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# FOURIER TRANSFORM RESTRICTION PHENOMENA FOR CERTAIN LATTICE SUBSETS AND APPLICATIONS TO NONLINEAR EVOLUTION EQUATIONS

## Part I: Schrödinger Equations

J. BOURGAIN

## 1. Introduction

The main purpose of this paper is to develop a harmonic analysis method for solving certain nonlinear periodic (in space variable) evolution equations, such as the nonlinear Schrödinger equations (Part I) and the KDV equation (Part II).

The initial value problem for the periodic nonlinear Schrödinger equation (NLSE).

$$\Delta_x u + i\partial_t u + u|u|^{p-2} = 0 \qquad (p \geq 3) \tag{1.1}$$

$$u = u(x,t) \text{ is 1-periodic in each coordinate of the } x\text{-variable} \tag{1.2}$$

with initial data

$$u(x,0) = \phi(x) \tag{1.3}$$

is our first concern.

My interest in this problem results from the paper [LeRSp] of Lebowitz, Rose and Speer, in particular the first problem raised in section 5 of [LeRSp]. We will obtain here local and global results on the well-posedness of (1.1)–(1.2) in one and several space dimensions for initial data $\phi \in H^\alpha(\mathbf{T}^n)$, $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ for essentially optimal $\alpha$, giving at least a partial answer to the question from [LeRSp]. In the non-periodic case (i.e. $\mathbf{R}^n$) the NLSE has been studied by a number of authors, such as Kato [K1,2], Ginibre-Velo [GiV], Tsutsumi [Ts], Cazenave-Weissler [CW] among others. It is known for instance in the $\mathbf{R}^n$-case that if $\alpha \equiv p-2 = \frac{4}{n-2s}$ then the NLSE (1.1),(1.3) has a global solution in time provided $\|\phi\|_{H^s}$ is sufficiently small, see [CW]. The method is based on solving the equivalent integral equation

$$u(\cdot,t) = U(t)\phi + i\int_0^t U(t-\tau)\big(|u(\cdot,\tau)|^{p-2}u(\cdot,\tau)\big)d\tau \quad \text{where} \quad U(t) = e^{it\Delta} \tag{1.4}$$

by Picard's fix point method.

The nonlinearity is controlled in the iteration process by invoking Strichartz's inequality

$$\|U(\cdot)\psi\|_{L^q(dxdt)} \le C\|\psi\|_{L^2(dx)} \quad \text{where} \quad q = \frac{2(n+2)}{n} . \tag{1.5}$$

The main idea here is to try to adjust this approach to the periodic case. The main difficulties are the following:

(i) The exact analogue of (1.5) where $L^q(dxdt)$ is replaced by $L^q(\mathbb{T}^{n+1})$ fails.

(ii) Inequalities of the form (1.5) may in the periodic case only hold locally in time (global solutions to (1.1), (1.2), (1.3) are not dispersive).

As far as (i) is concerned, consider the case $n = 1$ with (critical) exponent $q = 6$ given by (1.5). We will prove the inequality

$$\Big\| \sum_{|n|\le N} a_n \, e^{2\pi i(nx+n^2t)} \Big\|_{L^6(\mathbb{T}^2)} \ll N^\varepsilon \Big(\sum |a_n|^2\Big)^{1/2} . \tag{1.6}$$

The presence of the $N^\varepsilon$-factor in (1.6) is necessary as will be indicated later and constitutes a difficulty in performing the Picard iteration argument.

Inequality (1.6) is a statement on the so-called $\Lambda_q$-constant ($q = 6$) of the set of lattice points $\{(n, n^2) \mid |n| \le N\} \subset \mathbb{Z}^2$.

DEFINITION (cf. [Bo1]): Let $d \ge 1$ and $S$ a subset of $\mathbb{Z}^d$. Let $p > 2$. Define $K_p(S)$ as the smallest constant (possibly infinity) satisfying the inequality

$$\Big\| \sum_{\gamma\in S} a_\gamma \, e^{2\pi i\langle x,\gamma\rangle} \Big\|_{L^p(\mathbb{T}^d)} \le K_p(S) \Big(\sum |a_\gamma|^2\Big)^{1/2} \tag{1.7}$$

for all scalar sequences $(a_\gamma)_{\gamma\in S}$.

Thus (1.6) means that $K_6(\{(n,n^2) \mid |n| \le N\}) \ll N^\varepsilon$. In the higher dimensional case, we will need to consider sets $\{(\xi, |\xi|^2) \mid \xi \in \mathbb{Z}^n, |\xi| \le N\}$, $|\xi|^2 = \sum_{1\le i\le n} \xi_i^2$, and evaluate appropriate $\Lambda_p$-constants of these sets. Our method to achieve this will be either simple arithmetic or the analytic Weyl-sum approach. The results obtained here only cover certain ranges of $p$ and somehow leave the picture incomplete. Some remaining questions lead to interesting number theory problems.

In the non-periodic case global results are obtained in certain cases directly from the fix point method. In particular one gets global solutions for initial data $\phi \in H^s$ with $0 < s < 1$. The reason for this is the global nature of Strichartz's inequality. In the periodic case, the estimates are local in time and hence at this stage only a local solution is obtained. All global results obtained here result from combining

- existence of local solutions
- conservation laws.

A first conservation law is provided by the $L^2$-norm:
If $u = u(x,t)$ satisfies (1.1), (1.2), (1.3), then

$$\int_{\mathbf{T}^n} |u(x,t)|^2 dx = \int_{\mathbf{T}^n} |\phi(x)|^2 dx \equiv N(\phi) \text{ for all } t . \tag{1.8}$$

Next, consider the Hamiltonian

$$H(\psi) = \frac{1}{2}\int_{\mathbf{T}^n} |\nabla\psi|^2 dx - \frac{1}{p}\int_{\mathbf{T}^n} |\psi(x)|^p dx \tag{1.9}$$

which is preserved by $u(\cdot,t)$ for varying $t$, yielding a second conserved quantity.

These conservation laws are well studied in the literature and we do not elaborate on them here. From (1.8), it will be possible to deduce global existence from local results with $L^2$-data and from (1.9) global solutions for initial data $\phi$ controlled in $L^2$ and $H^1$ in a suitable way (there is a problem in this context because of the minus sign in (1.9)). In particular, no global results are obtained here for initial data $\phi \in H^s$ with fractional $s$. To achieve this in one of the main open problems.

The NLSE (1.1),(1.2),(1.3) in the case $n = 1$, $p = 4$ has been studied by Zakharov and Shabat [ZS] using the method of inverse scattering. In this case the NLSE is shown to be integrable and global solutions may be obtained for $L^2$-data. We will obtain here another (harmonic analysis) proof of this fact based on the method described above and the $L^2$-conservation law.

By "well-posedness" we mean existence, uniqueness and persistency of solutions in an appropriate function space in the $(x,t)$-variable. Here are a few examples of results obtained in the paper.

**THEOREM 1 ($n = 1$).** *The NLSE (1.1), (1.2), (1.3) is locally well-posed for $\phi \in H^s(\mathbf{T})$, provided $\alpha \equiv p - 2 < \frac{4}{1-2s}$.*

**THEOREM 2 ($n = 2$).** *The NLSE (1.1), (1.2), (1.3) is globally well-posed for $\alpha = 2$ with initial data $\phi \in H^1(\mathbf{T}^2)$ and sufficiently small $L^2$-norm. The same result holds for all $\alpha \geq 2$ for sufficiently small $H^1$-data.*

**THEOREM 3 ($n = 3$).** *The NLSE (1.1), (1.2), (1.3) is globally well-posed for $2 \leq \alpha < 4$ and sufficiently small $H^1$-data.*

**THEOREM 4 ($n \geq 4$).** *The NLSE (1.1), (1.2), (1.3) is locally well-posed for $2 \leq \alpha < \frac{4}{n-2s}$ and $s > \frac{3n}{n+4}$.*

The main feature of the approach followed here is an analysis on multiple Fourier series. This technique has lots of flexibility compared with other methods. A further result on global wellposedness for $n = 4$ and $\alpha = 1$ appears in [Bo2].

In Part II of this paper (to appear in the next issue) we discuss the Cauchy problem for the periodic KDV equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.10}$$

with initial data

$$u(x,0) = \phi(x) \ . \tag{1.11}$$

Results along these lines in the **R**-case were obtained by several authors, for instance in the papers of Kato [K3], Kenig, Ponce and Vega (see [KePoVe1,2]). In the periodic case, there seems to be a limited amount of literature on this issue. The paper of Sjöberg [Sj] deals with $H^3$-initial data. We will show here using the fix point method.

**THEOREM 5.** *The KDV-equation (1.10), (1.11) is in the periodic case globally well-posed (in a suitable space) for $\phi \in L^2(\mathbf{T})$ of prescribed mean $\int \phi$. Moreover, for data $\phi, \psi \in L^2(\mathbf{T})$, of same mean, the corresponding solutions $u, v$ satisfy $\|u(t) - v(t)\|_2 \leq e^{Ct}\|\varphi - \psi\|_2$, where $C$ depends on $\|\phi\|_2$, $\|\psi\|_2$.*

This argument permits also to give an alternative proof of the [Sj]-result and shows that solutions with real analytic data are spatially real analytic for all time (cf. [Tr]).

The conservation laws involved in the proof (for $L^2$ data) are simply

$$\int_{\mathbf{T}} u(x,t)dx \quad ; \quad \int_{\mathbf{T}} \left|u(x,t)\right|^2 dx \tag{1.12}$$

Theorem 5 seems of interest because precisely the $L^2$-norm is a naturally preserved quantity in this theory. The techniques of [MTr] may be reworked to prove that the solutions obtained in Theorem 5 are almost periodic in time. (This result is known for smooth data, see [L], [MTr].)

Coming back to the KDV equation in the **R**-case, the introduction of a similar norm as in the proof of Theorem 5 permits to obtain also

**THEOREM 6.** *The KDV-equation* (1.10), (1.11) *in the* **R***-case, is globally well-posed for data in* $L^2(\mathbf{R})$.

There has been a list of investigations on this issue by many authors and subsequent inprovements were obtained, the record up to now being the results of [KePoVe1], namely $s > \frac{3}{4}$ (local) and $s \geq 1$ (global) as regularity condition on the data $\phi \in H^s(\mathbf{R})$. Existence of *weak* solutions for $L^2$-data were proved in [K3] and [KruF].

Next we briefly summarize the organization of the paper. In the next section we will establish $L^4$ and $L^6$ estimates relative to lattice sets

$$\{(n, n^2) \mid n \in \mathbf{Z}\ ,\ |n| \leq N\} \tag{1.13}$$

and related multipliers. In section 3, we consider the higher dimensional analogues

$$\{(\xi, |\xi|^2) \mid \xi \in \mathbf{Z}^n\ ,\ |\xi| \leq N\} \tag{1.14}$$

for which we prove $L^p$-moment inequalities. For $n \geq 5$, the best results will be obtained using the method of exponential sums and the Hardy-Littlewood circle method. We will combine these ideas with the arguments of [T] and [St] in order to obtain the right $\Lambda_p$-constants of sets (1.14) for certain $p < 4$. I believe these investigations are of independent interest. In section 4, we treat the 1-dimensional NLSE (1.1),(1.2),(1.3) with $L^2$-data. Section 5 gives the general scheme on how to make estimates in 1 and higher dimensional setting for $H^s$-data, using the information from previous sections. In section 6 we list the consequences for existence of local (resp. global) solutions of the periodic NLSE and in particular prove Theorems 1,2,3,4. Section 7 deals with the KDV equation and the proof of Theorem 5. In section 8 we present further results on periodic KDV equations, with higher degree of nonlinearity based on the same method. In section 9, the almost periodicity of the KDV flow with periodic $L^2$ data is discussed. The proof of Theorem 6 is contained in section 10.

The author benefitted from discussions with C. Kenig, J. Ralston, C. Levermore and E. Trubowitz on the subject.

The reference list is by no means exhaustive.

## 2. One-dimensional Estimates

Here and in the sequel we denote by $c$ numerical constants.

