I_i(x),$$

where

$$x = (x_1, x_2, \dots, x_n); \quad f(t, x, y) = (f_1(t, x, y), \dots, f_n(t, x, y));$$

$$\varphi(t, s, x) = (\varphi_1(t, s, x), \dots, \varphi_m(t, s, x)); \quad I_i = (I_i^{(1)}, \dots, I_i^{(n)}); \quad t_i(x)$$

are scalar functions, $i = 0, \pm 1, \pm 2, \dots$.

An analogous problem has been considered in [1], but for systems of ordinary differential equations. The paper [6] is devoted to the problem of finding periodic solutions of integro-differential equations without impulses.

Let the following conditions (A) hold:

A1. The functions $f(t, x, y)$, $\varphi(t, s, x)$, $I_i(x)$ and $t_i(x)$ are defined and continuous with respect to their arguments in the region

$$G = \mathbf{R} \times \mathbf{R} \times D_1 \times D_2, \tag{2}$$

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where D_1 and D_2 are closed and bounded sets in the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, $\mathbf{R} = (-\infty, +\infty)$.

A2. The functions $f(t, x, y)$ and $\varphi(t, s, x)$ are periodic with respect to t, s with a period T.

A3. There exists a natural number p such that

$$I_{i+p}(x) = I_i(x), \qquad t_{i+p}(x) = t_i(x) + T \,. \tag{3}$$

A4. The functions $f(t, x, y)$, $\varphi(t, s, x)$ and $I_i(x)$ satisfy the inequalities

$$\begin{aligned} &\|f(t, x, y) - f(t, x', y')\| \le K_1 \|x - x'\| + K_2 \|y - y'\| \,, \\ &\|\varphi(t, s, x) - \varphi(t, s, x')\| \le K_3 \|x - x'\| \,, \\ &\|I_i(x) - I_i(x')\| \le K_4 \|x - x'\| \end{aligned} \tag{4}$$

in the region (2), uniformly in $t \in \mathbf{R}$, $s \in \mathbf{R}$, $i = 0, \pm 1, \pm 2, \ldots$, where K_j, $j = 1, 2, 3, 4$, are positive constants.

A5. The surfaces $t = t_i(x)$ are given by the continuously differentiable functions in D_1, and

$$\sup_{x \in D_1} \left\| \frac{\partial t(x)}{\partial x} \right\| \le N, \qquad N = \text{const.} > 0 \,. \tag{5}$$

Consider first the problem for existence of T-periodic solutions of the system (1) in the case when the instantaneous change of the state of the system occures at fixed moments, i.e. the hypersurfaces $t = t_i(x)$ are hyperplanes of the type $t = t_i$. Then for each two solutions the moments $t = t_i$ only the values of the jumps at these moments are different and the system (1) can be rewritten as

$$\begin{aligned} &\dot{x} = f\Big(t, x, \int_0^t \varphi(t, s, x(s))\, ds\Big) \,, \qquad t \ne t_i \,, \\ &\Delta x|_{t=t_i} = I_i(x) \,. \end{aligned} \tag{6}$$

As it was noted in [6], the periodic solutions of the integrodifferential equations have a specific character. A necessary condition for existence of periodic solution is the equality

$$f(0, \psi(0), 0) = f\Big(0, \psi(0), \int_0^T \varphi(T, s, \psi(s))\, ds\Big) \,. \tag{7}$$

Particularly, (7) will hold if the following relation holds for each t:

$$\int_0^T \varphi(t, s, \psi(s)) ds = 0 \,. \tag{8}$$

We need the following conditions (B) as well:

B1. There exists a nonempty closed set $D_0 \subseteq D_1$ contained in D_1 together with its $\frac{Mt}{2}\left(1+\frac{4p}{T}\right)$ neighbourhood, where

$$M = \sup_{\substack{t \in [0,T] \\ x \in D_1,\, y \in D_2}} \| f(t,x,y) \| + \max_{1 \le i \le p} \sup_{x \in D_1} \| I_i(x) \| \tag{9}$$

B2. The constants K_j, $j = 1, 2, 3, 4$ satisfy

$$\frac{T}{2}\left(K_1 + \frac{K_2 K_3 T}{2}\right) + 2p\,K_4 < 1\,. \tag{10}$$

Then the following theorem can be proved:

