Geometric and Functional Analysis
Vol. 1, No. 3 (1991)
1016-443X/91/0300231-22$1.50+0.20/0

DIFFEOMORPHISM FINITENESS FOR MANIFOLDS WITH RICCI CURVATURE AND $L^{n/2}$-NORM OF CURVATURE BOUNDED

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0. Introduction

The main purpose of this note is to prove the following finiteness theorem:

THEOREM 0.1. *For positive constants D, v, λ and Λ, the collection of closed, connected Riemannian n-manifolds M^n satisfying the bounds*

$$\begin{aligned} &\operatorname{diam}(M^n) \leq D \ , \\ &\operatorname{vol}(M) \geq v \ , \\ &|\operatorname{Ric}_{M^n}| \leq \lambda \ , \end{aligned} \tag{0.2}$$

and

$$\int_{M^n} |R|^{n/2} \leq \Lambda \ , \tag{0.3}$$

contains at most a finite number, $c = c(n, vD^{-n}, \lambda D^2, \Lambda)$ of diffeomorphism types.

Of course, Theorem 0.1 generalizes the finiteness theorem of the second author, [C2], which proves the same conclusion under the sectional curvature bound, $|R| \leq \Lambda$. But in contrast to the situation in that case, there is no smooth compactness theorem for the space of Riemannian manifolds satisfying (0.2), (0.3) (compare e.g. [GLP]). In fact, its closure in the Gromov-Hausdorff topology contains Riemannian spaces with isolated singular points. However, as we explain in Remark 3.2, the extent of the failure of the compactness theorem in our setup can be specified precisely in terms of a finite number of additional parameters.

Partially supported by NSF Grants.

Let us recall some previous results concerning the class of manifolds in Theorem 0.1. By [GLP], any sequence of closed n-manifolds (M_i, g_i) satisfying $\mathrm{diam}(M_i) \leq D$ and $\mathrm{Ric}_{M_i} \geq -\lambda$, has a subsequence which converges in the Gromov-Hausdorff metric to a length space M_∞, with $\mathrm{diam}(M_\infty) \leq D$. By [An2, Theorem 2.6], if in addition, the remaining hypotheses of Theorem 0.1 hold, then M_∞ is an orbifold with at most $N = N(n, vD^{-n}, \lambda D^2, \Lambda)$ singular points (the set of singular points is empty if $\Lambda \leq \delta(n, vD^{-n}, \lambda D^2)$ is sufficiently small). Away from these, the metrics converge in the $L^{2,p}$ topology (for all $p < \infty$), and hence in the $C^{1,\alpha}$, topology (for all $\alpha < 1$). At the set of points of M_i which lie close to the singular points of M_∞ (relative to a suitable Hausdorff approximation) the geometry is not uniformly bounded with respect to i. Thus, to prove Theorem 0.1, we must show that the degeneration of the metric can only happen in a tightly controlled fashion.

Building on the results in [An1] and [BKN], Bando has studied this degeneration for Einstein manifolds, satisfying (0.2),(0.3); see [B]. In this case, the degeneration can be modeled in terms of a finite sequence of rescalings of Ricci-flat orbifolds $V_1, \ldots, V_k$, each of which is complete and non-compact, but well-behaved at infinity (so called ALE or asymptotically locally Euclidean). In slightly more detail, as indicated above, given a sequence of compact Einstein manifolds (N_i, g_i) satisfying (0.2) and (0.3), a subsequence converges in the Gromov-Hausdorff topology to a compact Einstein orbifold V_0, with a finite number of singular points $\{q_j\}$. Near each singular point $q \in \{q_j\}$, one can find a suitable rescaling, $\{r_i\} \to 0$ of the metrics g_i, so that the new sequence of pointed manifolds $(M_i, r_i^{-2} g_i, x_i)$, with $x_i \to q$, converges in the pointed Gromov-Hausdorff topology to a complete, non-compact Ricci-flat orbifold V_1, with a finite number of singular points, and with

$$\int_{V_1} |R|^{n/2} \geq \theta \; , \tag{0.4}$$

for a fixed positive constant $\theta = \theta(n, vD^{-n}, \lambda D^2)$. If V_1 is not smooth, then one can find a second sequence of rescalings associated with each singular point of V_1, and so on. Using the bounds (0.3) and (0.4), one shows this process terminates after a finite number of iterations.

The topology of the Einstein manifolds N_i, for i sufficiently large, in a Hausdorff neighborhood of the limit singular point, $q \in V_0$, is then determined by a finite sequence of orbifolds, $V_1, \ldots, V_k$, as follows. A neighborhood of infinity of V_1 is diffeomorphic to an annulus about the singular point q. Thus one can remove a small ball about q and glue in the orbifold V_1. This process is repeated on the singular points of V_1, and so on. At the final stage, one is glueing in smooth manifolds. This determines the topology of N_i near q.

As it turns out, the above description, suitably formulated, is also valid for manifolds satisfying the bounds (0.2) and (0.3). (We note however that the explicit statement of Theorem 0.1 is new even for Einstein manifolds). To a significant extent, our argument is inspired by that of Bando [B]. However, partly due to the more general hypothesis $|\mathrm{Ric}_{M^n}| \leq \lambda$, technically it is somewhat different.

There are now numerous examples showing that orbifold singular limit spaces actually occur for sequences satisfying the bounds (0.2),(0.3); see [An1], [H], and Remark 3.6. On the other hand, the degeneration of the metric for such sequences is so well-behaved, that many geometric invariants are controlled almost as well as in the case of a family which is $C^{1,\alpha}$ compact.

For example, apart from at most a finite set of eigenvalues which are equal to or approach zero, the spectrum on p-forms of (M_i, g_i) can be shown to converge to that of M_∞ (the spectrum of M_∞ is well defined). Further, the number of such small eigenvalues can be bounded in terms of the bounds (0.2),(0.3).

1. The Neck Theorem

The proof of Theorem 0.1 will entail consideration of regions which are of three distinct types. These are the regions on which the curvature does not concentrate strongly, in $L^{n/2}$ norm, the regions on which it does concentrate, and the regions of transition between those of the previous two types.

The main purpose of this section, is to prove the technical result, Theorem 1.18, needed to handle the transition regions. We will make use of a result which concerns the connectedness of small annuli.

Let $B_p(r)$ (or just $B(r)$) denote the metric ball of radius r about p. In this paper, we will always be dealing with balls such that for $s < r$, the closure of $B_p(s)$ in $B_p(r)$ is complete. Let $S_p(r)$ (or $S(r)$) denote $\partial B_p(r)$, the set of points at distance r from p.