PROPOSITION 2.1. *The set $\{(n,n^2) \mid n \in \mathbb{Z}\}$ has bounded $\Lambda_4$-constant, i.e.*

$$\Big\| \sum_{n\in\mathbb{Z}} a_n \, e^{i(nx+n^2t)} \Big\|_{L^4(\mathbb{T}^2)} \leq c\Big(\sum_{n+\mathbb{Z}} |a_n|^2\Big)^{1/2} . \tag{2.2}$$

*Proof*: Letting $f = \sum a_n \, e^{i(nx+n^2t)}$, write $\|f\|_4 = \|f\cdot\overline{f}\|_2^{1/2}$, where

$$f\overline{f} = \sum |a_n|^2 + \sum_{n_1\neq n_2} a_{n_1}\overline{a}_{n_2} e^{i((n_1-n_2)x+(n_1^2-n_2^2)t)} . \tag{2.3}$$

Obviously, if $n_1 - n_2$ and $n_1^2 - n_2^2 = (n_1-n_2)(n_1+n_2)$ are specified, there is at most one choice for $n_1$, $n_2$ ($n_1 \neq n_2$). Hence, the $L^2$-norm of (2.3) is bounded by

$$\|f\|_2^2 + \Big(\sum_{n_1\neq n_2} |a_{n_1}\overline{a}_{n_2}|^2\Big)^{1/2} \leq 2\|f\|_2^2 . \tag{2.4}$$

Dualizing (2.1), it follows:

COROLLARY 2.5.

$$\Big(\sum |\widehat{f}(n,n^2)|^2\Big)^{1/2} \leq c\|f\|_{L^{4/3}(\mathbb{T}^2)} .$$

Instead of the restriction one may formulate following stronger multiplier inequality:

PROPOSITION 2.6. *Following estimate holds*

$$\|f\|_{L^4(\mathbb{T}^2)} \leq c\Big[\sum_{m,n\in\mathbb{Z}} (|n-m^2|+1)^{3/4} |\widehat{f}(m,n)|^2\Big]^{1/2} . \tag{2.7}$$

*Proof*: Write

$$f(x,t) = \sum_m e^{i(mx+m^2t)} \, f_m(t) \tag{2.8}$$

hence $\widehat{f}(m,n) = \widehat{f}_m(n-m^2)$ and the right number of (2.7) equals

$$\Big(\sum_m A_m^2\Big)^{1/2} \quad \text{where} \quad A_m = \Big(\sum_{n\in\mathbb{Z}} (|n|+1)^{3/4} \, |\widehat{f}_m(n)|^2\Big)^{1/2} . \tag{2.9}$$

One has

$$(f \cdot \overline{f})(x,t) = \sum_{\Delta \in \mathbf{Z}} e^{i\Delta x}\, e^{i\Delta^2 t} \sum_m e^{2im\Delta t}\, (f_m \overline{f}_{m+\Delta})(t) \tag{2.10}$$

$$\|f \cdot \overline{f}\|^2_{L^2(\mathbf{T}^2)} = \sum_{\Delta \in \mathbf{Z}} \int_{\mathbf{T}} \Big| \sum_{m \in \mathbf{Z}} e^{2im\Delta t} (f_m\, \overline{f}_{m+\Delta})(t) \Big|^2 dt\ . \tag{2.11}$$

Define for $j \geq 0$

$$f_{m,j}(t) = \sum_{|n| \sim 2^j} \widehat{f}_m(n)\, e^{int} \quad ; \quad f_m = \sum_j f_{m,j} \tag{2.12}$$

a Littlewood-Paley decomposition of $f_m$. Estimate from triangle inequality

$$\Big\| \sum_m e^{im\Delta t} f_m\, \overline{f}_{m+\Delta} \Big\|_{L^2(\mathbf{T})} \leq \sum_{j \geq k} \Big\| \sum_m e^{im\Delta t} f_{m,j}\, \overline{f}_{m+\Delta,k} \Big\|_2 \tag{2.13}$$

and distinguish the contibutions

$$\Delta^2 \leq 2^j \tag{2.14}$$

$$\Delta < 2^j \leq \Delta^2 \tag{2.15}$$

$$2^j \leq \Delta\ . \tag{2.16}$$

**Contribution of (2.14).** Evaluate pointwise by Cauchy-Schwartz

$$\Big| \sum_m e^{im\Delta t} f_{mj}\, \overline{f}_{m+\Delta,k} \Big| \leq \Big( \sum_m |f_{m,j}|^2 \Big)^{1/2} \Big( \sum_m |f_{m+\Delta,k}|^2 \Big)^{1/2} \tag{2.17}$$

which $L^2$-norm is at most

$$\begin{aligned}
&\Big( \sum_m \|f_{m,j}\|_2^2 \Big)^{1/2} \Big\| \Big( \sum_m |f_{m+\Delta,k}|^2 \Big)^{1/2} \Big\|_\infty \leq \\
&\leq 2^{k/2} \Big( \sum_m \|f_{m,j}\|_2^2 \Big)^{1/2} \Big( \sum_m \|f_{m+\Delta,k}\|_2^2 \Big)^{1/2} \\
&\leq 2^{k/8}\, 2^{-3j/8} \Big( \sum_m \sum_{|n| \sim 2^j} (|n|+1)^{3/4} |\widehat{f}_m(n)|^2 \Big)^{1/2} \cdot \\
&\quad \cdot \Big( \sum_m \sum_{|n| \sim 2^k} (|n|+1)^{3/4} |\widehat{f}_m(n)|^2 \Big)^{1/2}
\end{aligned} \tag{2.18}$$

from definition (2.12).

To evaluate (2.13), perform $\sum_{j\geq k}$ (2.18) and estimate the $k$-summation ($k \leq j$) by Cauchy-Schwartz. This gives

$$(2.11) \leq \sum_{\Delta\in\mathbb{Z}} \Big\{ \sum_{j, 2^j \geq \Delta^2} 2^{-j/4} \Big( \sum_m \sum_{|n|\sim 2^j} (|n|+1)^{3/4} |\widehat{f}_m(n)|^2 \Big)^{1/2} \Big\}^2 \cdot \Big( \sum_m A_m^2 \Big) . \tag{2.19}$$

Write $\sum_{\Delta\in\mathbb{Z}} = \sum_{\ell\geq 0} \sum_{|\Delta|\sim 2^\ell}$ and rewrite the first factor of (2.19) as

$$\sum_{\ell=0} \sum_{|\Delta|\sim 2^\ell} 2^{-\ell} \Big\{ \sum_{s\geq 0} 2^{-s/4} \Big( \sum_m \sum_{|n|\sim 2^{2\ell+s}} (|n|+1)^{3/4} |\widehat{f}_m(n)|^2 \Big)^{1/2} \Big\}^2 \tag{2.20}$$

$$\leq \sum_{\ell\geq 0} \Big[ \sum_s 2^{-s/4} \sum_m \sum_{|n|\sim 2^{2\ell+s}} (|n|+1)^{3/4} |\widehat{f}_m(n)|^2 \Big] \tag{2.21}$$

$$\leq \sum_s 2^{-s/4} \cdot \sum_m A_m^2 . \tag{2.22}$$

Hence the contribution of (2.14) to (2.11) is at most $c\big(\sum_m A_m^2\big)^2$.

**Contribution of (2.15).** Observe that by construction ($k \leq j$)

$$\operatorname{supp}(f_{mj}\, \overline{f}_{m+\Delta,k})^\wedge \subset [-2^{j+1}, 2^{j+1}] \tag{2.23}$$

while for increasing $m$ the frequency increment of $e^{im\Delta t}$ is $\Delta$. Splitting the $\sum_m$ in (2.13) in summations over arithmetic progressions of increment $\frac{2^j}{\Delta}$, say $\mathcal{M}$, one gets by orthogonality

$$\Big\| \sum_{m\in\mathcal{M}} e^{im\Delta t} f_{m,j}\, \overline{f}_{m+\Delta,k} \Big\|_2 \sim \Big( \sum_{m\in\mathcal{M}} \|f_{m,j}\, \overline{f}_{m+\Delta,k}\|_2^2 \Big)^{1/2} . \tag{2.24}$$

Summing over the progressions $\mathcal{M}$ yields then the bound

$$\Big\| \sum_m e^{im\Delta t} f_{m,j}\, \overline{f}_{m+\Delta,k} \Big\|_2 \lesssim \Big( \frac{2^j}{\Delta} \Big)^{1/2} \Big( \sum_m \|f_{mj}\, \overline{f}_{m+\Delta,k}\|_2^2 \Big)^{1/2} . \tag{2.25}$$

Estimate again

$$\|f_{mj}\cdot\overline{f}_{m+\Delta,k}\|_2 \le \|f_{mj}\|_2\,\|f_{m+\Delta,k}\|_\infty \lesssim 2^{k/2}\|f_{mj}\|_2\,\|f_{m+\Delta,k}\|_2\,. \tag{2.26}$$

So the contribution to (2.11) is at most

$$\sum_{\Delta\in\mathbb{Z}}\Big\{\sum_{\substack{j\ge k\\ \Delta<2^j<\Delta^2}}\Big(\frac{2^j}{\Delta}\Big)^{1/2}2^{k/2}\Big(\sum_m\|f_{mj}\|_2^2\,\|f_{m+\Delta,k}\|_2^2\Big)^{1/2}\Big\}^2 \tag{2.27}$$

$$\begin{aligned}\sim\sum_{\Delta\in\mathbb{Z}}\frac{1}{\Delta}\Big\{\sum_{\substack{j\ge k\\ \Delta<2^j<\Delta^2}}2^{j/8}2^{k/8}\Big[\sum_m\Big(\sum_{|n|\sim 2^j}\big(|n|+1\big)^{3/4}\big|\widehat{f}_m(n)\big|^2\Big)\cdot\\ \cdot\Big(\sum_{|n|\sim 2^k}\big(|n|+1\big)^{3/4}\big|\widehat{f}_{m+\Delta}(n)\big|^2\Big)\Big]^{1/2}\Big\}^2\end{aligned} \tag{2.28}$$

$$\le c\sum_{\Delta\in\mathbb{Z}}\Big(\sum_m A_m^2\,A_{m+\Delta}^2\Big)=\Big(\sum A_m^2\Big)^2\,. \tag{2.29}$$

**Contribution of (2.16).** From (2.23), one gets by orthogonality

$$\Big\|\sum_m e^{im\Delta t}f_{mj}\,\overline{f}_{m+\Delta,k}\Big\|_2\lesssim\Big(\sum_m\|f_{mj}\cdot\overline{f}_{m+\Delta,k}\|_2^2\Big)^{1/2} \tag{2.30}$$

and the previous calculation yields now

$$\begin{aligned}\sum_{\Delta\in\mathbb{Z}}\Big\{\sum_{j\ge k}2^{k/8}2^{-3j/8}\Big[\sum_m\Big(\sum_{|n|\sim 2^j}\big(|n|+1\big)^{3/4}\big|\widehat{f}_m(n)\big|^2\Big)\cdot\\ \cdot\Big(\sum_{|n|\sim 2^k}\big(|n|+1\big)^{3/4}\big|\widehat{f}_{m+\Delta}(n)\big|^2\Big)\Big]^{1/2}\Big\}^2\end{aligned} \tag{2.31}$$

instead of (2.28). One concludes similarly by applying the Cauchy-Schwartz inequality

$$\sum_{\Delta\in\mathbb{Z}}\Big[\sum_m A_m^2\,A_{m+\Delta}^2\Big]=\Big(\sum A_m^2\Big)^2\,.$$

Hence (2.11) is bounded by $\big(\sum A_m^2\big)^2$, which by (2.9) completes the proof.

Again by duality, Proposition 2.6 implies

COROLLARY 2.32.

$$\Big(\sum_{m,n\in\mathbb{Z}}\big(|n-m^2|+1\big)^{-3/4}\big|\widehat{f}(m,n)\big|^2\Big)^{1/2}\leq c\|f\|_{L^{4/3}(\mathbb{T}^2)}\ .$$

Combining (2.6), (2.32), one obtains following Fourier multiplier result

PROPOSITION 2.33. *Assume $\lambda=(\lambda_{m,n})_{m,n\in\mathbb{Z}}$ a multiplier satisfying*

$$|\lambda_{m,n}|\leq\big(1+|n-m^2|\big)^{-3/4}\ \text{ for all }\ m,n\ . \tag{2.34}$$

*Then $\lambda$ acts boundedly from $L^{4/3}(\mathbb{T}^2)$ to $L^4(\mathbb{T}^2)$, i.e.*

$$\Big\|\sum\lambda_{m,n}\ \widehat{f}(m,n)\ e^{i(mx+nt)}\Big\|_{L^4(\mathbb{T}^2)}\leq c\|f\|_{L^{4/3}(\mathbb{T}^2)}\ . \tag{2.35}$$

Inequality (2.35) will be used in the Picard approach to the NLSE (1.1), (1.2), (1.3) with $L^2$-data.

*Remark*: The exponent $\frac{3}{4}$ in (2.7) is sharp. Consider for instance the function

$$f=\sum_{\substack{|m|\leq N\\ |n|\leq N^2}}e^{i(mx+nt)}\ .$$

Then $\|f\|_2\sim N^{3/2}$ and $\|f\|_4\lesssim N^3N^{-3/4}=N^{9/4}\sim\big(\sum|n-m^2|^{3/4}\cdot\big|\widehat{f}(m,n)\big|^2\big)^{1/2}$.

Next, we consider $L^6$-estimates.