THEOREM 1. *Let the conditions* (A) *and* (B) *hold for the system* (6). *Then, if this system has a periodic solution* $x = \varphi(t)$ *with a period* T, *having value* $t = 0$ *at* $x_0 \in D_0$ *and such that* (8) *is fulfilled, this solution is a limit of a uniformly convergent sequence of periodic functions* $\{x_m(t, x_0)\}$ *given by the relations*

$$\begin{aligned} x_{m+1}(t,x_0) = x_0 + \int_0^t \Big\{ & f\Big(\tau, x_m(\tau,x_0), \int_0^\tau [\varphi(\tau,s,x_m(s,x_0)) - \\ & - \overline{\varphi(\tau,s,x_m(s,x_0))}]ds\Big) - \overline{f\Big(\tau, x_m(\tau,x_0), \int_0^\tau [\varphi(\tau,s,x_m(s,x_0)) -} \\ & \overline{- \overline{\varphi(\tau,s,x_m(s,x_0))}]\,ds\Big)} \Big\}\, d\tau + \sum_{0<t_i<t} I_i(x_m(t_i-0)) - t\,\overline{I(x_m(t_i-0))}\,; \end{aligned} \tag{11}$$

$$\overline{f\left(\tau, x(\tau), \int_0^\tau \varphi(\tau,s,x(s))\,ds\right)} = \frac{1}{T}\int_0^T f\left(\tau, x(\tau), \int_0^\tau \varphi(\tau,s,x(s))\,ds\right) d\tau\,,$$

$$\overline{\varphi(\tau,s,x(s))} = \frac{1}{T}\int_0^T \varphi(\tau,s,x(s))\,ds\,,$$

$$\overline{I(x(t_i-0))} = \frac{1}{T}\sum_{0<t_i<T} I_i(x(t_i-0))\,.$$

PROOF. Each of the functions of the sequence (11) is T-periodic with respect to t. For $x_0 \in D_0$ from Lemma 1 of [2] for $t \in [0,T]$ we get

$$\| x_m(t,x_0) - x_0 \| \le M\alpha(t) + 2pM \tag{12}$$

for each $m = 1, 2, 3, \ldots$ where

$$\alpha(t) = 2t\left(1 - \frac{t}{T}\right) \le \frac{T}{2}\,. \tag{13}$$

From (12) and (13) we obtain

$$\|x_m(t, x_0) - x_0\| \le \frac{MT}{2} + 2pM = \frac{MT}{2}\left(1 + \frac{4p}{T}\right). \tag{14}$$

From the last estimate it follows that for each $m = 1, 2, \ldots$; $t \in \mathbf{R}$ (from the periodicity) and for each $x_0 \in D_0$ the functions $x_m(t, x_0)$ exist and belong to the set D_1.

Now we prove the convergence of the sequence (11). For this purpose we estimate the expression

$$\|x_{m+1}(t, x_0) - x_m(t, x_0)\| .$$

For $m = 0$ from (12), (13) and (14) we have

$$\|x_1(t, x_0) - x_0\| \le M\alpha(t) + 2pM \le \frac{MT}{2}\left(1 + \frac{4p}{T}\right) = M_1 . \tag{15}$$

For $m = 1$, using Lemma 1 from [2] and the inequalities (4) and (15) we get

$$\|x_2(t, x_0) - x_1(t, x_0)\| \le \left(1 - \frac{t}{T}\right) \int_0^t \{K_1\|x_1(\tau, x_0) - x_0\| +$$

$$+ K_2\| \int_0^\tau [\varphi(\tau, s, x_1(s, x_0)) - \overline{\varphi(\tau, s, x_1(s, x_0))}]ds - \int_0^\tau [\varphi(\tau, s, x_0) -$$

$$- \overline{\varphi(\tau, s, x_0)}]\, ds\|\}\, d\tau + \frac{t}{T} \int_t^T \{K_1\|x_1(\tau, x_0) - x_0\| + K_2\| \int_0^\tau [\varphi(\tau, s, x_1(s, x_0)) -$$

$$- \overline{\varphi(\tau, s, x_1(s, x_0))}]\, ds - \int_0^\tau [\varphi(\tau, s, x_0) - \overline{\varphi(\tau, s, x_0)}]\, ds\|\}d\tau + 2pK_4M_1 \le$$