Let $A_{r_1,r_2}(p)$ (or A_{r_1,r_2}) denote a geodesic annulus. Thus, for some $p \in M$,

$$A_{r_1,r_2}(p) = B_p(r_2) - \overline{B}_p(r_1) . \tag{1.1}$$

LEMMA 1.2. *Given $n, \lambda, v > 0$ there exist $c_0 = c_0(n, v\lambda^{n/2})$, $R' = R'(n, v\lambda^{n/2}) > 0$ with the following property. Let M^n be a Riemannian manifold such that for all $q \in M^n$,*

$$\begin{aligned} &Ric_{B_q(r)} \geq -\lambda , \\ &vol\, B_q(r) \geq vr^n , \quad for \quad r < \operatorname{dist}(q, \partial M) . \end{aligned} \tag{1.3}$$

Then if

$$\begin{aligned} &s \leq c_0 \lambda^{-1/2} , \\ &R' < R , \\ &Rs < \operatorname{dist}(p, \partial M) , \end{aligned} \tag{1.4}$$

the annulus $A_{s,Rs}$ has at most one component whose intersection with $S_p(Rs)$ is nonempty.

Proof: By an obvious scaling argument, it suffices to assume $\lambda = 1$. Suppose there are at least two such components $\{D_i\}$. We may assume that if $C = D_1 \cap B_p(\frac{1}{3}Rs)$, then for any $i \neq 1$,

$$\operatorname{vol} C \leq \operatorname{vol} D_i \cap B_p(\tfrac{1}{3}Rs) . \tag{1.5}$$

In particular,

$$\operatorname{vol} A_{s,\frac{1}{3}Rs} \leq 2 \operatorname{vol}(B(\tfrac{1}{3}Rs) \setminus C) . \tag{1.6}$$

Now choose a point $q \in C \cap S(\frac{1}{3}Rs)$ and note that any minimal geodesic $\gamma(t)$ from q to a distinct component of $A_{s,\frac{1}{3}Rs}$ has length at most $\frac{2}{3}Rs$ and intersects $B(s)$ for some $t \in [\frac{1}{3}Rs - s, \frac{1}{3}Rs + s]$. Let $A^0_{u,v}(q)$ denote the set of points on such $\gamma(t)$, for $t \in [u, v]$. Thus,

$$B(\frac{1}{3}Rs) \setminus C \subset A^0_{\frac{1}{3}Rs - s, \frac{2}{3}Rs}(q) , \tag{1.7}$$

while, by the triangle inequality,

$$A^0_{\frac{1}{3}Rs-s,\frac{1}{3}Rs+s}(q) \subset B(3s) . \tag{1.8}$$

Putting these together, we obtain

$$\frac{\operatorname{vol} A_{s,\frac{1}{3}Rs}}{\operatorname{vol} B(3s)} \le 2\frac{\operatorname{vol} B((\frac{1}{3}Rs)\setminus C)}{\operatorname{vol} B(3s)} \le \frac{\operatorname{vol} A^0_{\frac{1}{3}Rs-s,\frac{2}{3}Rs}(q)}{\operatorname{vol} A^0_{\frac{1}{3}Rs-s,\frac{1}{3}Rs+s}(q)} \le C(n,c_0R)R , \tag{1.9}$$

where the last inequality follows from the proof of the relative volume comparison theorem. But the relative volume comparison theorem also gives

$$\frac{\operatorname{vol} A_{s,\frac{1}{3}Rs}}{\operatorname{vol} B(3s)} \ge \frac{\operatorname{vol} B_{\frac{1}{3}Rs}}{\operatorname{vol} B(3s)} - 1 \ge c(n,c_0)vR^n - 1 . \tag{1.10}$$

The lemma then follows by choosing $R' = R'(n, v\lambda^{n/2})$ sufficiently large, and say $c_0 = (R')^{-1}$. □

In the next corollary, the hypotheses and notation are as in the previous lemma.

COROLLARY 1.11. *Let $q_1, q_2 \in S_p(r)$, with $r < c_0R'\lambda^{-1/2}$. Then q_1, q_2 can be connected by a curve lying in $\overline{A_{r(R')^{-1},r}}$, of length at most $c_1(n, v\lambda^{n/2})r$.*

Proof: Given Lemma 1.1, this follows immediately from [AG, Proposition 1.1], or a similar argument given in [An1]. □

Before proceeding to Theorem 1.18, we introduce some notation and preliminary concepts.

Let $C(S^{n-1}/\Gamma)$ denote the metric cone on a spherical space form, S^{n-1}/Γ, where $\Gamma \subset O(n)$ is a finite subgroup, possibly trivial. Thus, $C(S^{n-1}/\Gamma)$ is isometric to R^n/Γ. Let $C_{r_1,r_2}(S^{n-1}/\Gamma)$ denote the annulus, $A_{r_1,r_2}(O) \subset C(S^{n-1}/\Gamma)$, with center O, the vertex of $C(S^{n-1}/\Gamma)$. In the course of proving Theorem 1.18, we will compare certain annuli in the manifold M^n, with corresponding "standard" annuli in $C(S^{n-1}/\Gamma)$.

If (M, g) is the open Riemannian manifold, the metric space completion of M is denoted by $\overline{M}$ and we put

$$M_\epsilon = \{x \in M \mid \operatorname{dist}(x, \partial M) > \epsilon\} , \tag{1.12}$$

where $\partial M = \overline{M} \setminus M$.

A diffeomorphism of Riemannian manifolds $f : (M,g) \to (N,h)$ is called an ϵ-quasi isometry if

$$e^{-\epsilon} g \leq f^*(h) \leq e^{\epsilon} g \ . \tag{1.13}$$

Note that this concept is scale-invariant, in the sense that if one rescales the metrics g and h by a fixed common factor, then f remains an ϵ-quasi isometry.

Recall that a sequence of Riemannian manifolds (M_i, g_i) converges to a Riemannian manifold (M,g) in the $C^{1,\alpha}$ topology if there are diffeomorphisms $F_i : M \to M_i$ such that the pullbacks, $F_i^* g_i$ converge to g in the $C^{1,\alpha}$ topology on M.

For fixed α, $\epsilon > 0$, we define the $C^{1,\alpha}$-harmonic radius at p, $r_H(\alpha, \epsilon, p)$, to be the largest r such that there exists a harmonic coordinate chart, $\phi : U \to B_0(r)$, with $\phi(p) = 0$, in which the metric satisfies

$$|g_{ij} - \delta_{ij}|_{C^{1,\alpha}} < \epsilon \ . \tag{1.14}$$

We now recall the following basic result which is behind the convergence theorem of [An2] employed below; see also [Ga] (Theorem 1.15 is not used *explicitly* until Section 2).