PROPOSITION 2.36. *Let $S_N=\big\{(n,n^2)\ \big|\ |n|\leq N\big\}$. Then*

$$K_6(S_N)<\exp c\,\frac{\log N}{\log\log N}\ . \tag{2.37}$$

*Hence, one has*

$$\Big\|\sum_{n\in\mathbb{Z}}a_ne^{i(nx+n^2t)}\Big\|_{L^6(\mathbb{T})}\ll N^{\varepsilon}\Big(\sum|a_n|^2\Big)^{1/2}\ ,\quad\forall\varepsilon>0\ . \tag{2.38}$$

*Proof*: Letting $f = \sum_1^N a_n e^{i(nx+n^2t)}$, one has by straight calculation

$$\int_{\mathbf{T}^2} |f|^6 = \|f^3\|_2^2 = \sum_{n,j} \Big| \sum_{n_1^2+n_2^2+(n-n_1-n_2)^2=j} a_{n_1} a_{n_2} a_{n-n_1-n_2} \Big|^2 \leq$$
$$\leq \max_{\substack{|n|\leq 3N \\ |j|\leq 3N^2}} r_{n,j} \cdot \Big( \sum_{n\leq N} |a_n|^2 \Big)^3 \tag{2.39}$$

defining

$$r_{n,j} = \#\{(n_1,n_2) \mid |n_i| \leq N \text{ and } n_1^2+n_2^2+(n-n_1-n_2)^2 = j\} \ . \tag{2.40}$$

It remains to show that $r_{n,j} < \exp c \frac{\log N}{\log\log N}$. The condition $n_1^2 + n_2^2 + (n-n_1-n_2)^2 = j$ may be rewritten as

$$n_1^2 + n_2^2 - nn_1 - nn_2 + n_1n_2 = \frac{j-n^2}{2}$$

$$\frac{3}{4}(n_1+n_2)^2 + \frac{1}{4}(n_1-n_2)^2 - n(n_1+n_2) = \frac{j-n^2}{2} \ . \tag{2.41}$$

Put $m_1 = n_1 + n_2$, $m_2 = n_1 - n_2$ to get

$$(3m_1 - 2n)^2 + 3m_2^2 = 6j - 2n^2 \ . \tag{2.42}$$

Consider the equation $X^2 + 3Y^2 = A$ $(X, Y, A \in \mathbf{Z})$. Denoting $\rho = e^{\frac{2\pi i}{3}} = \frac{1+i\sqrt{3}}{2}$, $\mathbf{Z} + \rho\mathbf{Z}$ is known to be an euclidean division domain. Consequently, the number of divisors of $A$ in $\mathbf{Z} + \rho\mathbf{Z}$ is at most $\exp c \frac{\log A}{\log\log A} < \exp c \frac{\log N}{\log\log N}$. Finally, observe that if $X, Y \in \mathbf{Z}$ satisfy $X^2 + 3Y^2 = A$, then $X + i\sqrt{3}Y$ is a divisor of $A$ in $\mathbf{Z} + \rho\mathbf{Z}$. This concludes the proof, since the pair $(3m_1 - 2n, m_2)$ defines $(n_1, n_2)$ uniquely.

*Remark 1*: Results on the number of lattice points on algebraic and real analytic curves are provided by the work of Bombieri and Pila [BP]. For instance (see [BP]), if $\Gamma$ is a real analytic image of the circle $S^1$, then for $t \to \infty$

$$|t\Gamma \cap \mathbf{Z}^2| \ll t^\varepsilon \tag{2.43}$$

which of course applies in the context above $\big(\Gamma = \{X^2 + 3Y^2 = 1\}\big)$.

*Remark 2*: The arithmetic analogue of Strichartz's inequality (1.5), i.e.

$$\left\| \int e^{i(\lambda x+\lambda^2 t)} \phi(\lambda) d\lambda \right\|_{L^6(dxdt)} \leq c\Big( \int |\phi(\lambda)|^2 d\lambda\Big)^{1/2} \tag{2.44}$$

is played by inequality (2.38). This estimate fails if one replaces $N^\varepsilon$ by a constant. In fact

$$\frac{1}{\sqrt{N}} \left\| \sum_{n=0}^{N} e^{i2\pi(nx+n^2 t)} \right\|_6 \to \infty \quad \text{for} \quad N \to \infty \tag{2.45}$$

(compare with the fact that the squares do not form a $\Lambda_4$-set). This may be seen as follows. Let $1 \leq a < q < N^{1/2}$, $(a,q) = 1$, $0 \leq b < q$ be integers and take

$$\left. \begin{aligned} \left| x - \frac{b}{q} \right| &= o\left(\frac{1}{N}\right) \\ \left| t - \frac{a}{q} \right| &= o\left(\frac{1}{N^2}\right) \end{aligned} \right\} . \tag{2.46}$$

It follows then from the theory of exponential sums (see [Vi]) that

$$\left| \sum_{n=0}^{N} e^{2\pi i(nx+n^2 t)} \right| \sim \frac{N}{\sqrt{q}} \tag{2.47}$$

in the context of (2.46). Denoting

$$f(x,t) = \sum_{n=0}^{N} e^{2\pi i(nx+n^3 t)} \tag{2.48}$$

and $\mathcal{M}(q,a,b)$ the region described by (2.46), it follows that

$$\int_{\mathcal{M}_0(q,a,b)} |f|^6 \, dxdt \sim \frac{N^3}{q^3} . \tag{2.49}$$

Hence,

$$\int |f|^6 \, dxdt \geq c \sum_{\substack{a<q \\ (a,q)=1}} \frac{N^3}{q^2} \geq c(\log N) \cdot N^3 \tag{2.50}$$

implying that

$$K_6(S_N) > c(\log N)^{1/6} . \tag{2.51}$$

**Problem.** Is $K_p(S_N)$ bounded for $p < 6$? This was shown to be the case if $p \leq 4$.

*Remark 3*: The exponent $\frac{3}{4}$ in inequality (2.7) may be "explained" by interpolating $T_s : L^2 \to L^p$ mapping $f$ to $\sum_{m,n} \widehat{f}(m,n)[1+|n-m^2|]^{-s}e^{i(mx+nt)}$ between $p = 2$, $p = 6$. If $p = 2$, $T_s$ is bounded for $Re s = 0$. If $p = 6$, $T_s$ is "almost" bounded for $Re s = \frac{1}{2}$. The interpolated value at $p = 4$ is then 3/8.

## 3. Higher Dimensional Estimates

Fix $d \geq 2$ and consider the subsets of $\mathbb{Z}^d$

$$\{(n_1, \dots, n_{d-1}, |\overline{n}|^2) \mid n_j \in \mathbb{Z} \ , \ |n_j| < N\} = S_{d,N} \tag{3.1}$$

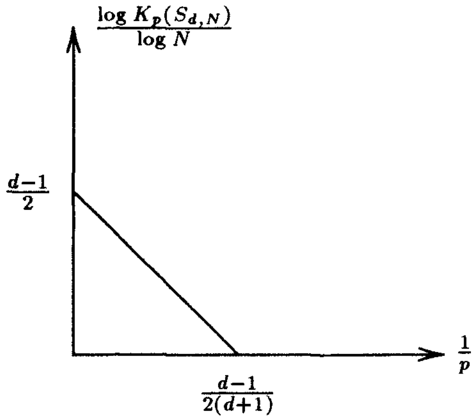
where $\overline{n} = (n_1, \dots, n_{d-1})$, $|\overline{n}|^2 = n_1^2 + \cdots + n_{d-1}^2$. These sets were considered in previous section for $d = 2$. It seems reasonable to conjecture that

$$\begin{cases} K_p(S_{d,N}) < c_p & \text{for} \quad p < \frac{2(d+1)}{d-1} & (3.2) \\ K_p(S_{d,N}) \ll N^\varepsilon & \text{for} \quad p = \frac{2(d+1)}{d-1} & (3.3) \\ K_p(S_{d,N}) < c_p \ N^{\frac{d-1}{2} - \frac{d+1}{p}} & \text{for} \quad p > \frac{2(d+1)}{d-1} \ . & (3.4) \end{cases}$$

For $d = 2$, (3.2) for $p \leq 4$ and (3.3) were proved in previous section. Interpolation with $L^\infty$ would yield (3.4) up to an $N^\varepsilon$-factor. From the estimates based on the Weyl-sum approach presented below, we will actually also deduce (3.4).

For $d = 3$, we will prove estimates (3.3), (3.4).

Starting from $d = 4$, even a rough understanding of the diagram (ignoring $N^\varepsilon$-factors)




is only partial.

For $d = 4$ we get (3.3) with $p = 4$ and (3.4) for $p > 4$. If $d \geq 5$, the Weyl-sum approach mentioned above yields (3.4) in the restricted range

$$p \geq \frac{2(d+3)}{d-1} \tag{3.5}$$

Also the (sharp) distributional inequality with $p = \frac{2(d+1)}{d-1}$ for level sets $[|f| > \lambda]$ may be obtained from this method provided $\lambda$ is large enough.

PROPOSITION 3.6.

$$K_4(S_{3,N}) \ll N^{\varepsilon} \tag{3.7}$$

$$K_4(S_{4,N}) \ll N^{\frac{1}{4}+\varepsilon} \tag{3.8}$$

$$K_4(S_{5,N}) \ll N^{\frac{1}{2}+\varepsilon} \tag{3.9}$$

$$K_4(S_{d,N}) < c_d\, N^{\frac{d-3}{4}} \quad \text{for} \quad d \geq 6 \tag{3.10}$$

(We will show later that the $N^{\varepsilon}$-factor in (3.9) is not necessary.)

*Proof*: The problem will reduce to estimating the number of representations of an integer as a sum of squares. Let

$$f(x,t) = \sum_{|\overline{n}|<N} a_{\overline{n}}\, e^{2\pi i(\langle \overline{n},x\rangle + |\overline{n}|^2 t)} \quad \text{with} \quad (x,t) \in \mathbf{T}^{d-1} \times \mathbf{T}\ . \tag{3.11}$$

Hence

$$f(x,t)^2 = \sum_{\overline{p}} e^{2\pi i \langle \overline{p}, x\rangle} \Big[ \sum_{\overline{n}} a_{\overline{n}} a_{\overline{p}-\overline{n}} e^{2\pi i (|\overline{n}|^2 + |\overline{p}-\overline{n}|^2)t} \Big] \tag{3.12}$$

$$\| f^2 \|^2_{L^2(\mathbf{T}^{d+1})} \leq \Big\{ \max_{\substack{|\overline{p}|\leq 2N \\ |j|\leq 2N^2}} r_{\overline{p},j} \Big\} \Big( \sum |a_{\overline{n}}|^2 \Big)^2 \tag{3.13}$$

denoting

$$r_{\overline{p},j} = \#\{ \overline{n} \in \mathbf{Z}^{d-1} \mid |\overline{n}| \leq N \ \text{ and } \ |\overline{n}|^2 + |\overline{p}-\overline{n}|^2 = j \}\ . \tag{3.14}$$

Rewrite the equation $|\overline{n}|^2 + |\overline{p}-\overline{n}|^2 = j$ as

$$(2n_1 - p_1)^2 + \cdots + (2n_{d-1} - p_{d-1})^2 = 2j - |p|^2 \tag{3.15}$$

so that $r_{\overline{p},j}$ may be bounded by the number of solutions of

$$X_1^2 + \cdots + X_{d-1}^2 = A\ ; \quad A = 2j - |p|^2\ . \tag{3.16}$$

Hence, there are the estimates

$$\left.\begin{array}{ll} A^{\varepsilon} & (d=3) \\ A^{\frac{1}{2}+\varepsilon} & (d=4) \\ A^{1+\varepsilon} & (d=5) \\ A^{\frac{d-3}{2}} & (d>5) \end{array}\right\} . \tag{3.17}$$

Since $|A| \leq N^2$, (3.13), (3.17) yield (3.6).

*Remark*: More details on representations of integers as sum of squares may be found in [Gr].

In the remainder of this section, we will develop a more analytical method. It is based on two ideas

- Tomas' proof of the restriction theorem for surfaces $S$ in $\mathbf{R}^d$
- The "major arc" description of exponential sums.

Roughly speaking, the surface $S$ in $\mathbf{R}^d$ is replaced by the set $S_{d,N}$ in $\mathbf{Z}^d$. Let $\sigma$ be the surface measure of $S$ (assumed compact). Tomas' argument consists then of analyzing the mapping properties of $f \mapsto f * \widehat{\sigma}$ by breaking up $\widehat{\sigma}$ in level sets and estimating their individual contribution by interpolation between $L^1$ and $L^2$. Following the same scheme, $\widehat{\sigma}$ becomes the (higher-dimensional) exponential sum

$$\sum_{|n_1|,\dots,|n_{d-1}|\le N} e^{2\pi i(\langle \overline{n},x\rangle + |\overline{n}|^2 t)}$$

which level sets on $\mathbf{T}^{d+1}$ correspond to the "major arc" description in the sense of Vinogradov [Vi]. This essentially explains our procedure.

We will use following Weyl type lemma. Its proof is classical. We include it for selfcontainedness sake.