$$\le \left(1 - \frac{t}{T}\right) \int_0^t \left\{K_1\|x_1(\tau, x_0) - x_0\| + K_2K_3\left(1 - \frac{\tau}{T}\right) \int_0^\tau \|x_1(s, x_0) - x_0\|\, ds +\right.$$

$$\left. + K_2K_3 \frac{\tau}{T} \int_\tau^T \|x_1(s, x_0) - x_0\|\, ds\right\} d\tau + \frac{t}{T} \int_t^T \left\{K_1\|x_1(\tau, x_0) - x_0\| +\right.$$

$$\left. + K_2K_3\left(1 - \frac{\tau}{T}\right) \int_0^\tau \|x_1(s, x_0) - x_0\|\, ds + K_2K_3 \frac{\tau}{T} \int_\tau^T \|x_1(s, x_0) - x_0\|\, ds\right\} d\tau +$$

$$+ 2pK_4M_1 \le \left(K_1 + \frac{K_2K_3T}{2}\right) M_1\alpha(t) + 2pK_4M_1 \le$$

$$\le \left(K_1 + \frac{K_2K_3T}{2}\right) \frac{M_1T}{2} + 2pK_4M_1 \equiv M_2 .$$

Hence

$$\|x_2(t, x_0) - x_1(t, x_0)\| \le \left(K_1 + \frac{K_2K_3T}{2}\right)\frac{M_1T}{2} + 2pK_4M_1 \equiv M_2\,. \tag{16}$$

Assume that for some m the following inequality holds:

$$\|x_m(t, x_0) - x_{m-1}(t, x_0)\| \le \left(K_1 + \frac{K_2K_3T}{2}\right)\frac{M_{m-1}T}{2} + 2pK_4M_{m-1} \equiv M_m\,. \tag{17}$$

For $m + 1$ we find, using Lemma 1 from [2] and the inequalities (4) and (17) that

$$\|x_{m+1}(t, x_0) - x_m(t, x_0)\| \le \left(1 - \frac{t}{T}\right)\int_0^t \Big\{K_1\|x_m(\tau, x_0) - x_{m-1}(\tau, x_0)\| +$$

$$+ K_2K_3\left(1 - \frac{\tau}{T}\right)\int_0^\tau \|x_m(s, x_0) - x_{m-1}(s, x_0)\|\,ds + K_2K_3\frac{\tau}{T}\int_\tau^T \|x_m(s, x_0) -$$

$$- x_{m-1}(s, x_0)\|\,ds\Big\}\,d\tau + \frac{t}{T}\int_t^T K_1\|x_m(\tau, x_0) - x_{m-1}(\tau, x_0)\| +$$

$$+ K_2K_3\left(1 - \frac{\tau}{T}\right)\int_0^\tau \|x_m(s, x_0) - x_{m-1}(s, x_0)\|\,ds + K_2K_3\frac{\tau}{T}\int_\tau^T \|x_m(s, x_0) -$$

$$- x_{m-1}(s, x_0)\|\,ds\Big\}\,d\tau + 2pK_4M_m \le \left(K_1 + \frac{K_2K_3T}{2}\right)\frac{M_mT}{2} +$$

$$+ 2pK_4M_m \equiv M_{m+1}\,.$$

By the method of the mathematical induction we conclude that, for each $m = 0, 1, 2, \ldots,$

$$\|x_{m+1}(t, x_0) - x_m(t, x_0)\| \le \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]M_m \equiv M_{m+1}\,,$$

where

$$M_1 = \frac{MT}{2}\left(1 + \frac{4p}{T}\right),$$

$$M_m = \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]^{m-1}\frac{MT}{2}\left(1 + \frac{4p}{T}\right).$$

Hence, the last inequality can be rewritten as

$$\|x_{m+1}(t, x_0) - x_m(t, x_0)\| \le \frac{MT}{2}\left(1 + \frac{4p}{T}\right)\left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]^m. \tag{18}$$

Then for $||x_{m+k}(t, x_0) - x_m(t, x_0)||$ we obtain the estimate

$$\|x_{m+k}(t, x_0) - x_m(t, x_0)\| \le \frac{MT}{2}\left(1 + \frac{4p}{T}\right)\left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]^m \times \tag{19}$$

$$\times \frac{1 - \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]}{1 - \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]},$$

from which, by the condition B2 the uniform in $(t, x_0) \in \mathbf{R} \times D_0$ convergence of the sequence (11) follows. If we denote $x_\infty(t, x_0) = \lim_{m \to \infty} x_m(t, x_0)$ then the following estimation holds:

$$\|x_\infty(t, x_0) - x_m(t, x_0)\| \le \frac{\frac{MT}{2}\left(1 + \frac{4p}{T}\right)\left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]^m}{1 - \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]}. \tag{20}$$