THEOREM 1.15. *Given $n, v, \lambda, a, \alpha, \epsilon > 0$, there exists $\delta = \delta(n, v, \lambda a^2, \alpha)$, $\rho = \rho(n, v, \lambda a^2, \epsilon a^{-(1+\alpha)}, \alpha)$), such that if (M^n, p) satisfy*

$$\mathrm{vol}(B_p(r)) \geq v r^n \ , \quad \textit{for} \quad r \leq \mathrm{dist}(p, \partial M) \ , \tag{1.16}$$

$$|Ric_{M^n}| \leq \lambda \ ,$$

$$\mathrm{dist}(p, \partial M) \geq a \ ,$$

$$\int_{M^n} |R|^{n/2} \leq \delta \ ,$$

then

$$r_H(\alpha, \epsilon a^{-(1+\alpha)}, p) \geq a\rho \ . \tag{1.17}$$

The following Neck Theorem provides control over the geometry and topology of the transition regions. A version of it is given for Ricci flat manifolds in [An1], [BKN].

NECK THEOREM 1.18. *Given $n, v, \lambda, a, \alpha, \epsilon > 0$, there exist $\delta_0 = \delta_0(n, v, \lambda a^2, \alpha, \epsilon)$, $c_2 = c_2(n, v, \lambda a^2, \alpha, \epsilon) > 0$ and $\#(v) \in \mathbb{Z}^+$, such that the following holds. Let M^n $(n \geq 3)$ satisfy*

$$\mathrm{vol}(B_p(r)) \geq v\, r^n \quad \textit{for} \quad r \leq \mathrm{dist}(p, \partial M) , \tag{1.19}$$

$$|Ric_M| \leq \lambda ,$$

$$a < \mathrm{dist}(p, \partial M) ,$$

and let $A_{r_1, r_2}(p) \subset M^n$, be such that

$$r_2 \leq c_2 a , \tag{1.20}$$

$$r_1 \leq \delta_0 r_2 ,$$

and

$$\int_{A_{r_1, r_2}} |R|^{n/2} \leq \delta_0 . \tag{1.21}$$

Then for some $\Gamma \subset O(n)$, acting freely on S^{n-1}, with $|\Gamma| \leq \#$, there is an ϵ-quasi isometry ψ, with

$$A_{(\delta_0^{-1/2}+\epsilon)r_1, \frac{1}{\#}r_2(1-\epsilon)} \subset \psi(C_{\delta_0^{-1/2}r_1, \frac{1}{\#}r_2}(S^{n-1}/\Gamma) \subset A_{(\delta_0^{-1/2}-\epsilon)r_1, \frac{1}{\#}r_2(1+\epsilon)} , \tag{1.22}$$

such that for all $C_{\frac{1}{2}r, r}(S^{n-1}/\Gamma) \subset C_{\delta_0^{-1/2}r_1, \frac{1}{\#}r_2}(S^{n-1}/\Gamma)$, in local normal coordinates (based on a frame), one has

$$|(\psi^*(r^{-2}g))_{i,j} - \delta_{i,j}|_{C^{1,\alpha}} \leq \epsilon . \tag{1.23}$$

Proof: By scaling, it suffices to consider the case $a = 1$.

The proof is divided into two claims.

Claim 1. There exist $\delta_0, c_2, \#$ such that for each r as above (1.22), (1.23) hold for some $\psi_r : C_{\frac{1}{2}r, r}(S^{n-1}/\Gamma_r) \to A_{\delta_0^{-1/2}r_1, \frac{1}{\#}r_2}$, where a priori ψ_r, Γ_r might depend on r.

Claim 2. The subgroup Γ_r is independent of r and (after slight modification) certain of the individual maps, ψ_r, can be pieced together to yield the map, ψ, of (1.22).

Claim 1 is proved by contradiction in the following four steps.

Step 1. (Construction of a flat model B_∞).

Suppose Claim 1 is false. Then for some ϵ_0, there exists a sequence of Riemannian manifolds (M_i^n, g_i) and annuli $A^i_{(\delta_0^i)^{-1/2}r_1^i, \frac{1}{\#}r_2^i} \subset (M_i, g_i)$ satisfying (1.20), (1.21) for some sequences $\{r_1^i\}$, $\{r_2^i\} \to 0$, $\{\delta_0^i\} \to 0$, such that some sub-annuli, $A_i = A_{\frac{1}{2}r_i, r_i}$, with rescaled metrics, $r_i^{-2} g_i$, are not ϵ_0-close, in the $C^{1,\alpha}$ topology, to annuli in any cone $C(S^{n-1}/\Gamma)$.

For each such M_i, we pick a base point $x_i \in S(r_i)$. Then there exists a subsequence of rescaled manifolds $(M^n, r_i^{-2}g, x_i)$ which converges in the pointed Gromov-Hausdorff sense, to a *complete* length space, B_∞. The arguments of [An1] or [BKN] show that B_∞ is a Ricci flat manifold with at most a finite number of singular points, $q_1 \cdots q_N$.

Let p_∞ be the limit of the center points, p_i, of the annuli. Possibly one has $p_\infty \in \{q_j\}$ but in any case, there are no other singular points in $B_{p_\infty}(2)$. Away from the singular points, the convergence is actually in $C^{1,\alpha} \cap L^{2,p}$, for all p. By taking $p = n/2$, we see that B_∞ is flat at points of $B_{p_\infty}(\frac{1}{\#}) \setminus p_\infty$. From the unique continuation property for Einstein metrics, it follows that $B_\infty \setminus \cup q_j$ is a flat manifold.

We will prove in Steps 2–4 that B_∞ is isometric to the cone on a spherical space form, $C(S^{n-1}/\Gamma)$ with $|\Gamma| \leq \#(v)$ (in particular there is just one singular point, $q_j = q_1$). By a standard (and obvious) argument this leads to a contradiction, thereby proving Claim 1.

Step 2. (Local connectedness of $B_\infty \setminus q_j$).

It follows immediately from Corollary 1.11 that for all $q_j \in B_\infty$, there exists a neighborhood basis of open sets, V, such that $V \setminus q_j$ is connected.

Step 3. (Tangent cone analysis).

We will prove that for any singular point, q_j, there exists $U \subset B_\infty \setminus q_j$, with $q_j \in \overline{U}$, such that $\pi_1(U)$ is finite. In fact,

$$|\pi_1(U)| \leq \#(v) \; . \tag{1.24}$$

To prove (1.24) we examine the tangent cone to B_∞ at some q_j. First, note that since B_∞ is a rescaled limit of smooth manifolds satisfying (1.20), it follows that for any geodesic ball, $B_x(r) \subset B_\infty$,

$$c_3^{-1} \leq \frac{\operatorname{vol} B_x(r)}{r^n} \leq c_3 \; , \tag{1.25}$$

where $c_3 = c_3(v, a^2\lambda) > 0$. Similarly, the ratio,

$$v(r) = \frac{\operatorname{vol} B_x(r)}{r^n} , \tag{1.26}$$

is monotone nonincreasing in r, since (by the Bishop comparison theorem) the ratio is almost monotone on each $(M_i, r_i^{-2} g_i)$.

Now let T be a tangent cone of B_∞ at q_j. Thus, T is a limit as $s_i \to 0$, (in the pointed Gromov-Hausdorff topology) of a pointed sequence of rescaled spaces $(B_\infty, q_j, s_j^{-2} g_{B_\infty})$. Of course, any such T is again a flat manifold, with isolated singular point O, and the convergence to T is smooth away from O (compare [An2]).