LEMMA 3.18. *Let $\{\sigma_n\}$ be a multiplier satisfying*

$$\begin{cases} 0 \le \sigma_n \le 1\ ,\ \sigma_n = 1 \ \text{ on } \ [-N,N] \\ \{\sigma_{n+1} - \sigma_n\} \text{ is bounded by } \frac{1}{N} \text{ and has variation bounded by } \frac{1}{N} \\ \operatorname{supp} \sigma_n \subset [-2N, 2N] \end{cases} \tag{3.19}$$

*and let*

$$f(x,t) = \sum \sigma_n\, e^{2\pi i(nx+n^2 t)}\ . \tag{3.20}$$

*If $0 < a < q < N$, $(a,q) = 1$ and $\|t - \frac{a}{q}\| < \frac{1}{qN}$, then*

$$\big|f(x,t)\big| < c\, \frac{N}{\sqrt{q}\left(1 + N\|t - \frac{a}{q}\|^{1/2}\right)} \tag{3.21}$$

*(the role of $\sigma = \{\sigma_n\}$ is to avoid logarithmic factors in $N$ and plays only a technical role).*

*Proof*: One has

$$\big|f(x,t)\big|^2 = \sum_{n_1,n_2} \sigma_{n_1}\ \sigma_{n_2} e^{2\pi i[(n_1-n_2)x + (n_1-n_2)(n_1+n_2)t]}$$

and letting $k = n_1 - n_2$, $\ell = n_1 + n_2$

$$\big|f(x,t)\big|^2 \le \sum_{\ell} \Big| \sum_{k \equiv \ell(2)} \sigma_{\frac{k+\ell}{2}}\ \sigma_{\frac{\ell-k}{2}}\ e^{2\pi i k(x+\ell t)} \Big|\ . \tag{3.22}$$

Writing $k = 2k_1$ (if $\ell$ even), $k = 2k_1 + 1$ (if $\ell$ odd), each multiplier $\tau = \{\tau_{k_1} = \sigma_{[\frac{\ell+1}{2}]+k_1}\ \sigma_{[\frac{\ell}{2}]-k_1}\}$ satisfies (3.19) (up to factor 2). Therefore (3.22) is bounded by

$$c \sum_{|\ell|\leq 4N} \frac{1}{N(\ \| \ 2x + 2\ell t \ \| + \frac{1}{N})^2} \ . \tag{3.23}$$

Write $t = \frac{a}{q} + \tau$, $|\tau| < \frac{1}{Nq}$. Hence $2x + 2\ell t = 2x + 2\ell \frac{a}{q} + 2\ell\tau$. Assume $-2x \in [\ell^* \frac{a}{q}, (\ell^* + 1)\frac{a}{q}] = I$ and consider the orbit in $\mathbf{R}/\mathbf{Z}$ of $2x + \ell\frac{a}{q} + \ell\tau$. One easily verifies that the contribution (3.23) away from the interval $I$ is

$$\leq c\, \frac{N}{q} \sum_{0<r<q} \frac{1}{N\left(\frac{r}{q} + \frac{1}{N}\right)^2} < cq \ . \tag{3.24}$$

The contribution around $I$ may be estimated as

$$\leq \min\left\{\frac{N}{q} \cdot N \ , \ N + \sum_{r<\frac{N}{q}} \frac{1}{N\big(rq|\tau| + \frac{1}{N}\big)^2}\right\} \tag{3.25}$$

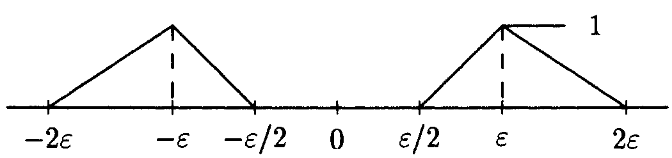
$$\leq \min\left\{\frac{N^2}{q}, \frac{1}{q|\tau|}\right\} \ . \tag{3.26}$$

Collecting estimates, (3.21) immediately follows.

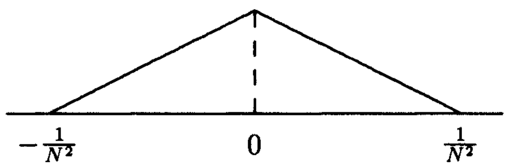
The (one-dimensional) major arcs appear on neighbourhoods as sets of rationals

$$\mathcal{R}_Q = \left\{\frac{a}{q} \mid 1 \leq a < q \ , \ (a,q) = 1 \ , \ Q \leq q < 2Q\right\} \ . \tag{3.26}$$

More precisely, define for $\frac{1}{N^2} < \varepsilon \leq \frac{1}{N}$ the function $\omega_\varepsilon$



and $\omega_{\frac{1}{N^2}}$




so that for $1 \leq Q \leq N$

$$\sum_{N \geq 2^s \geq Q} \omega_{\frac{1}{N 2^s}} = 1 \quad \text{on} \quad \left[-\frac{1}{NQ}\, , \, \frac{1}{NQ}\right] \tag{3.27}$$

$$\sum_{N \geq 2^s \geq Q} \omega_{\frac{1}{N 2^s}} \quad \text{is supported by} \quad \left[-\frac{2}{NQ}\, , \, \frac{2}{NQ}\right]\, . \tag{3.28}$$

Let $N_1 = o(N)$ and define $\mathcal{R} = \bigcup_{Q \leq N_1} \mathcal{R}_Q$.

Observe that for $Q_1 \leq Q_2 \leq N_1$

$$\left(\mathcal{R}_{Q_1} + \left[-\frac{2}{NQ_1}\, , \, \frac{2}{NQ_1}\right]\right) \cap \left(\mathcal{R}_{Q_2} + \left[-\frac{2}{NQ_2}\, , \, \frac{2}{NQ_2}\right]\right) = \emptyset \tag{3.29}$$

(major-arc disjointness property).

Write ($\delta_x$ = Dirac measure at point $x$)

$$1 = \sum_{\substack{Q \leq N_1 \\ Q \text{ dyadic}}} \sum_{Q \leq 2^s \leq N} \left[\left(\sum_{x \in \mathcal{R}_Q} \delta_x\right) * \omega_{\frac{1}{N 2^s}}\right] + \rho\, . \tag{3.30}$$

We recall Dirichlet's lemma.

LEMMA 3.31. *Given $t \in \mathbf{T}$, there is fraction $\frac{a}{q}$, $(a,q) = 1$, $q \leq N$ such that $\left|t - \frac{a}{q}\right| < \frac{1}{qN}$.*

Observe that if for given $t \in \mathbf{T}$, the first term in (3.30) differs from 1, i.e. $\rho(t) \neq 0$, then $q$ given by (3.31) has to satisfy $q > N_1$, so that by Lemma (3.18), for all $x \in \mathbf{T}$

$$\left|\rho(t)\Big(\sum \sigma_n\, e^{2\pi i(nx+n^2 t)}\Big)\right| < c\sqrt{N}\, . \tag{3.32}$$

Our next aim is to evaluate the Fourier transform of $\sum_{x \in \mathcal{R}_Q} \delta_x$ (which is a function on $\mathbf{Z}$).

LEMMA 3.33. *Denote $d(n;Q)$ the number of divisors of $n$ less than $Q$. Then for $n \neq 0$*

$$\left| \Big( \sum_{x \in \mathcal{R}_Q} \delta_x \Big)^{\wedge} (n) \right| \ll d(n;Q)\ Q^{1+\varepsilon} \tag{3.34}$$

*and obviously*

$$\Big( \sum_{x \in \mathcal{R}_Q} \delta_x \Big)^{\wedge} (0) \sim Q^2 \ . \tag{3.35}$$

*Proof*: We have to evaluate

$$\sum_{Q \leq q < 2Q} \Bigg( \sum_{\substack{a<q \\ (a,q)=1}} e^{2\pi i \frac{a}{q} n} \Bigg) \ . \tag{3.36}$$

Fix $q$ with prime decomposition $q = p_1^{r_1}\ p_2^{r_2} \cdots$. Hence

$$\frac{a}{q} = \frac{a_1}{p_1^{r_1}} + \frac{a_2}{p_2^{r_2}} + \cdots \qquad 0 \leq a_j < p_j^{r_j} \tag{3.37}$$

and the condition $(a,q)=1$ is equivalent to $(a_j, p_j) = 1$ for each $j$. Therefore

$$\sum_{\substack{a<q \\ (a,q)=1}} e^{2\pi i n \frac{a}{q}} = \prod_j \Bigg[ \sum_{a=0}^{p_j^{r_j}-1} e^{2\pi i \frac{a}{p_j^{r_j}} n} - \sum_{0}^{p_j^{r_j-1}-1} e^{2\pi i \frac{a}{p_j^{r_j-1}} n} \Bigg] \tag{3.38}$$

$$= \prod_j \Big[ p_j^{r_j}\ c(n, p_j^{r_j}) - p_j^{r_j-1}\ c(n, p_j^{r_j-1}) \Big] \tag{3.39}$$

where one denotes

$$c(n,q) = 1 \ \text{ if } \ q \mid n \ \text{ and } \ c(n,q) = 0 \ \text{ otherwise} \ . \tag{3.40}$$

It follows from (3.39) that

$$\left| \sum_{\substack{a<q \\ (a,q)=1}} e^{2\pi i n \frac{a}{q}} \right| = q \prod_j \left| c(n, p_j^{r_j}) - \frac{1}{p_j}\ c(n, p_j^{r_j-1}) \right| \ . \tag{3.41}$$

If $p_j \mid n$, estimate $c(n, p_j^{r_j}) - \frac{1}{p_j}\ c(n, p_j^{r_j-1})$ by 1. Otherwise

$$c(n, p_j^{r_j}) - \frac{1}{p_j}\ c(n, p_j^{r_j-1}) = 0 \ \text{ if } \ r_j \geq 2$$

$$= -\frac{1}{p_j} \ \text{ if } \ r_j = 1$$

so that for $(n,p_j)=1$

$$\left|c(n,p_j^{r_j}) - \frac{1}{p_j}\, c(n,p_j^{r_j-1})\right| \le \frac{1}{p_j^{r_j}} \ . \tag{3.42}$$

Write $q=q'q''$ where the prime factors of $q'$ divide $n$ and $(n,q'')=1$. It follows then from the preceding that

$$\left| \sum_{a<q,(a,q)=1} e^{2\pi i n \frac{a}{q}} \right| \le \frac{q}{q''} \ . \tag{3.43}$$

Consequently

$$\sum_{Q\le q<2Q} \left| \sum_{a<q,(a,q)=1} e^{2\pi i n \frac{a}{q}} \right| < Q \sum_{\substack{q'<Q \text{ with prime} \\ \text{factors dividing } n}} \ \sum_{\substack{q''\sim \frac{Q}{q'} \\ (n,q'')=1}} \frac{1}{q''}$$

$$< Q\#\{q<Q \mid \text{ prime factors of } q \text{ divide } n\} \ . \tag{3.44}$$

If $p_1,p_2,\dots,p_k$ are the prime factors of $n$, an element $q$ appearing in (3.34) has the form

$$q=p_1^{r_1}\cdots p_k^{r_k} \quad \text{with} \quad r_j\ge 0 \tag{3.45}$$

and

$$\sum_{j=1}^{k} r_j \log p_j \le \log Q \ . \tag{3.46}$$

Hence their number may be estimated by $\exp c\,\frac{\log Q}{\log\log Q}\cdot d(n;Q)$. This proves (3.34).

Related to the quantities $d(n,Q)$, we recall Lemma 4.28 from [Bo1].

LEMMA 3.47. *$\#\{0\le n\le N \mid d(n;Q)>D\} < c_{\tau,B}(D^{-B}Q^{\tau}N+Q^{B})$ whenever $\tau>0$, $B<\infty$ are given constants.*

*Proof*: For $2\le q<Q$, define the function $\mathcal{J}_q$ on $[0,N]$, putting

$$\left.\begin{aligned} \mathcal{J}_q(n) &= 1 \quad \text{if } q\mid n \\ &= 0 \quad \text{otherwise} \end{aligned}\right\} \ . \tag{3.48}$$

Fixing an integer power $B\ge 1$, write from Tchebychev's inequality

$$\#\{0\le n\le N \mid d(n;Q)>D\} \le D^{-B}\sum_{1}^{N}\Big(\sum_{2\le q\le Q}\mathcal{J}_q(n)\Big)^{B} \ . \tag{3.49}$$

Denote $[q_1, q_2, \dots, q_B]$ the smallest common multiple. Expanding the $B$-power in (3.49), one finds

$$\frac{1}{N}\sum_1^N \Big[\sum_{q\leq Q} \mathcal{J}_q(n)\Big]^B \sim \sum_{q_1\cdots q_B\leq Q} \Big([q_1,\dots,q_B]^{-1} + \frac{1}{N}\Big) \leq$$
$$\sum_{q\leq Q^B} \frac{1}{q}\, d(q)^B + \frac{Q^B}{N} < \exp\Big(CB^2 \frac{\log Q}{\log\log Q}\Big) + \frac{Q^B}{N}\,. \tag{3.50}$$

Thus (3.49) is bounded by

$$N\cdot D^{-B}\cdot \exp\Big(CB^2\, \frac{\log Q}{\log\log Q}\Big) + Q^B \tag{3.51}$$

and the lemma follows.

Define following function on $\mathbb{Z}^d$

$$K = \sum_{n_1,\dots,n_{d-1}} \sigma_{n_1}\, \sigma_{n_2} \cdots \sigma_{n_{d-1}}\, \delta_{(n_1,\dots,n_{d-1},n_1^2+\cdots+n_{d-1}^2)} \tag{3.52}$$

where $\{\sigma_n\}$ is the sequence of weights considered in (3.19).