Tending to the limit in (11) $m \to \infty$ we get that $x_\infty(t, x_0)$ is a periodic solution of the equation

$$x(t, x_0) = x_0 + \int_0^t \Big\{f\Big(\tau, x(\tau, x_0), \int_0^\tau [\varphi(\tau, s, x(s, x_0)) - \overline{\varphi(\tau, s, x(s, x_0))}]\, ds\Big) -$$

$$- \overline{f\Big(\tau, x(\tau, x_0), \int_0^\tau [\varphi(\tau, s, x(s, x_0))) - \overline{\varphi(\tau, s, x(s, x_0))}]\, ds\Big)}\Big\}\, d\tau + \tag{21}$$

$$+ \sum_{0<t_i<t} I_i(x(t_i - 0)) - t\overline{I(x(t_i - 0))}.$$

On the other hand, since the function $\psi(t)$ is a periodic solution of the system (6), for which (8) holds, then from Lemma 1 of [1], the function will satisfy the condition:

$$\frac{1}{T}\int_0^T f\left(\tau, \psi(\tau), \int_0^\tau \varphi(\tau, s, \psi(s))\, ds\right) d\tau + \frac{1}{T}\sum_{0<t_i<T} I_i(\psi(t_i - 0)) = 0.$$

Hence $\psi(t)$ is a solution of the equation (21) as well.

In order to complete the proof we have to show the uniqueness of the solution of the equation (21). Assume the opposite. Let $x(t, x_0)$ and $z(t, x_0)$ be two solutions of (21). Then for their difference we have

$$\|x(t, x_0) - z(t, x_0)\| \leq \left(1 - \frac{t}{T}\right) \int_0^t \left\{K_1\|x(\tau, x_0) - z(\tau, x_0)\| + \right.$$

$$+ K_2K_3\left(1 - \frac{\tau}{T}\right) \int_0^\tau \|x(s, x_0) - z(s, x_0)\| \, ds + K_2K_3\frac{\tau}{T} \int_\tau^T \|x(s, x_0) -$$

$$\left. - z(s, x_0)\| \, ds\right\} d\tau + \frac{t}{T} \int_t^T \left\{K_1\|x(\tau, x_0) - z(\tau, x_0)\| + K_2K_3\left(1 - \frac{\tau}{T}\right) \times \right.$$

$$\left. \times \int_0^\tau \|x(s, x_0) - z(s, x_0)\| \, ds + K_2K_3\frac{\tau}{T} \int_\tau^T \|x(s, x_0) - z(s, x_0)\| \, ds\right\} d\tau +$$

$$+ \left\| \sum_{0<t_i<t} I_i(x(t_i - 0)) - I_i(z(t_i - 0)) - t\,\overline{I(x(t_i - 0))} + t\,\overline{I(z(t_i - 0))} \right\|.$$

Introduce the notation

$$\|x(t, x_0) - z(t, x_0)\| = r(t), \quad |r(t)|_0 = \max_t |r(t)|$$

Then (22) can be rewritten as

$$r(t) \leq \left(1 - \frac{t}{T}\right) \int_0^t \left\{K_1|r(t)|_0 + \frac{K_2K_3T}{2}|r(t)|_0\right\} d\tau + \tag{23}$$

$$+ \frac{t}{T} \int_t^T \left\{K_1|r(t)|_0 + \frac{K_2K_3T}{2}|r(t)|_0\right\} d\tau + 2pK_4|r(t)|_0 \leq$$

$$\leq \left(K_1 + \frac{K_2K_3T}{2}\right) \frac{T}{2} |r(t)|_0 + 2pK_4|r(t)|_0 =$$

$$= \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right] |r(t)|_0 .$$

If we replace the right hand side of the inequality (22) with the right hand side of the inequality (23) and continue this process further then, after the m-substitution we have

$$r(t) \leq \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]^{m-1} |r(t)|_0 ,$$

which implies:

$$|r(t)|_0 \leq \left[\frac{T}{2}\left(K_1 + \frac{K_2K_3T}{2}\right) + 2pK_4\right]^{m-1} |r(t)|_0 .$$

Tending to the limit with $m \to \infty$ in the last inequality from B2, we get $|r(t)|_0 = 0$, i.e., $r(t) = 0$. The proof of the theorem is complete.