As in Step 2 above, $T \setminus O$ is connected.

From (1.23), one immediately sees that T is a volume cone, in the sense that

$$\frac{\operatorname{vol} B_O^T(r)}{r^n} = v_0 , \tag{1.27}$$

for some v_0 and all $r > 0$. One may then apply for instance the argument of [BKN, Lemma 5.13] to conclude that T is a Euclidean cone, i.e. T is isometric to $C(S^{n-1}/\Gamma(T))$, for some $\Gamma(T) \subset O(n)$. Essentially, the argument of [BKN, Lemma 5.3] goes as follows. If $\tilde{\rho}$ denotes the distance to $\{O\}$ in T, then from (1.20) and (1.26), one finds that $\Delta\tilde{\rho} \leq \frac{n-1}{\tilde{\rho}}$, while the fact that T is a volume cone implies that $\Delta\tilde{\rho} = \frac{n-1}{\tilde{\rho}}$, weakly in $L^{1,2}_{loc}$ on T. Elliptic regularity then implies that ρ is smooth, and from this it follows easily that T is a Euclidean cone.

In the next paragraph, some details are omitted, since the situation is essentially the same as in the proof of Claim 2 below, where these details are given.

Since the rescaled convergence to any such T is smooth, away from q_j, it follows that a punctured neighborhood U of q_j in $B_\infty \setminus q_j$, is covered by a sequence of domains C_i (for i large) each of which is diffeomorphic to, and, after rescaling, $C^{1,\alpha}$ close to a standard domain, say, $C_{\frac{1}{2},2}(S^{n-1}/\Gamma_i)$. Further, $C_i \subset A_{2^{-i-2},2^{-i+2}}(0)$ and $C_i \cap C_{i-1}$ is diffeomorphic to, and after rescaling, $C^{1,\alpha}$ close to $C_{\frac{1}{2},1}(S^{n-1}/\Gamma_i)$ and to $C_{\frac{1}{2},1}(S^{n-1}/\Gamma_{i-1})$ as well. In particular, $\Gamma_i = \Gamma$ is independent of i. Finally, the punctured neighborhood U is diffeomorphic and quasi isometric to $C_{0,2}(S^{n-1}/\Gamma)$. In particular, in view of (1.27), the fundamental group of U satisfies (1.24).

Step 4. (Developing map).

Consider the universal cover $\tilde{U} \xrightarrow{\pi} U$. The developing map β gives an isometric immersion $\beta : \tilde{U} \to \mathbf{R}^n$. Since U is quasi-isometric to $C_{0,2}(S^{n-1}/\Gamma)$, there is a unique point $\tilde{q}_j$ in the completion, $\overline{\tilde{U}}$, such that the canonical extension of β to $\overline{\tilde{U}}$ carries $\tilde{q}_j$ to $0 \in R^n$. The map β preserves the lengths of geodesics, and it follows that β extends to a continuous map $\beta : \overline{\tilde{U}} \to \mathbf{R}^n$.

If $\tilde{S}_{\tilde{q}_j}(s)$ is the universal cover of $S_{q_j}(s)$, then we see that the developing map sends $\tilde{S}_{\tilde{q}_j}(s)$ into $S_0(s)$, the sphere of radius s about $0 = \beta(\tilde{q}_j)$. Clearly, β is an isometric immersion of $\tilde{S}_{\tilde{q}_j}(s)$ into $S_0(s)$, and since $\tilde{S}_{\tilde{q}_j}(s)$ is compact, $\beta|\tilde{S}_{q_j}(s)$ is a covering map onto $S_0(s)$. Since $n \geq 3$, it follows that $\beta|\tilde{S}_{\tilde{q}_j}(s)$ is actually an isometry. Hence β is an isometry of $\overline{\tilde{U}}$ into $\mathbf{R}^n$. Thus, q_j is an orbifold singular point of B_∞, i.e. q_j has a neighborhood isometric to $C_{0,2}(S^{n-1}/\Gamma)$. In particular, B_∞ is a complete flat orbifold.

According to [H, 13.2.2], a complete flat orbifold, Z, of dimension ≥ 3, with singular set, $\{z_j\}$, is isometric to $\mathbf{R}^n/\Gamma$, for some discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbf{R}^n)$. (To prove this one observes that the developing map of the universal cover of $Z \setminus \cup z_j$ maps isometrically and bijectively onto $\mathbf{R}^n$ punctured at a countable, discrete set of pints $\{x_k\}$. Thus, its metric completion is isometric to $\mathbf{R}^n$). By applying this result to B_∞ and recalling that B_∞ has Euclidean volume growth (see (1.25)), we conclude that Γ is a finite group.

We claim that if Γ is not trivial then it has a unique fixed point. To see this, note that the center of mass of any orbit is a fixed point x_0 of Γ. If $x \neq x_0$ is also fixed under the subgroup generated by some $h \in \Gamma$, then so is the line ℓ from x to x_0. But then ℓ/Γ lies on the singular set in $B_\infty = \mathbf{R}^n/\Gamma$, contradicting the fact that the singularities are isolated. Thus, x_0 is the only fixed point; in fact it is the only point whose isotropy subgroup is nontrivial. This proves our assertion that B_∞ is isometric to $C(S^{n-1}/\Gamma)$. Moreover, either p_∞ or a point lying outside $B_{p_\infty}(2)$ corresponds to the vertex. In the latter case the injectivity radius at points of $S_{p_\infty}(1)$ is at least $\frac{1}{\#}$. In either case Claim 1 follows (note that in the second case the annulus one obtains is isometric to a standard annulus in R^n, compare also Remark 1.29).

Claim 2. We have proved that for any $\epsilon > 0$, there exist $\delta_0, c_2, \#$ such that for any r, with $r_1\delta^{-1/2} \leq r \leq \frac{1}{\#}r_2$, the annulus, $A_{\frac{1}{2}r,r}$ is ϵ-quasi isometric and $\epsilon r^{-(1+\alpha)}$ $C^{1,\alpha}$-close to an annulus, $C_{\frac{1}{2}r,r}(S^{n-1}/\Gamma_r)$. To see that Γ_r is independent of r, take $\epsilon < \epsilon(v)$ sufficiently small, and for fixed r, consider $\bar{r}$ very close to r. Since, on its maximal domain of definition, the map $\psi_{\bar{r}}^{-1}\psi_r$ is a 2ϵ-quasi-isometry, it follows directly that Γ_r is locally constant, and hence independent of r.