Thus $\operatorname{supp} K \subset [-2N, 2N]^{d-1} \times [0, 4dN^2]$. Also $\widehat{K}(x,t)$ on $\mathbb{T}^{d-1}\times\mathbb{T}$ appears as the product

$$\widehat{K}(x,t) = \prod_{j=1}^{d-1} \Big(\sum \sigma_n\, e^{2\pi i(nx_j + n^2 t)}\Big)\,. \tag{3.53}$$

From (3.30), (3.35) one has

$$1 \leq \sum_{\substack{Q\leq N_1\\ Q \text{ dyadic}}} \sum_{Q\leq 2^s\leq N} \frac{Q^2}{N2^s} + \widehat{\rho}(0)$$

$$\big|\widehat{\rho}(0) - 1\big| < \frac{N_1}{N}\,. \tag{3.54}$$

Define coefficients $\alpha_{Q,s}$ such that

$$\Big[\Big(\sum_{x\in\mathcal{R}_Q} \delta_x\Big) * \omega_{\frac{1}{N2^s}}\Big]^\wedge(0) = \alpha_{Q,s}\, \widehat{\rho}(0) \tag{3.55}$$

hence by (3.54)

$$\alpha_{Q,s} < c\,\frac{Q^2}{N2^s}\ . \tag{3.56}$$

Based on (3.30), one has the identity

$$\widehat{K}(x,t) =$$

$$= \sum_{\substack{Q<N_1\\ Q \text{ dyadic}}} \sum_{Q\leq 2^s\leq N} \widehat{K}(x,t)\cdot\Big[\Big(\Big(\sum_{x\in\mathcal{R}_Q}\delta_x\Big) * \omega_{\frac{1}{N2^s}}\Big) - \alpha_{Q,s}\cdot\rho\Big](t) \tag{3.57}$$

$$+\Big[1+\sum_{\substack{Q\leq N_1\\ Q \text{ dyadic}}} \sum_{Q\leq 2^s\leq N} \alpha_{Q,s}\Big]\widehat{K}(x,t)\rho(t)\ . \tag{3.58}$$

Define the multipliers

$$\Lambda_{Q,s}(x,t) = \widehat{K}(x,t)\cdot\Big[\Big(\Big(\sum_{x\in\mathcal{R}_Q}\delta_x\Big) * \omega_{\frac{1}{N2^s}}\Big) - \alpha_{Q,s}\cdot\rho\Big](t)\ . \tag{3.59}$$

It follows from Lemma 3.18, (3.32), (3.53), (3.56) that

$$|\Lambda_{Q,s}| \lesssim \left[\frac{N}{\sqrt{Q}\Big(1+N\big(\frac{1}{N2^s}\big)^{1/2}\Big)}\right]^{d-1} + \frac{Q^2}{N2^s}(\sqrt{N})^{d-1}\ . \tag{3.60}$$

Since $2^s > Q$, there is the pointwise estimate

$$|\Lambda_{Q,s}| \lesssim \left(\frac{N\cdot 2^s}{Q}\right)^{\frac{d-1}{2}}\ . \tag{3.61}$$

Next, estimate $\widehat{\Lambda}_{Q,s}$. From (3.59)

$$\widehat{\Lambda}_{Q,s} = K * \Big[\Big(\sum_{x\in\mathcal{R}_Q}\delta_x\Big)^{\wedge}\cdot\widehat{\omega}_{\frac{1}{N2^s}} - \alpha_{Q,s}\cdot\widehat{\rho}\Big]\ . \tag{3.62}$$

By definition (3.55) of $\alpha_{Q,s}$, the second factor vanishes outside $\big\{(0,\dots,0)\in \mathbb{Z}^{d-1}\big\}\times\big(\mathbb{Z}\backslash\{0\}\big)$. Hence

$$\widehat{\Lambda}_{Q,s}(z_1,\dots,z_d) = \sum_{\overline{n}\in\mathbb{Z}^{d-1}} \sigma_{n_1}\cdots\sigma_{n_{d-1}}\Big[\Big(\sum_{x\in\mathcal{R}_Q}\delta_x\Big)^{\wedge}\cdot\widehat{\omega}_{\frac{1}{N2^s}} - \alpha_{Q,s}\,\widehat{\rho}\Big]\cdot \tag{3.63}$$

$$\cdot\big(z_1-n_1,\dots,z_{d-1}-n_{d-1},z_d-|\overline{n}|^2\big)$$

$$= \sigma_{z_1}\cdots\sigma_{z_{d-1}}\Big[\Big(\sum_{x\in\mathcal{R}_Q}\delta_x\Big)^{\wedge}\cdot\widehat{\omega}_{\frac{1}{N2^s}} - \alpha_{Q,s}\widehat{\rho}\Big]\cdot \tag{3.64}$$

$$\cdot\big(z_d - z_1^2 - \cdots - z_{d-1}^2\big)\ .$$

If $z_d = z_1^2 + \cdots + z_{d-1}^2$, (3.64) vanishes. Otherwise (3.34), (3.56) imply

$$\left|\widehat{\Lambda}_{Q,s}(z_1,\ldots,z_d)\right| \ll \frac{Q^{1+\varepsilon}}{N2^s}\, d(z_d - z_1^2 - \cdots - z_{d-1}^2; Q) + \\ + \frac{Q^2}{N2^s}\left|\widehat{\rho}(z_d - z_1^2 - \cdots - z_{d-1}^2)\right| . \tag{3.65}$$

It follows from (3.30), (3.34) that for $n \neq 0$, $|n| \leq N^2$

$$\left|\widehat{\rho}(n)\right| \leq \sum_{\substack{Q \leq N_1 \\ Q \text{ dyadic}}} \sum_{Q \leq 2^s \leq N} \frac{d(n,Q)Q^{1+\varepsilon}}{N2^s} \ll \frac{N^\varepsilon}{N} \tag{3.66}$$

so that by (3.65)

$$\left|\widehat{\Lambda}_{Q,s}(z_1,\ldots,z_d)\right| \ll \frac{Q}{N2^s}\left[Q^\varepsilon d(z_d - z_1^2 - \cdots - z_{d-1}^2; Q) + \frac{Q}{N^{1-\varepsilon}}\right] \tag{3.67}$$

$$\ll \frac{Q}{N2^s} . \tag{3.68}$$

We assume here $|z_1|,\ldots,|z_{d-1}| \leq N$ and $|z_d| < cN^2$.

Inequalities (3.61) and (3.67), (3.68) are the key estimates in what will follow. From (3.61) one gets

$$\|f * \Lambda_{Q,s}\|_{L^\infty(\mathbf{T}^d)} \leq \|f\|_1\, \|\Lambda_{Q,s}\|_\infty \leq c\left(\frac{N\cdot 2^s}{Q}\right)^{\frac{d-1}{2}} \|f\|_1 . \tag{3.69}$$

From (3.68)

$$\|f * \Lambda_{Q,s}\|_{L^2(\mathbf{T}^d)} \leq \|f\|_2 \|\widehat{\Lambda}_{Q,s}\|_{\ell^\infty(\mathbf{Z}^d)} \ll \frac{Q}{N2^s} N^\varepsilon \|f\|_2 . \tag{3.70}$$

In fact, more precisely

$$\|f * \Lambda_{Q,s}\|_2 \leq \frac{Q}{N2^s}\left[\sum_{z_d \neq z_1^2+\cdots+z_{d-1}^2} \left|\widehat{f}(z_1,\ldots,z_d)\right|^2 d(z_d - z_1^2 - \cdots - z_{d-1}^2; Q)^2\right]^{1/2} \\ + \frac{Q^2}{2^s N^{2-\varepsilon}} \parallel f \parallel_2 . \tag{3.71}$$

It follows easily from Lemma 3.47 that for given $D$ and constants $\tau, B$

$$\begin{aligned}\#\{(z_1,\dots,z_d) \mid |z_1|,\dots,|z_{d-1}| \le N\ ,\ |z_d| < CN^2 \\ \text{and}\ \ d(z_d - z_1^2 - \dots - z_{d-1}^2; Q) > D\} \\ \le C_{\tau,B}(D^{-B}\, Q^\tau\, N^2 + Q^B)N^{d-1}\ .\end{aligned} \tag{3.72}$$

Hence, combined with (3.71) and using the trivial estimate

$$\big|\widehat{f}(z_1,\dots,z_d)\big| \le \|f\|_1 \tag{3.73}$$

$$\begin{aligned}\|f * \Lambda_{Q,s}\|_2 \le \frac{Q\cdot D}{N2^s}\|f\|_2 + \frac{Q}{N2^s}\cdot Q\cdot C_{\tau,B}(D^{-B/2}Q^\tau N + Q^{B/2})\cdot \\ \cdot N^{\frac{d-1}{2}}\ \|\ f\ \|_1 + \frac{Q^2}{2^s N^{2-\varepsilon}}\|f\|_2\ .\end{aligned} \tag{3.74}$$

Take $M > 1$ and $D = MQ^\tau$ and assume

$$B > \frac{6}{\tau}\ \ \text{and}\ \ N > (MQ)^B \tag{3.75}$$

(3.74) then yield

$$\|f * \Lambda_{Q,s}\|_2 \le \frac{Q^{1+\tau}\cdot M}{N2^s}\|f\|_2 + C_{\tau,B}\ 2^{-s}\cdot M^{-B/2}\ N^{\frac{d-1}{2}}\|f\|_1\ . \tag{3.76}$$

Consider $p_0 = \frac{2(d+1)}{d+3}$ (which is the conjugate of the critical exponent) and interpolate between (3.69) and (3.70). Thus with $\frac{1}{p_0} = \frac{1-\theta_0}{1} + \frac{\theta_0}{2}$ $\left(\theta_0 = \frac{d-1}{d+1}\right)$

$$\|f * \Lambda_{Q,s}\|_{L^{p_0'}(\mathbf{T}^d)} \ll N^\varepsilon \|f\|_{L^{p_0}(\mathbf{T}^d)}\ . \tag{3.77}$$

If (3.75) holds, one may also write by (3.69), (3.76) for $\theta < \frac{d-1}{d+1+2\tau}$, $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$

$$\begin{aligned}\|f * \Lambda_{Q,s}\|_{p'} \le Q^{-\sigma}\ M^\theta (N2^s)^{\frac{d-1}{2}(1-\theta)-\theta}\|f\|_p + \\ + C_{\tau,B}Q^{-1/4}2^{s(\frac{d-1}{2}(1-\theta)-\theta)}M^{-B/6}\ N^{\frac{d-1}{2}}\|f\|_1\end{aligned} \tag{3.78}$$

$$\begin{aligned}\le Q^{-\sigma}M(N2^s)^{(d+1)(\frac{1}{p}-\frac{1}{p_0})}\|f\|_p + \\ + C_{\tau,B}\ Q^{-1/4}\ M^{-B/6}\ N^{\frac{d-1}{2}}\cdot 2^{s(d+1)(\frac{1}{p}-\frac{1}{p_0})}\|f\|_1\end{aligned} \tag{3.79}$$

where

$$\sigma = \sigma(\tau, p) > 0\ \ \text{for}\ \ p\ \text{ as above}\ .$$

Finally interpolation between (3.69) and (3.70) also yield for $p < p_0$

$$\|f * \Lambda_{Q,s}\|_{p'} \ll N^{\varepsilon} \left(\frac{N2^s}{Q}\right)^{(d+1)\left(\frac{1}{p}-\frac{1}{p_0}\right)} \|f\|_p \,. \tag{3.80}$$

We will use inequalities (3.77), (3.79), (3.80) in the proof of the following distributional properties for polynomials of the form

$$F(x,t) = \sum_{|n_1|,\dots,|n_{d-1}|\le N} a_{\overline{n}}\, e^{2\pi i(\langle x,\overline{n}\rangle + t|\overline{n}|^2)} \tag{3.81}$$

where $\overline{n} = (n_1,\dots,n_{d-1})$, $d \ge 2$. This statement is the main result of this section.

PROPOSITION 3.82. *Let $F$ be given by (3.81), $\|F\|_2 = 1$*

(i) $\operatorname{mes}\left\{(x,t) \in \mathbf{T}^d \mid |F(x,t)| > \lambda\right\} \ll N^{\varepsilon}\, \lambda^{-\frac{2(d+1)}{d-1}}$ *for* $\lambda > N^{\frac{d-1}{4}}$

(ii) $\operatorname{mes}\left\{(x,t) \in \mathbf{T}^d \mid \left|F(x,t)\right| > \lambda\right\} < C_q\, N^{\frac{d-1}{2}q-(d+1)}\, \lambda^{-q}$

*for* $\lambda > N^{\frac{d-1}{4}}$, $q > \frac{2(d+1)}{d-1}$.

*Proof*: Consider a set $\mathcal{E}$ of $\left(\frac{1}{N} \times \cdots \times \frac{1}{N} \times \frac{1}{N^2}\right)$ disjoint intervals in $\mathbf{T}^d$ exhausting the level set $[|F| > \lambda]$. Denote $f$ a $\pm 1$-valued function on the union of these $\mathcal{E}$-intervals such that

$$\left|\langle F, f\rangle\right| \lesssim \lambda \cdot |\mathcal{E}|\; N^{-(d+1)} \,. \tag{3.83}$$

Recall the definition (3.52) of $K$. Thus from definition (3.19) of the weights $\{\sigma_n\}$ one has

$$F = F * \left(\sqrt{K}\right)^{\wedge} \tag{3.84}$$

and by (3.83)

$$\begin{aligned} \lambda^2\, |\mathcal{E}|^2\, N^{-2(d+1)} &\le \left|\langle F, f * (\sqrt{K})^{\wedge}\rangle\right|^2 \le \left\|f * (\sqrt{K})^{\wedge}\right\|_2^2 = \\ &= \left\langle f, f * (\sqrt{K})^{\wedge} * (\sqrt{K})^{\wedge}\right\rangle = \langle f, f * \widehat{K}\rangle \,. \end{aligned} \tag{3.85}$$

Consider the representation (3.57), (3.58) and define

$$\Lambda = \sum_{\substack{Q<N_1\\ Q \text{ dyadic}}} \sum_{Q<2^s<N} \Lambda_{Q,s} \tag{3.86}$$

where $\Lambda_{Q,s}$ is given by (3.59). Writing

$$\widehat{K} = \Lambda + \left(\widehat{K} - \Lambda\right) \tag{3.87}$$

estimate (3.85) as

$$\left|\langle f, f * \Lambda\rangle\right| + \left|\langle f, f * (\widehat{K} - \Lambda)\rangle\right| \tag{3.88}$$

$$\leq \|f\|_p \, \|f * \Lambda\|_{p'} + \|f\|_1^2 \, \|\widehat{K} - \Lambda\|_\infty \, . \tag{3.89}$$

Taking $p = p_0$ and summing inequalities (3.77) over $Q$ and $s$, the first term of (3.89) is bounded by

$$\|f\|_{p_0} \sum_{Q,s} \|f * \Lambda_{Q,s}\|_{p_0'} \ll N^\varepsilon \|f\|_{p_0}^2 = N^\varepsilon \big[|\mathcal{E}| \; N^{-(d+1)}\big]^{\frac{2}{p_0}} \, . \tag{3.90}$$

The expression $\widehat{K} - \Lambda$ corresponds with (3.58), which by (3.32), (3.53), (3.56) is bounded by

$$\|\widehat{K} - \Lambda\|_\infty \leq cN^{\frac{d-1}{2}} \, . \tag{3.91}$$

The second term of (3.89) is therefore bounded by

$$CN^{\frac{d-1}{2}} \big[|\mathcal{E}| \cdot N^{-(d+1)}\big]^2 \, . \tag{3.92}$$

Collecting inequalities, it follows

$$\lambda^2 \cdot |\mathcal{E}|^2 \cdot N^{-2(d+1)} \ll N^\varepsilon \; |\mathcal{E}|^{\frac{2}{p_0}} \; N^{-(d+1)\frac{2}{p_0}} + |\mathcal{E}|^2 \; N^{-\frac{3d+5}{2}} \tag{3.93}$$

from (3.93), it follows immediately that if we assume $\lambda > CN^{\frac{d-1}{4}}$, then

$$|\mathcal{E}| \ll N^\varepsilon \; N^{d+1} \; \lambda^{-\frac{2(d+1)}{d-1}} . \tag{3.94}$$

This proves (3.82), (i).