Consider the problem of existence of periodical solutions of the system (6). Denote by $\Delta(x_0)$ the expression

$$(24) \qquad \Delta(x_0) = \frac{1}{T}\int_0^T f\left(t, x_\infty(t, x_0), \int_0^t \varphi(t, s, x_\infty(s, x_0))\, ds\right) dt + \\ + \frac{1}{T}\sum_{0<t_i<T} I_i(x_\infty(t_i - 0), x_0)$$

where $x_\infty(t, x_0)$ is the limit of the sequence (11).

Since $x_\infty(t, x_0)$ is a periodic solution of the equation (21), then for $\Delta(x_0) = 0$ and

$$\overline{\varphi(t, s, x_\infty(s, x_0))} = 0$$

the function $x_\infty(t, x_0)$ is a periodical solution of the system (6). In such a way the existence of periodic solutions of the system (6) is connected with the existence of zeroes of the function $\Delta(x_0)$ and with the relation

$$\overline{\varphi(t, s, x_\infty(s, x_0))} = 0 .$$

However, to find the function $\Delta(x_0)$ is practically impossible. Then the following problem arises: how, using the function

$$(25) \qquad \Delta_m(x_0) = \frac{1}{T}\int_0^T f(t, x_m(t, x_0), \int_0^t \varphi(t, s, x_m(s, x_0))\, ds)\, dt + \\ + \frac{1}{T}\sum_{0<t_i<T} I_i(x_m(t_i - 0, x_0)) ,$$

to conclude about the existence of zeroes of the function $\Delta(x_0)$.

The following result holds:

THEOREM 2. *Let the following conditions hold:*

1. *The conditions* (A) *and* (B) *are fulfilled.*

2. *For some integer* $m \geq 0$ *the function* $\Delta_m(x_0)$ *has an isolated singular point*: $\Delta_m(x^0) = 0$.

3. *The index of this point is different from zero.*

4. *For each* $x_0 \in D_0$ *uniformly in* $t \in [0, T]$ *there exists*

$$\lim_{m\to\infty} \int_0^T \varphi(t, s, x_m(s, x_0))\, ds = 0 .$$

For this m for which Condition 2 holds, the condition

$$\overline{\varphi(t, s, x_m(s, x_0))} = 0$$

is fulfilled.

5. *There exists a closed convex region $D \subseteq D_0$ with a unique singular point x^0 such that on its boundary the following inequality holds*:

$$\inf_{x \in \Gamma_D} \|\Delta_m(x)\| \geq$$

$$\geq \frac{\frac{MT}{2}\left(1+\frac{4p}{T}\right)\left(K_1+\frac{K_2K_3T}{2}+2pK_4\right)\left[\frac{T}{2}\left(K_1+\frac{K_2K_3T}{2}\right)+2pK_4\right]^m}{1-\left[\frac{T}{2}\left(K_1+\frac{K_2K_3T}{2}\right)+2pK_4\right]}.$$

Then the system (6) *has a T-periodic solution $x = x(t)$, for which $x(0) \in D$.*

The proof of this theorem is analogous to the proof of Theorem 1 from [3] and uses the inequality

$$\|\Delta(x_0) - \Delta_m(x_0)\| <$$

$$< \frac{\frac{MT}{2}\left(1+\frac{4p}{T}\right)\left(K_1+\frac{K_2K_3T}{2}+2pK_4\right)\left[\frac{T}{2}\left(K_1+\frac{K_2K_3T}{2}\right)+2pK_4\right]^m}{1-\left[\frac{T}{2}\left(K_1+\frac{K_2K_3T}{2}\right)+2pK_4\right]}.$$

Consider now the equations with a general impulse effect. Let the impulse effect occurs when the mapping point reaches the hypersurface $t = t_i(x)$. Then the system (1) differs essentially from the system (6). In this case we cannot use an iteration process since at each step the function $x_m(t)$ has discontinuities different, in general, from the discontinuities of the function $x_{m-1}(t)$. Moreover, in the system (1) it is possible to get "rolling" of some of its solutions on the surface $t = t_i(x)$, i.e., reaching of the same surface by the same solution several times. We assume that there is no "rolling" of the solutions of the system (1) on the surface $t = t_i(x)$, i.e., the solution goes through every surface only once. For that purpose we need some additional ssumptions for the functions $t_i(x)$ and $I_i(x)$ which are given by the following lemma:

LEMMA 1 (see [4]). *Let the function $I_i(x)$* and $t_i(x)$ *in the system of equations* (1) *satisfy the conditions* (A) *and* (B) *and the inequality*

$$\sup_{0 \leq \sigma \leq 1} \left\langle \frac{\partial t_i(x+\sigma I_i(x))}{\partial x}, I_i(x) \right\rangle \leq 0 \tag{26}$$

for all $i = 1, 2, \ldots, p$ and $x \in D_1$. Then for sufficiently small N the solutions of the equation (1) *pass every surface $t = t_i(x)$, $i = 1, 2, \ldots, p$ only once.*

Further we consider such equations only, for which the conditions of the Lemma 1 hold.

For constructing a periodic solution of the system (1), we proceed as in [1]. We fix p points $z^{(j)} \in D_1$, $j = 1, 2, \ldots, p$, and build a sequence of T-periodic functions

$$(27)\quad x_{m+1}(t, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)}) = x_0 + \int_0^t \{f(\tau, x_m(\tau, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)}),$$

$$\int_0^\tau [\varphi(\tau, s, x_m(s, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)})) - \overline{\varphi(\tau, s, x_m(s, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)}))}]\, ds -$$

$$\overline{- f(\tau, x_m(\tau, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)}), \int_0^\tau [\varphi(\tau, s, x_m(s, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)})) -}$$

$$\overline{- \overline{\varphi(\tau, s, x_m(s, x_0, z^{(1)}, z^{(2)}, \ldots, z^{(p)}))}]\, ds)}\}\, d\tau + \sum_{0 < t_i(z^{(i)}) < t} I_i(z^{(i)}) - t\overline{I(z^{(i)})}\,.$$

If this sequence converges uniformly in $t \in [0, T]$ then the limit function $x_\infty(t, x_0, z^{(1)}, z^{(2)} \ldots, z^{(p)})$ is a T-periodic solution of the system of equations

$$(28)\qquad \dot{x} = f(t, x(t), \int_0^t [\varphi(t, s, x(s)) - \overline{\varphi(t, s, x(s))}]\, ds) -$$

$$- \overline{f(t, x(t), \int_0^t [\varphi(t, s, x(s)) - \overline{\varphi(t, s, x(s))}]\, ds)} - \frac{1}{T} \sum_{i=1}^p I_i(z^{(i)}), \quad t \neq t_i(z^{(i)})$$

$$\Delta x|_{t=t_i(z^{(i)})} = I_i(z^{(i)})\,.$$

If we choose $x_0, z^{(1)}, z^{(2)} \ldots, z^{(p)}$ from the conditions

$$(29)\quad \overline{f(t, x_\infty(t, x_0, z^{(1)}, \ldots, z^{(p)}) \int_0^t [\varphi(t, s, x_\infty(s, x_0, z^{(1)}, z^2, \ldots, z^{(p)})) -}$$

$$\overline{- \overline{\varphi(t, s, x_\infty(s, x_0, z^{(1)}, \ldots, z^{(p)}))}]\, ds)} + \frac{1}{T} \sum_{i=1}^p I_i(z^{(i)}) = 0\,,$$

$$z^{(i)} = x_\infty(t_i(z^{(i)}), x_0, z^{(1)}, \ldots, z^{(p)})\,, \qquad i = 1, 2, \ldots, p$$

where

$$\overline{\varphi(t, s, x_\infty(s, x_0, z^{(1)}, \ldots, z^{(p)}))} = 0$$

identically in t, then the limit function $x_\infty(t, x_0, z^{(1)}, \ldots, z^{(p)})$ will be the desired periodical solution of the system (1).