Now, let $t_i = \frac{1}{\#}\,(\frac{2}{3})^{i-1}r_2$ and choose fixed maps, $\psi_i = \psi_{t_i}$. Let $\tau_a : C(S^{n-1}/\Gamma) \to C(S^{n-1}/\Gamma)$, be defined by $(r,\theta) \to (ar,\theta)$. After successively modifying the maps $\psi_2, \psi_3, \ldots$, by isometries, we can assume that the map $(\psi_{i+1}\tau_{t_i})^{-1}\circ\psi_i\circ\tau_{t_i}$ is 2ϵ-close in the $C^{1,\alpha}$ sense to the identity on its maximal domain of definition. Then using the isotopy extension theorem we can modify $(\psi_{i+1}\tau_{t^i})\,|\,C_{\frac{1}{3},\frac{7}{12}}(S^{n-1}/\Gamma)$ so that it agrees with $\psi_i\circ\tau_i$ on $C_{\frac{7}{12},\frac{1}{2}}(S^{n-1}/\Gamma)$ and is left unchanged on $C_{\frac{5}{12},\frac{1}{3}}(S^{n-1}/\Gamma)$; compare [C1]. Call the resulting map $\tilde{\psi}_{i+1}\tau_i$. The maps $\tilde{\psi}_{i+1}$ fit together to define the map ψ that we are seeking. Moreover, it is clear that $\{\tilde{\psi}_i\}$ can be chosen so that (1.23) holds with ϵ replaced by some $\epsilon' = \epsilon'(n,\epsilon)$, where $\epsilon' \to 0$ as $\epsilon \to 0$. By interchanging, the roles of ϵ, ϵ', the theorem follows. □

Remark 1.28: We mention that it is also possible to show by more elementary purely geometric arguments, that (1.27) implies that T is isometric to $C(S^{n-1}/\Gamma)$. Briefly, a volume comparison argument shows that each $y \in T \setminus O$ lies on a unique ray emanating from $\{O\}$. Then the basic construction of totally convex sets for complete manifolds of non-negative curvature can be used to show that geodesic balls are totally convex. This implies that there are no geodesic loops on the vertex, from which the claim easily follows. In particular, one may bypass the use of elliptic regularity.

Remark 1.29: Let (N^4, g) be an ALE manifold. As $i \to \infty$, the sequence of (suitably) pointed manifolds, $(N^4, i^{-4}g)$, converges in the Gromov-Hausdorff topology to a cone, $C(S^3/\Gamma)$. Thus, for i large, we may speak of points in $(N^4, i^{-4}g)$ which lie at distance approximately i^{-1} from the vertex. By considering such a sequence, $p_i \in (N^4, i^{-4}g)$, it becomes clear why the outer radius of the annulus in Theorem 1.18 must be chosen to be (roughly) $\leq \frac{1}{\#(v)}r_2$. More-or-less equivalently, the second possibility at the end of Step 4 above cannot be ruled out. However, this case does not occur in the application to the proof of Theorem 0.1 given in §2, where the points p_i are points of "maximal curvature concentration".

Remark 1.30: The proof of Theorem 1.18 uses the rigidity of flat structures in dimensions ≥ 3, in Step 4. However, it is straightforward to formulate and prove a version of Theorem 1.18 valid in dimension 2.

2. Finiteness and Controlled Degeneration

Let M^n be a manifold satisfying the conditions of Theorem 0.1. To prove Theorem 0.1, we will decompose M^n into a definite number of regions of two different types. These are (mutually disjoint) connected regions, y_i^j, whose geometry and size become bounded after suitable rescaling of the metric, and neck regions, e_ℓ^j (as in Theorem 1.18) which connect pairs of regions $y_\ell^j, y_{\ell'}^{j-1}$.

The pattern of connections is that of an ordered tree whose vertices and edges, denoted $\bar{y}_\ell^j$ and $\bar{e}_\ell^j$, are in 1–1 correspondence with the y_ℓ^j, e_ℓ^j. Here, $\bar{y}_\ell^j$ is the final vertex of $\bar{e}_\ell^j$ and y_ℓ^j intersects the smaller of the two boundary components of the neck e_ℓ^j. Also, $\bar{y}_{\ell'}^{j-1}$ is the initial vertex of $\bar{e}_\ell^j$ and $y_{\ell'}^{j-1}$ intersects the larger of the two boundary components of e_ℓ^j.

The region y^0, corresponding to the initial vertex, $\bar{y}^0$, has bounded geometry and a definite size. The rescaling factor for the region y_ℓ^j is the smaller radius r_1, of the neck e_ℓ^j. Alternatively, this factor can be taken as the larger radius, r_2, of any neck $e_{\ell'}^{j+1}$, such that $\bar{y}_\ell^j$ is the initial vertex of $\bar{e}_{\ell'}^{j+1}$.

To obtain the above decomposition, we proceed as follows.

Fix $\alpha, \epsilon > 0$. Let $\delta = \delta(n, v, \lambda D^2, \alpha)$ be as in Theorem 1.15. Let $\delta_0(n, v, \lambda D^2, \alpha, \epsilon)$, $c_2(n, v, \lambda D^2, \alpha, \epsilon)$ be as in Theorem 1.18. Put

$$\delta_1 = \frac{1}{3} \min(\delta, \delta_0) \ . \tag{2.1}$$

Let $X \subset M^n$ be open. Assume

$$\int_X |R|^{n/2} > 2\delta_1 \ , \tag{2.2}$$

and

$$\int_{T_\eta(\partial X)} |R|^{n/2} \leq \delta_1 \ , \tag{2.3}$$

where

$$T_\eta(\partial X) = \{p \in M^n \mid \operatorname{dist}(x, \partial X) < \eta\} . \tag{2.4}$$

Let $d(p) = \operatorname{dist}(p, \partial X)$ and let $r(p)$ be the largest radius $\leq d(p)$, satisfying

$$\int_{B_p(r(p))} |R|^{n/2} \leq \int_X |R|^{n/2} - \delta_1 . \tag{2.5}$$

Further, let $s(p)$ be defined by the equality

$$\int_{B_p(s(p))} |R|^{n/2} = \delta_1 . \tag{2.6}$$

Fix σ, to be determined later, with $0 < \sigma \leq \eta/4$. Then the X belongs to exactly one of the following three cases.

Case 1. $s(p) > \sigma$, for all $p \in X$.

If, on the other hand, $s(p) \leq \sigma$ for some p, then (with the notation of (1.12)) for any such p,

$$\begin{aligned} &p \in X_{\frac{3}{4}\eta} \\ &s(p) \leq \frac{1}{4}\eta \end{aligned} \tag{2.7}$$

and by (2.2),

$$s(p) \leq r(p) . \tag{2.8}$$

Case 2. For some $p_0 \in X$, $s(p_0) < \sigma$, while $r(p_0) \geq 2\sigma$.