The proof of (ii) is a bit more delicate. It is clear from (i) that if $q > \frac{2(d+1)}{d-1}$ is fixed, then (ii) will hold unless $\lambda$ is "large", i.e.

$$\lambda > N^{\frac{d-1}{2} - \varepsilon} \, . \tag{3.95}$$

Fix constants $\tau$, $B > \frac{6}{\tau}$ and consider $Q$ satisfying (3.75), i.e.

$$N > (MQ)^B \, . \tag{3.96}$$

Take $p$ as in inequality (3.79). Summing (3.79) over $s$ and dyadic $Q$ in the range (3.96), we get

$$\begin{aligned} \|f * \Lambda_1\|_{p'} \leq C_{p,\tau} M N^{2(d+1)(\frac{1}{p} - \frac{1}{p_0})} \|f\|_p + \\ + C_{\tau,B} \; M^{-B/6} \; N^{\frac{d-1}{2} + (d+1)(\frac{1}{p} - \frac{1}{p_0})} \|f\|_1 \end{aligned} \tag{3.97}$$

denoting

$$\Lambda_1 = \sum_{\substack{Q \text{ dyadic} \\ Q < Q_1}} \sum_{Q \leq 2^s \leq N} \Lambda_{Q,s} \tag{3.98}$$

and $Q_1$ the largest $Q$-value satisfying (3.96).

For the values $Q > Q_1$, use (3.80) with same $p$. Hence

$$\left\| f * (\Lambda - \Lambda_1) \right\|_{p'} \ll N^{\varepsilon} \left( \frac{N^2}{Q_1} \right)^{(d+1)(\frac{1}{p} - \frac{1}{p_0})} \|f\|_p \tag{3.99}$$

(3.97), (3.99) give an estimate on $\|f * \Lambda\|_{p'}$ and proceeding as before with $p$ instead of $p_0$, (3.89) yields

$$\lambda^2 \, |\mathcal{E}|^2 \, N^{-2(d+1)} \ll C_{p,\tau} \, N^{2(d+1)(\frac{1}{p} - \frac{1}{p_0})} \left( M + \frac{N^{\varepsilon}}{Q_1^{(d+1)(\frac{1}{p} - \frac{1}{p_0})}} \right) \cdot$$

$$\cdot \left[ |\mathcal{E}| N^{-(d+1)} \right]^{\frac{2}{p}} \tag{3.100}$$

$$+ C_{\tau,B} \, M^{-B/6} \, N^{\frac{d-1}{2} + (d+1)(\frac{1}{p} - \frac{1}{p_0})} \left[ |\mathcal{E}| \, N^{-(d+1)} \right]^{1 + \frac{1}{p}} \tag{3.101}$$

$$+ C N^{\frac{d-1}{2}} \left[ |\mathcal{E}| \, N^{-(d+1)} \right]^2 . \tag{3.102}$$

For $\lambda > C N^{\frac{d-1}{4}}$, the last term (3.102) may be dropped.

Assume $Q_1 = N^{\delta}$ where $\delta > 0$ has to satisfy (3.96)

$$(M N^{\delta})^B < N . \tag{3.103}$$

Also $(d+1)(\frac{1}{p} - \frac{1}{p_0}) > \sigma(\tau) > 0$. Hence

$$(3.100) < C_{p,\tau} \, N^{2(d+1)(\frac{1}{p} - \frac{1}{p_0})} \cdot M \left[ |\mathcal{E}| \cdot N^{-(d+1)} \right]^{\frac{2}{p}} \tag{3.104}$$

and it follows that

$$|\mathcal{E}| < C_{p,\tau} \, M^{1/2} \, N^{\frac{d-1}{2} p'} \, \lambda^{-p'} + C_{\tau,B} \, M^{-Bp'/6} \, N^{(d-1)p'} \, \lambda^{-2p'} . \tag{3.105}$$

Choose

$$M = \left[ \frac{N^{\frac{d-1}{2}}}{\lambda} \right]^{\tau} \tag{3.106}$$

and

$$B = \frac{12}{\tau} . \tag{3.107}$$

Thus (3.105) yields

$$|\mathcal{E}| < C_{p,\tau} \left[\frac{N^{\frac{d-1}{2}}}{\lambda}\right]^{p'+\tau} . \tag{3.108}$$

(3.103) becomes because of the assumption (3.95) on $\lambda$ and (3.107)

$$(N^{\varepsilon} \cdot N^{\delta})^{\frac{12}{\tau}} < N \tag{3.109}$$

(for $\varepsilon$ choosen arbitrarily small).

Assume $\tau > 0$ given, $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$, $\theta < \frac{d-1}{d+1+2\tau}$, let $\delta = \frac{\tau}{20}$ so that (3.109) holds. The value $q = p' + \tau$ satisfies (3.82, ii) by (3.108) and may clearly be taken any exponent $> \frac{2(d+1)}{d-1}$.

This completes the proof of (3.82).

Coming back to the statement (3.5)

PROPOSITION 3.110. *For $d \geq 5$ and $p \geq \frac{2(d+3)}{d-1}$*

$$K_p(S_{d,N}) < CN^{\frac{d-1}{2} - \frac{d+1}{p}} .$$

*Proof*: Write with $F$ given by (3.81)

$$\int_{\mathbf{T}^d} |F|^p \sim \int \lambda^{p-1} \operatorname{mes}\left[|F| > \lambda\right] \geq \int_{\lambda < N^{\frac{d-1}{4}}} + \int_{\lambda > N^{\frac{d-1}{4}}}$$
$$\lesssim N^{\frac{d-1}{4}(p-2)} \int |F|^2 + N^{\frac{d-1}{2}q-(d+1)} \left( \int_{N^{\frac{d-1}{4}}}^{N^{\frac{d-1}{2}}} \lambda^{p-1}\, \lambda^{-q}\, d\lambda \right) \tag{3.111}$$

choosing $\frac{2(d+1)}{d-1} < q < p$ and applying (3.82, ii). Thus

$$\int |F|^p \lesssim N^{\frac{d-1}{4}(p-2)} + N^{\frac{d-1}{2}p-(d+1)} \tag{3.112}$$

and the first term is at most the second by assumption on $p$.

PROPOSITION 3.113. *Assume $p_2 > p_1 \geq p_0 = \frac{2(d+1)}{d-1}$ and $K_{p_1}(S_{d,N}) \ll N^{\frac{d-1}{2} - \frac{d+1}{p_1} + \varepsilon}$. Then $K_{p_2}(S_{d,N}) \leq C_{p_2}\, N^{\frac{d-1}{2} - \frac{d+1}{p_2}}$.*

*Proof*: Write again

$$\int |F|^{p_2} \sim \int \lambda^{p_2-1} \operatorname{mes}\left[|F| > \lambda\right] .$$

From the assumption on $K_{p_1}$ only $\lambda > N^{\frac{d-1}{2}-\varepsilon}$ has to be considered. In this case (3.82), (ii) holds and we conclude similarly as above, letting $p_0 < q < p_2$.

As a corollary of (3.113) and (2.35), (3.8), we get

PROPOSITION 3.114.

$$K_p(S_{2,N}) = K_p(S_N) < C_p\, N^{\frac{1}{2}-\frac{3}{p}} \quad \text{for} \quad p > 6 \tag{3.115}$$

$$K_p(S_{3,N}) < C_p\, N^{1-\frac{4}{p}} \quad \text{for} \quad p > 4 \tag{3.116}$$

$$K_p(S_{4,N}) < C_p\, N^{\frac{3}{2}-\frac{5}{p}} \quad \text{for} \quad p > 4 \tag{3.117}$$

*Remark*: From (3.110), it follows that

$$K_4(S_{5,N}) < CN^{1/2} \tag{3.118}$$

improving on (3.9). Observe that the arithmetic approach amounts to counting the number of representations of an integer $n$ as sum of 4 squares, which may be at least $cn \log\log n \gg n$ (see [Gr], p. 121).

## 4. Proof of the 1-Dimensional $L^2$-Theorem

Consider the NLSE

$$\Delta u + i\partial_t u + u|u|^\alpha = 0 \qquad \alpha \equiv p-2 \ , \ p > 2 \tag{4.1}$$

with initial condition

$$u(x,0) = \phi(x) \tag{4.2}$$

$$u \text{ is periodic in } x\text{-variable}$$

and the equivalent integral equation (setting $w = u|u|^\alpha$)

$$u(\cdot,t) = U(t)\phi + i\int_0^t U(t-\tau)\, w(\cdot,\tau)d\tau \ \ ; \ \ U(t) = e^{it\Delta} . \tag{4.3}$$

We will seek for a solution of (4.3) local in time, i.e. consider a function $0 \leq \psi_1 \leq 1$, $\psi_1 = 1$ on a neighborhood $[-\delta, \delta]$ of 0 and $\operatorname{supp} \psi_1 \subset [-2\delta, 2\delta]$. Using a fixpoint argument, we will construct a function $u$ satisfying

$$u(\cdot, t) = \psi_1(t)\ U(t)\phi + i\psi_1(t) \int_0^t U(t-\tau)\ w(\cdot, \tau) d\tau\ . \tag{4.4}$$

Write $\phi, u, w$ as Fourier series

$$\phi(x) = \sum_{\xi \in \mathbb{Z}^{d-1}} \widehat{\phi}(\xi)\ e^{2\pi i \langle x, \xi\rangle} \tag{4.5}$$

$$u(x,t) = \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i \langle x, \xi\rangle} \int_{-\infty}^{\infty} e^{2\pi i \lambda t}\ \widehat{u}(\xi, \lambda) d\lambda \tag{4.6}$$

$$w(x,t) = \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i \langle x, \xi\rangle} \int_{-\infty}^{\infty} e^{2\pi i \lambda t}\ \widehat{w}(\xi, \lambda) d\lambda \tag{4.7}$$

(4.4) then becomes

$$\begin{aligned} u(x,t) = \psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} \widehat{\phi}(\xi) e^{2\pi i(\langle x, \xi\rangle + t|\xi|^2)} + \\ + \frac{1}{2\pi}\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i(\langle x, \xi\rangle + t|\xi|^2)} \int_{-\infty}^{\infty} \frac{2^{2\pi i(\lambda - |\xi|^2)t} - 1}{\lambda - |\xi|^2}\ \widehat{w}(\xi, \lambda) d\lambda\ . \end{aligned} \tag{4.8}$$

In the integral appearing in (4.9), we distinguish $\int_{|\lambda - |\xi|^2| < B}$ and $\int_{|\lambda - |\xi|^2| > B}$.