In such a way, for proving the existence of T-periodic solutions of the system (1) we need the following:

1. Prove the uniform convergence of the sequence (27).
2. Solve the system (29).

For getting

$$\overline{\varphi(t, s, x_\infty(s, x_0, z^{(1)}, \dots, z^{(p)}))} = 0$$

it is sufficient to assume that for every point from D_1 uniformly in $t \in [0, T]$

$$\lim_{m\to\infty} \frac{1}{T}\int_0^T \varphi(t, s, x_m(s, x_0, z^{(1)}, \dots, z^{(p)}))\, ds = 0 .$$

In order to simplify the exposition we assume that $p = 1$ in the system (1). Then

$$I_i(x) = I(x), \qquad t_i(x) = t(x) + iT , \tag{30}$$

and the sequence (27) can be written as

$$\begin{aligned} x_{m+1}(t, x_0, z) = x_0 + \int_0^t \Big\{ f\Big(\tau, x_m(\tau, x_0, z), \int_0^\tau [\varphi(\tau, s, x_m(s, x_0 z)) - \\ - \overline{\varphi(\tau, s, x_m(s, x_0, z))}]\, ds\Big) - \overline{f\Big(\tau, x_m(\tau, x_0, z), \int_0^\tau [\varphi(\tau, s, x_m(s, x_0, z)) -} \\ \overline{- \overline{\varphi(\tau, s, x_m(s, x_0, z))}]\, ds\Big)} \Big\}\, d\tau + \sum_{0<t(z)<t} I(z) - \frac{t}{T} I(z), \qquad 0 \le t \le T \end{aligned} \tag{31}$$

The conditions (29) can be written as

$$\Delta(x_0, z) = \overline{f\left(t, x_\infty(t, x_0, z), \int_0^t \varphi(t, s, x_\infty(s, x_0, z))\, ds\right)} + \frac{1}{T} I(z) = 0 , \tag{32}$$

$$z = x_\infty(t(z), x_0, z)$$

if the condition

$$\overline{\varphi(t, s, x_\infty(s, x_0, z))} = 0$$

holds.

From the proof of Theorem 1 it follows that the sequence (31) converges uniformly for every $x_0 \in D_0$ and $z \in D_1$ to the T-periodic limit function $x_\infty(t, x_0, z)$.

We establish some properties of the functions $x_m(t, x_0, z)$ and x_∞.

LEMMA 2. *There exists a positive constant $K' = K'(K_1, K_2, K_3, K_4)$ such that for every $y, z \in D_1$, $t(y) \le t(z)$ the following inequality holds:*

$$\|x_m(t, x_0, y) - x_m(t, x_0, z)\| \le K' \|y - z\| \tag{33}$$

uniformly in $0 \le t < t(y)$, $t(z) < t \le T$ for each $m = 1, 2, \dots$.

PROOF. From (31) for $m = 0$ we have

$$\|x_1(t, x_0, y) - x_1(t, x_0, z)\| \leq \left(1 - \frac{t}{T}\right) \int_0^t \Big\{K_1\|y - z\| + K_2K_3\left(1 - \frac{\tau}{T}\right)\times$$

$$\times \int_0^\tau \|y - z\|\, ds + K_2K_3\frac{\tau}{T}\int_\tau^T \|y - z\|\, ds\Big\}d\tau + \frac{t}{T}\int_t^T K_1\|y - z\| + K_2K_3\left(1 - \frac{\tau}{T}\right)\times$$

$$\times \int_0^t \|y - z\|\, ds + K_2K_3\frac{\tau}{T}\int_\tau^T \|y - z\|\, ds\Big\}\, d\tau + K_4\|y - z\| \leq$$

$$\leq \left(1 - \frac{t}{T}\right)\int_0^t \left(K_1 + \frac{K_2K_3T}{2}\right)\|y - z\|\, d\tau + \frac{t}{T}\int_t^T \left(K_1 + \frac{K_2K_3T}{2}\right)\|y - z\|\, d\tau +$$

$$+ K_4\|y - z\| \leq \left[\left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2} + K_4\right]\|y - z\|$$

when $0 \leq t < t(y)$ or $t(z) < t \leq T$.

By the method of the mathematical induction we get that for each $m = 1, 2, \ldots.$

$$\|x_m(t, x_0, y) - x_m(t, x_0, z)\| \leq$$

$$\leq \left\{\left[\left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2}\right]^m + K_4\frac{1 - \left[\left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2}\right]^m}{1 - \left[\left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2}\right]}\right\}\|y - z\|.$$

From the condition B2 we conclude that

$$\left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2} < 1.$$

Thus for completing the proof it is sufficient to take

$$K' = \frac{K_4}{1 - \left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2}} + \left(K_1 + \frac{K_2K_3T}{2}\right)\frac{T}{2}.$$

COROLLARY. *The functions $x_m(t(z), x_0, z)$ satisfy the Lipschitz condition with respect to z.*