We consider the annulus $A_0 = A_{\sigma,2\sigma}(p_0) \subset X$ and note that

$$\int_{A_0} |R|^{n/2} \leq \int_X |R|^{n/2} - 2\delta_1 . \tag{2.9}$$

Set $\Lambda_0 = \int_{A_0} |R|^{n/2}$ and let $[x]$ denote the greatest integer contained in x. The annulus A_o can be divided into $1 + [\frac{\Lambda_0}{\delta_1}]$ concentric annuli $\{A^i\}$, $i = 1, \ldots, 1 + [\frac{\Lambda_0}{\delta_1}]$, of inner and outer radii $\sigma(1 + \frac{i-1}{1+[\Lambda_0/\delta_1]})$ and $\sigma(1 + \frac{i}{1+[\Lambda_0/\delta_1]})$, respectively. It follows that there is (at least) one annulus $A^j \in \{A^i\}$ such that

$$\int_{A^j} |R|^{n/2} \leq \delta_1 . \tag{2.10}$$

We may then divide X into domains X_1 and X_2 with

$$\begin{aligned} X_1 &= \{x \in X : \mathrm{dist}(x, p_0) \leq \sigma\big(1 + \frac{j - \frac{1}{2}}{1 + [\Lambda_0/\delta_1]}\big) \\ X_2 &= \{x \in X : \mathrm{dist}(x, p_0) \geq \sigma\big(1 + \frac{j - \frac{1}{2}}{1 + [\Lambda_0/\delta_1]}\big) . \end{aligned} \tag{2.11}$$

Note that

$$\int_{X_i} |R|^{n/2} \leq \int_X |R|^{n/2} - \delta_1 , \tag{2.12}$$

for $i = 1, 2$ and

$$\int_{T_b(\partial X_i)} |R|^{n/2} \leq \delta_1 , \tag{2.13}$$

where

$$b = \frac{1}{2}\sigma(1 + [\Lambda_0/\delta_1])^{-1} . \tag{2.14}$$

Case 3. For all $p \in X$, $s(p) \leq \sigma$ and $r(p) \leq 2\sigma$.

Note that in this case,

$$B_p(r(p)) \subset X_{\frac{1}{4}\eta} . \tag{2.15}$$

LEMMA 2.16. *There exist constants* $\omega(n, v, \lambda\eta^2)$, $\theta(n, v, \lambda g^2) > 0$ *and* $\mu(n, v, \lambda\eta^2) > 1$, *such that if* $r(p)$ *takes its minimum at* p_1, *and* $r(p_1) \leq \omega\eta$, *then*

$$\int_{X \setminus B_{p_1}(\mu\, r(p_1))} |R|^{n/2} \geq \theta \tag{2.17}$$

Proof: By scaling, it suffices to consider the case $\eta = 1$. The argument is essentially the same as one already given in the proof Claim 1 of the Neck Theorem, to which we refer for further details.

The proof is by contradiction. If the lemma is not true, then there exist sequences (X_i, p_1^i, g_i) as above, with $\omega_i \to 0$, $\theta_i \to 0$, $\mu_i \to 1$, such that

$$\int_{X_i \setminus B_{p_1^i}(\mu_i r(p_1^i))} |R|^{n/2} < \theta_i . \tag{2.18}$$

By definition, this implies that

$$\int_{A_{r(p_1^i),\mu_i r(p_1^i)}(p_1^i)} |R|^{n/2} \geq \delta_1 - \theta_i , \tag{2.19}$$

so that curvature is concentrating on the sphere of radius $r(p_1^i)$ about p_1^i.

Consider the sequence $(X_i, p_1^i, r(p_1^i)^{-2} g_i)$. A subsequence converges to $(X_\infty, p_\infty, g_\infty)$, which is a complete Ricci-flat manifold with a finite number of isolated orbifold singularities.

By (2.18) and [An2], the convergence is smooth on the interior Ω of $X_\infty \setminus B_{p_\infty}(1)$, so that in fact

$$\int_\Omega |R|^{n/2} = 0 \ . \tag{2.20}$$

By the unique continuation property of Einstein metrics, this implies that X_∞ is actually flat away from its singular points, all of which are located in the closure of $B_{p_\infty}(1) \subset X_\infty$.

Since the flat orbifold, X_∞, has Euclidean volume growth, X_∞ has at most one singular point, q_∞.

There is at least one singular point on $S_{p_\infty}(1)$. For if not, by [An2, Remark 3.3] the convergence of X_i to X_∞ would be smooth in a neighborhood of $S_{p_\infty}(1)$. Thus, the $L^{n/2}$ norm of curvature on neighborhoods of $S_{p_1^i}(r(p_1^i))$ would converge to 0, contradicting (2.19).

Since the convergence of X_i to the flat orbifold X_∞ is smooth away from its singular point q_∞, it follows that *all* of the $L^{n/2}$ norm of the curvature of X_i is concentrating near q_∞. This however contradicts the definition of $\{p_1^i\}$ (converging to p_∞), as the sequence of points of 'maximal' curvature concentration. □

We now fix σ by putting

$$\sigma = \frac{1}{10}\eta \cdot \min(\omega, \delta_0 c_2) \ , \tag{2.21}$$

where δ_0, c_2 are as in (1.20) of Theorem 1.18.

We now divide the region X into domains Z_1 and Z_2 with

$$\begin{aligned} Z_1 &= \{x \in X : \operatorname{dist}(x, p_1) \le \mu\, r(p_1)\} \ , \\ Z_2 &= \{x \in X : \operatorname{dist}(x, p_1) \ge \mu\, r(p_1)\} \ . \end{aligned} \tag{2.22}$$

Since by (2.8), $s(p_1) \le r(p_1)$, the above discussion implies

$$\delta_1 \le \int_{Z_1} |R|^{n/2} \le \int_X |R|^{n/2} - \theta \ , \tag{2.23}$$

and

$$\theta \le \int_{Z_2} |R|^{n/2} \le \delta_1 \; . \tag{2.24}$$

Note that

$$\int_{T_{b'}(\partial Z_i)} |R|^{n/2} \le \delta_1 \; , \tag{2.25}$$

where

$$b' = (\mu - 1)\, r(p_1) \; . \tag{2.26}$$

We now return to consideration of the manifold M^n, itself, which we decompose as follows.

If $\int_M |R|^{n/2} \le 2\delta_1$ or if M^n is as in Case 1, no further decomposition is required.

If M^n is as in Case 2, we write $M^n = X_1 \cup X_2$ as in (2.11). We set aside those X_i $(i = 1, 2)$ which are as in Case 1. If there are any X_i $(i = 1, 2)$ which are as in Case 2, we again subdivide their interiors as in (2.11). It follows from (2.12), that by continuing in this way, after at most $N(\delta_1, \Lambda)$ steps, we obtain a decomposition of M^n into closed domains, with disjoint interiors, each of which is either of the type considered in Case 1 or in Case 3. A typical domain is denoted F^0 if it is as in Case 1 and by V^0 if it is as in Case 3.

Suppose that M itself is as in Case 3 or that M is as in Case 2 and we have arrived at the decomposition

$$M = (\cup_k F_k^0) \cup (\cup_k V_k^0) \tag{2.27}$$

of the previous paragraph. For each domain, V^0, write

$$V^0 = Z_1^1 \cup Z_2^0 \; , \tag{2.28}$$

as in (2.22).