Thus consider another cutoff function $\psi_2$, $0 \leq \psi_2 \leq 1$, $\psi_2 = 1$ on $[-B, B]$ and $\operatorname{supp} \psi_2 \subset [-2B, 2B]$. Assume

$$B < \frac{1}{100\delta}\ . \tag{4.9}$$

Write

$$\psi_1(t) \int_{-\infty}^{\infty} \frac{e^{2\pi i(\lambda - |\xi|^2)t} - 1}{\lambda - |\xi|^2}\ \widehat{w}(\xi, \lambda) d\lambda =$$

$$\sum_{k \geq 1} \frac{(2\pi i)^k}{k!}\ \psi_1(t) t^k \int \psi_2(\lambda - |\xi|^2) \big(\lambda - |\xi|^2\big)^{k-1}\ \widehat{w}(\xi, \lambda) d\lambda \tag{4.10}$$

$$+ \psi_1(t) \int (1 - \psi_2)(\lambda - |\xi|^2) \frac{e^{2\pi i(\lambda - |\xi|^2)t}}{\lambda - |\xi|^2}\ \widehat{w}(\xi, \lambda) d\lambda \tag{4.11}$$

$$-\ \psi_1(t) \int (1 - \psi_2)(\lambda - |\xi|^2) \frac{\widehat{w}(\xi, \lambda)}{\lambda - |\xi|^2}\ d\lambda\ . \tag{4.12}$$

Thus in order to control the right member of (4.8), we have to consider following contributions

$$\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} \widehat{\phi}(\xi)\ e^{2\pi i(\langle x,\xi\rangle + t|\xi|^2)} \tag{4.13}$$

$$\frac{1}{2B} \sum_{k \geq 1} \frac{(2\pi i)^k}{k!}\ (2Bt)^k\ \psi_1(t)$$
$$\left\{ \sum_{\xi} \left[ \int \psi_2(\lambda - |\xi|^2) \left( \frac{\lambda - |\xi|^2}{2B} \right)^{k-1} \widehat{w}(\xi,\lambda) d\lambda \right] e^{2\pi i(\langle x,\xi\rangle + t|\xi|^2)} \right\} \tag{4.14}$$

$$\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i\langle x,\xi\rangle} \int \frac{(1-\psi_2)(\lambda - |\xi|^2)}{\lambda - |\xi|^2}\ e^{2\pi i\lambda t}\ \widehat{w}(\xi,\lambda) d\lambda \tag{4.15}$$

$$\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i(\langle x,\xi\rangle + t|\xi|^2)} \int \frac{(1-\psi_2)(\lambda - |\xi|^2)}{\lambda - |\xi|^2}\ \widehat{w}(\xi,\lambda) d\lambda\ . \tag{4.16}$$

Consider the one-dimensional case, i.e. $d = 2$ and $\alpha \leq 2$ in (4.1).

CLAIM. *Given sufficiently large $M$, the map naturally defined by (4.8) is a contraction of the ball of radius $M$ in $L^4(\mathbb{T} \times [-1,1])$ into itself, provided $\delta, B$ are well choosen.*

The estimates needed are provided by the $L^4$-estimates of section 2. Because they are local in $t$, the inequalities for $\mathbb{T}^2$ are applicable here.

(i) (4.13) becomes

$$\psi_1(t) \sum_{n \in \mathbb{Z}} \widehat{\phi}(n)\ e^{2\pi i(nx + n^2 t)} \tag{4.17}$$

which $L^4(dxdt)$-norm is bounded by

$$c\Big( \sum |\widehat{\phi}(n)|^2 \Big)^{1/2} = c\|\phi\|_2 \tag{4.18}$$

invoking (2.2).

(ii) Because of (4.9), the $L^4$-norm of (4.14) is bounded by

$$c\delta \sup_{k \geq 1} \left\| \sum_n \left[ \int \psi_2(\lambda - n^2) \left( \frac{\lambda - n^2}{2B} \right)^{k-1} \widehat{w}(n,\lambda) d\lambda \right] e^{2\pi i(nx + n^2 t)} \right\|_4 \tag{4.19}$$

and again by (2.2),

$$c\delta \sup_{k\geq 1} \left( \sum_n \left| \int \psi_2(\lambda - n^2) \left( \frac{\lambda - n^2}{2B} \right)^{k-1} \widehat{w}(n,\lambda) d\lambda \right|^2 \right)^{1/2} . \tag{4.20}$$

From the definition of $\psi_2$ and (2.5), (4.20) may be estimated by

$$c\delta B \|w\|_{L^{4/3}(dxdt)} = c\delta B \left[ \int |u|^{(1+\alpha)\frac{4}{3}} \right]^{3/4} \leq c\delta B \|u\|_4^{1+\alpha} \tag{4.21}$$

assuming $\alpha \leq 2$.

**(iii)** (4.15) becomes

$$\psi_1(t) \sum_{n\in\mathbb{Z}} e^{2\pi i n x} \int \frac{(1-\psi_2)(\lambda - n^2)}{\lambda - n^2} e^{2\pi i \lambda t} \widehat{w}(n,\lambda) d\lambda . \tag{4.22}$$

We use here Proposition 2.31. Thus the $(n,\lambda)$-multiplier is given by $\frac{(1-\psi_2)(\lambda-n^2)}{\lambda-n^2}$ which is clearly bounded by $B^{-1/4}(1+|\lambda-n^2|)^{-3/4}$. Hence the $L^4$-norm of (4.22) is at most

$$CB^{-1/4}\|w\|_{4/3} \leq CB^{-1/4}\|u\|_4^{1+\alpha} . \tag{4.23}$$

**(iv)** (4.16) becomes

$$\psi_1(t) \sum_{n\in\mathbb{Z}} e^{2\pi i(nx+n^2t)} \int \frac{(1-\psi_2)(\lambda - n^2)}{\lambda - n^2} \widehat{w}(n,\lambda) d\lambda \tag{4.24}$$

which $L^4$-norm is bounded by

$$c\left( \sum_n \left| \int \frac{(1-\psi_2)(\lambda - n^2)}{\lambda - n^2} \widehat{w}(n,\lambda) d\lambda \right|^2 \right)^{1/2} \tag{4.25}$$

by (2.2). Estimating again $\frac{(1-\psi_2)(\lambda-n^2)}{\lambda-n^2}$ by $B^{-1/4}(1+|\lambda-n^2|)^{-3/4}$ from definition of $\psi_2$, (2.30) yields the bound

$$CB^{-1/4}\|w\|_{4/3} \leq CB^{-1/4}\|u\|_4^{1+\alpha} . \tag{4.26}$$

Denote $T$ the transformation defined by (4.9), i.e.

$$\begin{aligned}(Tu)(x,t) = \psi_1(t) \sum_{n\in\mathbb{Z}} \widehat{\phi}(n) e^{2\pi i(nx+n^2t)} + \\ +\frac{1}{2\pi} \psi_1(t) \sum_{n\in\mathbb{Z}} e^{2\pi i(nx+n^2t)} \int_{-\infty}^{\infty} \frac{e^{2\pi i(\lambda-n^2)t} - 1}{\lambda - n^2} (u|u|^\alpha)^\wedge(n,\lambda) d\lambda .\end{aligned} \tag{4.27}$$

We proved that

$$\|Tu\|_4 \le c_1\big\{\|\phi\|_2 + \delta B\|u\|_4^{1+\alpha} + B^{-1/4}\|u\|_4^{1+\alpha}\big\} \ . \tag{4.28}$$

Hence

$$\|u\|_4 \le M \Rightarrow \|Tu\|_4 \le M \tag{4.29}$$

provided

$$C_1\big(\|\phi\|_2 + \delta BM^{1+\alpha} + B^{-1/4}\ M^{1+\alpha}\big) < M \ . \tag{4.30}$$

If $M$ is sufficiently large, (4.30) may clearly be achieved for suitable $B$, $\delta$. Consider next $Tu - Tv$. The first term in (4.27) disappears and in the second, $u|u|^\alpha$ has to be replaced by $w = u|u|^\alpha - v|v|^\alpha$. Since now by Hölder's inequality

$$\|w\|_{4/3} \le c\big(\|u\|_4 + \|v\|_4\big)^\alpha \|u-v\|_4 \tag{4.31}$$

($\alpha \le 2$), the contribution of (ii), (iii), (iv) above is estimated by

$$\begin{aligned}\|Tu - Tv\|_4 &\le c_1(\delta B + B^{-1/4})\big(\|u\|_4 + \|v\|_4\big)^\alpha \|u-v\|_4 \le \\ &\le 2c_1 M^\alpha(\delta B + B^{-1/4})\|u-v\|_4 \ .\end{aligned} \tag{4.32}$$

Hence, for suitable $B$, $\delta$

$$\|Tu - Tv\|_4 \le \tfrac{1}{2}\|u-v\|_4 \ . \tag{4.33}$$

This establishes the claim made above. Picard's theorem yields a function $u \in L^4(\mathbf{T} \times [-1,1])$ satisfying $Tu = u$, hence (4.4). Moreover this solution is unique and persistent. At this point, we proved local well-posedness (in generalized sense) of (4.1) for $n = 1$ ($d = 2$) and $\alpha \le 2$ ($p \le 4$).

To derive the global result from previous fact and the $L^2$-conservation law

$$\int_{\mathbf{T}} \big|u(x,t)\big|^2 dx \tag{4.34}$$

is a routine procedure.

In the previous construction of local solutions, one may consider norms of the form

$$|||u||| = \|u\|_{L^4(dxdt)} + \sum_{k\ge 1}^{s} \rho_k \|\partial_x^{(k)} u\|_{L^4(dxdt)} \tag{4.35}$$

where $\rho_k \ge 0$ are some weight, assuming the data $\phi \in H^(\mathbf{T})$, i.e.

$$\sum_{k=0}^{s} \|\phi^{(k)}\|_{L^2(\mathbf{T})} < \infty \ . \tag{4.36}$$

In performing the estimates (ii), (iii), (iv) above for $x$-derivatives, one will have to replace the function $w = u|u|^\alpha$ or $w = u|u|^\alpha - v|v|^\alpha$ by $\partial_x^{(k)} w$. Assume here

$$s \leq 1 + \alpha \quad \text{or} \quad \alpha = 2 \,. \tag{4.37}$$

Since one may bound for $w = u|u|^\alpha - v|v|^\alpha$

$$\begin{gathered} \|\partial_x^{(k)} w\|_{4/3} \leq \\ c\Big\{ \big\|\partial_x^{(k)}(u-v)\big\|_4 \big(\|u\|_4 + \|v\|_4\big)^\alpha + \\ \|u-v\|_4 \big(\|\partial_x^{(k)} u\|_4 + \|\partial_x^{(k)} v\|_4\big) \big(\|u\|_4 + \|v\|_4\big)^{\alpha-1} \Big\} \end{gathered} \tag{4.38}$$

we get

$$|||Tu - Tv||| \leq c(\delta B + B^{-1/4}) \Big\{ \|u-v\|_4 \big(\|u\|_4 + \|v\|_4\big)^\alpha + \sum_{k=1}^{s} \rho_k (4.38) \Big\} \tag{4.39}$$

$$\begin{aligned} \leq c(\delta B + B^{-1/4}) \Big\{ &\big(\|u\|_4 + \|v\|_4\big)^\alpha |||u-v||| + \\ &\big(\|u\|_4 + \|v\|_4\big)^{\alpha-1} \big(|||u||| + |||v|||\big) \|u - v\|_4 \Big\} \end{aligned} \tag{4.40}$$

$$\leq c(\delta B + B^{-1/4}) \big(|||u||| + |||v|||\big)^\alpha |||u - v||| \,. \tag{4.41}$$

Start by fixing $M$ such that

$$\|\phi\|_2 + \sum_{k=1}^{\overline{k}} \rho_k \|\phi\|_{H^k} < M \tag{4.42}$$

and consider the set of functions on $\mathbf{T} \times [-1, 1]$ satisfying

$$|||u||| < 10M \,. \tag{4.43}$$

By letting $B$ and $\delta$ satisfy

$$(\delta B + B^{-1/4}) M^{\alpha-1} < c \tag{4.44}$$

($c$ is sufficiently small constant).

(4.41) implies that $T$ is a contraction on (4.43) and hence the fixpoint argument applies to get a local unique solution $u$ of (4.3) of bounded $|||\ \ |||$-norm. Observe at this point that the time interval $\delta$ is independent of the weights $\rho_k$ in (4.35). This observation permits in particular the carrying out of a regularization procedure on the initial data $\phi$ in order to justify the conservation law (4.34) for the "generalized NLSE". This requires taking $\overline{k} \geq 2$, assuming $\alpha \geq 1$.

The conclusion is the following.

**THEOREM 4.45.** *The generalized periodic one-dimensional NLSE* (4.1) *with $p \leq 4$ is globally well-posed in the space $L^4(\mathbf{T} \times \mathbf{R}_{\text{loc}})$ for $L^2$-data. For $p = 4$ and data $\phi \in H^s(\mathbf{T})$ ($s$ integer) this solution $u$ will satisfy moreover*

$$\partial_x^k u \in L^4(\mathbf{T} \times \mathbf{R}_{\text{loc}}) \quad k \leq s \, . \tag{4.46}$$

*Remarks*: (i) In the previous statement, one has in fact a uniform estimate

$$\sup_{t \in \mathbf{R}} \int_{\mathbf{T}} \int_t^{t+1} \left|u(x, t')\right|^4 dt' \, . \tag{4.47}$$

Statement (4.46) is understood as

$$\int_{\mathbf{T}} \int_0^T \left|\partial_x^{(k)} u(x,t)\right|^4 dt < \infty \text{ for all } T \tag{4.48}$$

(the stronger uniform statement in the form (4.47) is valid also but its proof requires the use of higher order conservation laws for $\alpha = 2$, see [ZS]).

(ii) Some comments on the derivation of (4.38). For $k = 1$, the argument is straightforward. The case $k > 1$ is based on following inequality.