PROOF. Let $z \in D_1$, $z' \in D_1$ and $t(z) < t(z')$. Then

$$(34)\qquad \|x_m(t(z), x_0, z) - x_m(t(z'), x_0, z')\| \leq x_m(t(z), x_0, z) - \\ - x_m(t(z), x_0, z')\| + \|x_m(t(z), x_0, z') - x_m(t(z'), x_0, z')\| \leq \\ \leq K'\|z - z'\| + \|x_m(t(z), x_0, z') - x_m(t(z'), x_0, z')\| \leq \\ \leq K'\|z - z'\| + MN\left(2 + \frac{1}{T}\right)\|z - z'\| = \left[K' + MN\left(2 + \frac{1}{T}\right)\right]\|z - z'\|$$

for each $m = 1, 2, \ldots$.

LEMMA 3. *The function $x_m(t(z), x_0, z)$ satisfies the Lipschitz condition with respect to z with a constant*

$$N' = K' + MN\left(2 + \frac{1}{T}\right).$$

The proof follows from the uniform convergence of the sequence $x_m(t, x_0, z)$ and from the corollary of Lemma 2.

If $N' < 1$, the equation $z = x_\infty(t(z), x_0, z)$ is solvable in the form $z = z(x_0)$. Then the problem of existence of T-periodical solutions of the equation (1) is transformed to the problem of existence of zeroes of the function $\Delta(x_0, z(x_0))$ and satisfying the condition

$$\overline{\varphi(t, s, x_\infty(x_0, s, z(x_0)))} = 0 .$$

In many cases this problem can be solved using the function $\Delta_m(x_0, z_m(x_0))$ under the condition that

$$\lim_{m \to \infty} \frac{1}{T} \int_0^T \varphi(t, s, x_m(s, x_0, z_m(x_0))) \, ds = 0$$

uniformly in $t \in [0, T]$ where $z_m(x_0)$ is the solution of the equation $z = x_m(t(z), x_0, z)$.

The following theorem is true:

THEOREM 3. *Suppose that the following conditions hold:*

1. *The conditions* (A), (B) *and* (26) *hold and* $N' < 1$.
2. *For some integer $m \geq 0$ the mapping*

$$\Delta_m(x_0, z_m(x_0)) \colon D_0 \to \mathbf{R}^n$$

has an isolated singular point with a nonzero index.

3. *For each* $x_0 \in D_0$ *uniformly in* $t \in [0, T]$ *there exists*

$$\lim_{m\to\infty} \frac{1}{T} \int_0^T \varphi(t, s, x_m)\, ds = 0$$

and for this m for which Condition 2 *is fulfilled, it holds*

$$\overline{\varphi(t, s, x_m(s, x_0, z_m(x_0))} = 0 \, .$$

4. *There exists a closed convex region* $D \subseteq D_0$ *having an isolated singular point such that on its boundary* Γ_D *the following inequality holds*:

$$\inf_{x\in\Gamma_D} \|\Delta_m(x_0, z_m(x_0)\| \geq$$

$$\geq \frac{\dfrac{MT}{2}\left(1 + \dfrac{4p}{T}\right)\left(K_1 + \dfrac{K_2K_3T}{2} + 2pK_4\right)\left[\dfrac{T}{2}\left(K_1 + \dfrac{K_2K_3T}{2}\right) + 2pK_4\right]^m}{1 - \left[\dfrac{T}{2}\left(K_1 + \left(\dfrac{K_2K_3T}{2}\right) + 2pK_4\right]\right.} \, .$$

Then the system (1) *has a T-periodic solution* $x = x(t)$, $x(0) \in D_0$, *and this solution can be found as a limit of the sequence* (31).

The proof of this result is analogous to the proof of Theorem 1 from [3] and uses the inequality

$$\|\Delta(x_0) - \Delta_m(x_0)\| <$$

$$< \frac{\dfrac{MT}{2}\left(1 + \dfrac{4p}{T}\right)\left(K_1 + \dfrac{K_2K_3T}{2} + 2pK_4\right)\left[\dfrac{T}{2}\left(K_1 + \dfrac{K_2K_3T}{2}\right) + 2pK_4\right]^m}{1 - \left[\dfrac{T}{2}\left(K_1 + \dfrac{K_2K_3T}{2}\right) + 2pK_4\right]} \, .$$

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