We temporarily set aside the domains Z_1^1. Each domain, Z_2^0, can be decomposed as

$$Z_2^0 = G^1 \cup e^1 \cup W^0 \; . \tag{2.29}$$

Here, e^1 is a maximal neck as constructed in Theorem 1.18. Also, G^1 is the component of $Z_2^0 \backslash e^1$ which intersects the corresponding sphere, $S_{p_1}(\mu\, r(p_1))$

(see (2.22)) and W^0 is the remaining component of $Z_2^0 \setminus e^1$. Note that W^0 is of the type considered in Case 1. Put

$$y^0 = (\cup_k F_k^0) \cup (\cup W_k^0) . \tag{2.30}$$

It is easy to see that y^0 is connected.

Now, starting with each of the domains Z_1^1, we repeat the process just described. Thus, we decompose each Z_1^1 as a union of domains of type F^1, $Z_2^1 = G^2 \cup e^2 \cup W^1$ and Z_1^2. We write

$$\cup_\ell y_\ell^1 = (\cup_k G_k^1) \cup (\cup_k F_k^1) \cup (\cup_k W_k^1) , \tag{2.31}$$

where by definition, the sets y_ℓ^1 are the components of the set on the right-hand side of (2.31).

By continuing in this fashion, we obtain a decomposition of M into a definite number of domains, y_ℓ^j, and necks, e_ℓ^j, where

$$Z_2^j = G^{j+1} \cup e^{j+1} \cup W^j , \tag{2.32}$$

and

$$\cup_\ell y_\ell^j = \big(\cup_k G_k^{j+1}\big) \cup (\cup_k F_k^j) \cup (\cup_k W_k^j) . \tag{2.33}$$

Using (2.12), (2.23) and induction, it follows that the total number of domains so obtained is at most, $N(\delta_1, \Lambda_0)$. It is clear by inspection that for $j > 0$, each y_ℓ^j meets a unique neck, e_ℓ^j, a possibly empty set of necks, $e_{\ell'}^{j+1}$, and no others.

We are now in a position to prove Theorem 0.1.

We will say that the $C^{1,\alpha}$ geometry of a subset, $K \subset M$, is (Q, ϵ, C, r) controlled if

i) there exist $C^{2,\alpha}$ coordinate charts, $\phi_j : U_j \to B_0(r)$, Q in number, in which the metric satisfies

$$|g_{ij} - \delta_{ij}|_{C^{1,\alpha}} \le \epsilon , \tag{2.34}$$

ii) The transitions maps are uniformly bounded by C in the $C^{2,\alpha}$ norm in these coordinates.

iii) $K \subset \cup_j \phi_j^{-1}(B_0(\frac{1}{2}r))$.

Proof of Theorem 0.1: By using Theorem 1.15 (see also (2.13), (2.25)), a standard covering argument (compare [An2]) and induction, we easily obtain the following. There exist constants $Q = Q(n, vD^{-n}, \lambda D^2, \alpha, \epsilon)$, $C = C(n, v, \lambda D^2, \alpha, \epsilon)$, $\nu = \nu(n, v, \lambda D^2, \alpha, \epsilon)$ such that with respect to the rescaled metric, $r_1^{-2}g$, the region y^0 is (Q, ϵ, C, ν)-controlled, and the regions, y^j_ℓ, are (Q, ϵ, C, ν)-controlled. Here, $r_1 = r_1(e^j_\ell)$ is the smaller radius corresponding to the neck, e^j_ℓ. Also, if $e^{j+1}_{\ell'}$ is any neck meeting y^j_ℓ, it is clear that

$$r_2(e^{j+1}_{\ell'}) \geq \rho r_1(e^j_\ell) , \tag{2.35}$$

for some $\rho = \rho(n, v, \lambda D^2, \alpha, \epsilon)$. Thus, the above control could be expressed in terms of $r_2(e^{j+1}_{\ell'})$ as well.

Finally, let ψ denote the map of (1.22) parametrizing the neck e^j_ℓ and let ϕ denote a harmonic chart corresponding to the covering providing the control of y^j_ℓ. Then with respect to the rescaled metric, $[r_1(e^j_\ell)]^{-2}g$, on its maximal domain of definition, the map, $\psi\phi^{-1}$ satisfies

$$|\psi\phi^{-1}|_{C^{2,\alpha}} \leq C , \tag{2.36}$$

($C = C(n, v, \lambda D^2, \alpha, \epsilon)$). Similarly, if ψ parametrizes a neck $e^{j+1}_{\ell'}$ meeting y^j_ℓ, then with respect to the rescaled metric, $[r_2(e^{j+1}_{\ell'})]^{-2}g$,

$$|\psi\phi^{-1}|_{C^{2,\alpha}} \leq C . \tag{2.37}$$

These estimates follow directly from elliptic regularity theory (and the construction of ψ) (compare [An2]).

Now the conclusion of Theorem 0.1 follows by an obvious modification of the standard proof of the finiteness theorem (compare [C1]). The point here is that M is covered by a definite number of balls and necks and that all change of coordinates maps are $C^{2+\alpha}$-bounded after rescaling by the inverses of the radii of the balls. □

3. Further Remarks

Let (M_i, g_i) be a sequence of metrics satisfying (0.2) and (0.3), converging in the Gromov-Hausdorff topology to an orbifold (V_0, g), as in the introduction.

Remark 3.1: To see how the Ricci flat ALE orbifolds arise in this degeneration, choose a singular point $q \in \{q_k\} = (V_0)_{sing}$. As in the discussion of Case 3 and Lemma 2.16 above, there are points $p_i \in M_i$, (converging to q) and a sequence $r_i = r_i(p_i) \to 0$ such that the rescaled sequence $(M_i, p_i, r_i^{-2} g_k)$ converges to a complete Ricci-flat ALE orbifold V_1 $(= V_1(q))$, whose end is asymptotic to $C(S^{n-1}/\Gamma)$, for some Γ, and whose singular points lie in the closure of $B_{p_\infty}(1)$, with $p_\infty = \lim p_i$, in the Gromov-Hausdorff topology. By the discussion of Case 3 above, small annuli about $p_i \in M_i$ are also diffeomorphic to an annulus in $C(S^{n-1}/\Gamma)$, so that, to a first approximation, the topology of M_i near $\{q\}$ is that of V_1. Now for each singular point $q^2 \in \{q_k^2\} \subset B_{p_\infty}(1) \subset V_1$, define the sequence r_i^2 as previously to replace a neighborhood of q^2 in V_1 by another ALE orbifold V_2 $(= V_2(q^2))$; this gives a 2^{nd} order approximation to the topology of M near q^2; etc. The proof of Theorem 0.1 shows that this process terminates after a finite number of iterations. In fact, at the last stage, one is glueing in a smooth Ricci-flat ALE manifold V_m.

Following the work of Sacks-Uhlenbeck on minimal 2-spheres, each ALE orbifold V_j arising from a singular point of a previous orbifold V_{j-1} is known as a bubble. To each bubble is associated a *scaling sequence*, $\{r_i^j\}$, or more briefly, a *scale*, with $\{r_i^j\} \to 0$ and $r_i^j / r_i^{j-1} \to 0$.

This process not only determines the topological description of the degeneration, but also much of the metric description (see also Remark 3.2). For example, the convergence of M_i to V_0 is in $C^{1,\alpha} \cap L^{2,p}$, for any $p < \infty$, away from the singular points of V_0. Since the integral $\int |R|^{n/2}$ is lower semi-continuous in this topology, one finds from (0.3) that $\int_{V_0} |R|^{n/2} \leq \Lambda$. By scale invariance, the same is true for each of the orbifolds V_k constructed above. From the proof of Theorem 0.1, especially the Neck Theorem 1.18, one then easily obtains that

$$\lim_{i\to\infty} \int_{M_i} |R|^{n/2} = \sum_{k=0}^{m} \int_{V_k} |R|^{n/2} , \tag{3.1}$$

as well as

$$\lim_{i\to\infty} \int_{M_i} P_\chi(R) = \sum_{k=0}^{m} \int_{V_k} P_\chi(r) , \tag{3.2}$$

where $P_\chi(r)$ is the Chern-Gauss-Bonnet integrand. Here, the summation is over all the ALE orbifolds associated with each singular point of V_0. Similar formulas hold also for other scale-invariant curvature integrals.

Remark 3.2: Recall that in the situation considered in [C1], [GLP], condition (0.2) is replaced by the stronger condition, $|R| \leq \Lambda$. In this case, one actually has a $C^{1,\alpha}$-compactness theorem. Alternatively, if one thinks in terms of total boundedness (or more particularly in terms of ϵ-quasi-isometry type), then one has the following formulation (in which we assume $D = \text{diam}(M^n, g)$).

Given ϵ, there exist at most $N(n, vD^{-n}, \lambda D^2, \epsilon)$ representative manifolds, (M_i^n, g_i), with $\text{diam}(M_i^n, g_i) = 1$, such that for every (M^n, g) as above, $(M^n, D^{-2}g)$ is ϵ-quasi-isometric to some (M_i^n, g_i).

Inspection of the proof of Theorem 0.1 leads immediately to the conclusion that in the present setup, this statement has a natural generalization.

Given ϵ, there exist $N_1(n, vD^{-n}, \lambda D^2, \epsilon)$ families of manifolds, $(M_i^n, g_i^{(r)})$, with $\text{diam}(M_i^n, g_i^{(r)}) = 1$, each depending on at most $N_2(n, vD^{-n}, \lambda D^2, \epsilon)$ additional parameters, $0 < r_1(e_\ell^j) < 1$ (where $(r) = (r_1(e_1^1), \cdots))$ such that for any (M^n, g) as in Theorem 0,1, $(M^n, D^{-2}g)$ is ϵ-quasi-isometric to some $(M_i^n, g_i^{(r)})$.

Remark 3.3: Let M be close, in the Gromov-Hausdorff topology, to an orbifold V_0 as above. It is an interesting open question whether a neighborhood in M, sufficiently close to a neighborhood of a singular point $q \in V_0$, admits a complete Ricci-flat ALE metric. For instance this is the case in the examples discussed in Remark 3.6 below. In the special case where M is a 4-dimensional Kahler-Einstein manifold, Bando [B] has shown the answer to be affirmative.

Remark 3.4: We also point out that for orientable manifolds, if the dimension n is odd, then the limit (V, g) is a $C^{1,\alpha}$ Riemannian manifold and one has $C^{1,\alpha}$ convergence above; see [An2, Corollary 2.8]. In particular, the family of orientable manifolds satisfying (0.1) and (0.2) with n odd, is compact in the $C^{1,\alpha}$ topology. This follows since for n odd, there are no non-trivial flat cones $C(S^{n-1}/\Gamma)$, with $\Gamma \neq \{e\}$, provided S^{n-1}/Γ is orientable. (For a much deeper result of this type for Kähler Einstein manifolds of complex dimension ≥ 3 and nonzero first Chern class, see [Ti]).

Note that in any dimension, if the orbifold limit V_0 is in fact a smooth manifold, then one has $C^{1,\alpha}$ convergence of (M_i, g_i) to (V_0, g), see also [An2, Remark 3.3].

Remark 3.5: It is interesting to note that while a bound on the number of singular points of V requires control on the total energy $\int |R|^{n/2}$, the number of scales at each singular point of V may be controlled by the lower bound on the volume, $\mathrm{vol}(M) \geq v$ (in the presence of arbitrary bounds on the remaining quantities in (0.2), (0.3)). This is obtained from the fact that there are only finitely many cones $C(S^{n-1})/\Gamma$ with $\mathrm{vol}(S^{n-1}/\Gamma) \geq v > 0$. More precisely, since the region of transition between scales (as in Remark 3.1) is determined as in the Neck Theorem 1.18, one is reduced to verifying that if Y is a complete Ricci-flat, asymptotically locally Euclidean orbifold, and there is a point $q \in Y$ such that

$$\lim_{r \to 0} \frac{\mathrm{vol}(B_q(r))}{r^n} = \lim_{r \to \infty} \frac{\mathrm{vol}(B_q(r))}{r^n}$$

then Y is a cone on a spherical space form. This follows easily from standard volume comparison arguments.

Remark 3.6: We point out that there are numerous examples where this hierarchy of scales near a singular point does in fact occur. Perhaps the simplest examples are given by the Gibbons-Hawking metrics. This is a family, $\mathcal{F}$, of complete, Ricci-flat ALE manifolds $(M, g_{\mathcal{F}})$, parametrized by the configurations of a finite set of distinct points, the center points, $\{p_i\}_1^N \in R^3$, see [GLP]. By letting these points converge to a strictly smaller set of points $\{q_k\} \in R^3$, one produces the above set of scales. For example, for any $\epsilon \leq 1$, let $p_i = \epsilon^i(1,0,0)$, $i = 1, \ldots, N$. Then the corresponding family (M, g_ϵ) has N distinct scales, each degenerating to a Ricci flat orbifold V_j with a single singular point; V_0 is the cone $C(S^3/\mathbf{Z}_N)$ while V_{N-1} is the Eguchi-Hanson metric on the tangent bundle $T(S^2)$ of S^2.

Remark 3.7: Finally, we note that Theorem 0.1 is false if one relaxes the bound $|\mathrm{Ric}_M| \leq \lambda$ to the bound $|\mathrm{Ric}_M|_{L^p} \leq \lambda$, for some $p < \infty$. This follows from the construction of graph manifolds of infinite topological type, as in [CG]; see [Y] for further details.

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Submitted: September 13, 1990