LEMMA 4.49. *Let $s = s_1 + s_2 + \cdots + s_m$ be integers and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $1 < p_\ell < \infty$. Let $\int_{\mathbf{T}} f_\ell = 0$. Then*

$$\left\| \prod_{\ell=1}^m f_\ell^{(s_\ell)} \right\|_p \leq c \sum_{\ell=1}^m \prod_{\ell' \neq \ell} \|f_{\ell'}\|_{p_{\ell'}} \cdot \|f_\ell^{(s)}\|_{p_\ell} \, . \tag{4.50}$$

*Proof*: Take $m = 2$. The general argument is similar. Write $f = \sum_j Q_j(f)$ for the Littlewood-Paley decomposition of $f$. Denote $f^*$ the usual Hardy-Littlewood maximal function of $f$. One has

$$Q_j(f)^{(s)} \leq c \, 2^{js} \, Q_j(f)^* \sim Q_j\left(f^{(s)}\right)^* \, . \tag{4.51}$$

Write

$$|f_1^{(s_1)} f_2^{(s_2)}| \lesssim \sum_{j_1 \geq j_2} 2^{j_1 s_1} \, Q_{j_1}(f_1)^* \, 2^{j_2 s_2} \, Q_{j_2}(f_2)^* + \sum_{j_1 \leq j_2} \, . \tag{4.52}$$

Consider the first term in (4.52) (second term is similar). One gets by (4.51)

$$\sum_{d \geq 0} 2^{-ds_2} \sum_{j_1} 2^{j_1 s} \, Q_{j_1}(f_1)^* \, Q_{j_1 - d}(f_2)^* \leq$$
$$\leq \sum_{d \geq 0} 2^{-ds_2} \Big( \sum_j [Q_j(f_1^{(s)})^*]^2 \Big)^{1/2} \Big( \sum_j [Q_j(f_2)^*]^2 \Big)^{1/2} . \tag{4.53}$$

We assume $s_2 > 0$ (otherwise the statement is obvious). Estimate (4.53) by Hölder's inequality

$$\sum_{d\geq 0} 2^{-ds_2} \Big\| \Big( \sum_j [Q_j(f_1^{(s)})^*]^2 \Big)^{1/2} \Big\|_{p_1} \Big| \Big( \sum_j [Q_j(f_2)^*]^2 \Big)^{1/2} \Big|_{p_2} . \tag{4.54}$$

The proof is then completed by invoking following standard facts

$$\Big\| \Big( \sum_j [Q_j(f)^*]^2 \Big)^{1/2} \Big\|_p \leq$$

$$c_p \Big\| \Big( \sum_j [Q_j\, f]^2 \Big)^{1/2} \Big\|_p \sim \tag{4.55}$$

$$\|f\|_p \tag{4.56}$$

for $1 < p < \infty$.

Similar estimates may be found in the appendix of [KePoVe2].

## 5. Estimates in $H^s$, $s > 0$

In this section we describe a method to obtain (local) solutions for the generalized NLSE (4.1) with $H^s$-initial data. The approach is the same as in previous section and we need to introduce appropriate function spaces to perform the iteration (they will replace the space $L^4(dxdt)$ and are in fact defined from Fourier transform properties). The exponent $s$ may be fractional here. There will be no significant difference between the one and higher dimensional situation as far as the method is concerned. We consider the equation $\Delta u + i\partial_t u + u|u|^\alpha = 0$ with $\alpha \geq 2$. Call an exponent $p$ "admissible" if (if section 3)

$$p \geq \frac{2(d+1)}{d-1} \quad \text{and} \quad K_p(S_{d,N}) \ll N^\varepsilon N^{\frac{d-1}{2} - \frac{d+1}{p}} . \tag{5.1}$$

The dependence of $K_p(S_{d,N})$ on $N$ is the reason for most of the complications in what follows.

Consider the sets ($K$, $N$ positive integers)

$$\Lambda_{K,N} = \big\{ \zeta = (\xi, \lambda) \in \mathbf{Z}^n \times \mathbf{R} \;\big|\; N \leq |\xi| < 2N \quad \text{and} \quad K \leq \big|\lambda - |\xi|^2\big| < 2K \big\} . \tag{5.2}$$

If $I$ is an interval in $\mathbb{Z}^n$, let

$$\Lambda_{K,I} = \left\{\zeta \in I \times \mathbf{R} \,\middle|\, K \leq \left|\lambda - |\xi|^2\right| < 2K\right\} . \tag{5.3}$$

For a function

$$u = \sum_{\xi \in \mathbb{Z}^n} \int d\lambda \; \widehat{u}(\zeta) \; e^{2\pi i(\langle \xi, x\rangle + \lambda t)} \tag{5.4}$$

in $L^2(\mathbf{T}^n \times \mathbf{R})$, define

$$|||u||| = \sup_{K,N} (K+1)^{1/2} \; (N+1)^s \left(\int_{\Lambda_{K,N}} \left|\widehat{u}(\zeta)\right|^2 d\zeta\right)^{1/2} . \tag{5.5}$$

Fixing an interval $[-\delta, \delta]$, we will in fact consider the restriction norm $|||u||| \equiv \inf |||\widetilde{u}|||$ where the infimum is taken over all function $\widetilde{u}$ coinciding with $u$ on $\mathbf{T}^n \times [-\delta, \delta]$ (in order to avoid technical difficulties). Thus time restriction acts as a contraction wrt this norm.

Let $p_0$ be admissible. Then , by (3.113), for $p > p_0$

$$\left\| \sum_{|\xi| \leq N} a_\xi \; e^{i(\langle x,\xi\rangle + t|\xi|^2)} \right\|_{L^p(\mathbf{T}^d)} \leq N^{\frac{d-1}{2} - \frac{d+1}{p}} \left(\sum |a_\xi|^2\right)^{1/2} . \tag{5.6}$$

Let $I$ be a $(d-1)$-interval (or ball) of size $N$ in $\mathbb{Z}^{d-1}$, centered at $\xi_0$. Write

$$\langle x, \xi\rangle + t|\xi|^2 = \langle x, \xi_0\rangle + t|\xi_0|^2 + \langle x + 2t\,\xi_0, \xi - \xi_0\rangle + t|\xi - \xi_0|^2 . \tag{5.7}$$

A change of variable $x' = x + 2t\,\xi_0$, $t' = t$ immediately yields that also

$$\left\| \sum_{\xi \in I} a_\xi \; e^{i(\langle x,\xi\rangle + t|\xi|^2)} \right\|_p \lesssim N^{\frac{d-1}{2} - \frac{d+1}{p}} \left(\sum_{\xi \in I} |a_\xi|^2\right)^{1/2} . \tag{5.8}$$

It follows from (5.8), (5.3) and triangular inequality, writing $\lambda = |\xi|^2 + k$, $|k| < K$, that the map

$$L^2_{\Lambda_{K,I}} \longrightarrow L^p(\mathbf{T}^n \times \mathbf{R}_{\text{loc}}) : \{a_\zeta\}_{\zeta \in \Lambda_{K,I}} \longrightarrow \int_{\Lambda_{K,I}} a_\zeta \; e^{2\pi i(\langle x,\xi\rangle + t\lambda)} \, d\zeta \tag{5.9}$$

has norm bounded by $K^{1/2} N^{\frac{d-1}{2} - \frac{d+1}{p}}$.

Obviously the map (5.9) from $L^2_{\Lambda_{K,I}}$ to $L^2(\mathbf{T}^n \times \mathbf{R}_{\text{loc}})$ is of bounded norm. Interpolation between these last two facts implies

LEMMA 5.10. *Let $p_1 > p_0$, $p_1 > p_2 > 2$, $\frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{2}$. Then the map (5.9) ranging into $L^{p_2}(\mathbf{T}^n \times \mathbf{R}_{\rm loc})$ has norm bounded by*

$$K^{\frac{1}{2}(1-\theta)}\, N^{(\frac{d-1}{2} - \frac{d+1}{p_1})(1-\theta)}\ . \tag{5.11}$$

It is our aim to prove the existence of a solution of (4.5) by performing a fixpoint argument with respect to the norm $|||\ \ |||$. We first show how to control the expressions (4.13), (4.14), (4.15), (4.16) in $|||\ \ |||$ by $|||u|||$. Minor modifications will yield the contractive property, assuming $\alpha \geq 2$.

We introduce some further notation.

For dyadic $M$, define

$$u_M = \sum_{|\xi| \leq M} e^{2\pi i \langle \xi, x\rangle} \int \widehat{u}(\xi, \lambda)\, e^{2\pi i \lambda t}\, d\lambda \tag{5.12}$$

$$\Delta_M u = u_M - u_{\frac{M}{2}}\ . \tag{5.13}$$

If $I$ is an interval in $\mathbf{Z}^n$, let

$$\Delta_I u = \sum_{\xi \in I} e^{2\pi i \langle \xi, x\rangle} \int \widehat{u}(\xi, \lambda)\, e^{2\pi i \lambda t}\, d\lambda \tag{5.14}$$

$$= \sum_{K \text{dyadic}} \int_{\Lambda_{K,I}} \widehat{u}(\zeta)\, e^{2\pi i (\langle \xi, x\rangle + \lambda t)}\, d\zeta\ . \tag{5.15}$$

The control of (4.13) in $|||\ \ |||$-norm is clear, assuming $\phi \in H^s(\mathbf{T}^n)$ ($n = d-1$). The $\psi_1$ multiplication is harmless. Observe that even letting $\psi_1 = \chi_{[-\delta,\delta]}$, the condition

$$\Big(\sum_{|k| \sim K} \big|\widehat{\psi}_1(k)\big|^2\Big)^{1/2} \lesssim K^{-1/2} \tag{5.16}$$

is fulfilled. But it is convenient to consider cutoff functions with more smoothness properties in order to get better localization properties of the Fourier transform.

To evaluate (4.14), (4.15), (4.16) one mainly need an estimate on

$$K^{-1/2}\, N^s \Big(\int_{\Lambda_{K,N}} |\widehat{w}(\zeta)|^2 d\zeta\Big)^{1/2}\ . \tag{5.17}$$

Write with notation (5.12)

$$w = u|u|^\alpha = \sum \big(u_M |u_M|^\alpha - u_{\frac{M}{2}} |u_{\frac{M}{2}}|^\alpha\big)\ . \tag{5.18}$$

In the analysis of (5.17), one gets contributions of (5.18) terms for

$$M \geq N \,.^{(*)} \tag{5.19}$$

Since $\alpha \geq 1$, one may write for complex $z$, $w$

$$z|z|^{\alpha} - w|w|^{\alpha} = (z-w)\ \phi_1(z,w) + (\overline{z} - \overline{w})\ \phi_2(z,w) \tag{5.20}$$

where $\phi_1$, $\phi_2$ satisfy

$$\big|\nabla\phi_i(z,w)\big| \leq C\big(|z| + |w|\big)^{\alpha-1} \,. \tag{5.21}$$

Substitute in (5.20)

$$z = u_M \quad , \quad w = u_{\frac{M}{2}}$$

to get

$$u_M|u_M|^{\alpha} - u_{\frac{M}{2}}|u_{\frac{M}{2}}|^{\alpha} = \Delta_M u \cdot \phi_1(u_M, u_{\frac{M}{2}}) + \overline{\Delta_M u} \cdot \phi_2(u_M, u_{\frac{M}{2}}) \,. \tag{5.22}$$

The estimates of both terms in (5.22) are identical and we only consider the first.

Defining

$$v_M = \phi_1(u_M, u_{\frac{M}{2}}) \tag{5.23}$$

write again

$$v_M = (v_M - v_{\frac{M}{2}}) + (v_{\frac{M}{2}} - v_{\frac{M}{4}}) + \cdots = \sum_{\substack{M_1 \text{ dyadic} \\ M_1 < M}} (v_{M_1} - v_{\frac{M_1}{2}}) \,. \tag{5.24}$$

Since $\phi_1$ is Lipschitz, one has by (5.21)

$$\begin{aligned} v_{M_1} - v_{\frac{M_1}{2}} &= \Delta_{M_1} u \cdot \psi_1(u_{M_1}, u_{\frac{M_1}{2}}, u_{\frac{M_1}{4}}) \\ +\overline{\Delta_{M_1} u} \cdot \psi_2(\cdots) &+ \Delta_{\frac{M_1}{2}} u \cdot \psi_3(\cdots) + \overline{\Delta_{\frac{M_1}{2}} u} \cdot \psi_4(\cdots) \end{aligned} \tag{5.25}$$

where

$$\big|\psi_i(u_{M_1}, u_{\frac{M_1}{2}}, u_{\frac{M_1}{4}})\big| < c\big(|u_{M_1}| + |u_{\frac{M_1}{2}}| + |u_{\frac{M_1}{4}}|\big)^{\alpha-1} \,. \tag{5.26}$$

---

(*) This is obvious if $\alpha$ is an even integer. In general some more technicalities are needed which we skip for the sake of the exposition.

Considering (5.22), (5.25), we have to evaluate (5.17) with $w$ replaced by

$$\Delta_M u \cdot \Delta_{M_1} u \cdot \psi(u_{M_1}, u_{\frac{M_1}{2}}, u_{\frac{M_1}{4}}) \tag{5.27}$$

where $M_1 < M$, $M > N$.

Partition $\frac{M}{2} < |\xi| \leq M$ is intervals of size $M_1$ and write with notation (5.14)

$$\Delta_M u = \sum \Delta_I u. \tag{5.28}$$

The functions

$$w_I = \Delta_I u \cdot \Delta_{M_1} u \cdot \psi(u_{M_1}, u_{\frac{M_1}{2}}, u_{\frac{M_1}{4}}) \tag{5.29}$$

have essentially disjointly supported Fourier transform for varying $I$.(**)

Thus the contribution of (5.27) to (5.17) becomes

$$K^{-1/2} \cdot N^s \cdot \left( \sum_I \int_{\Lambda_{K,I}} |\widehat{w}_I(\zeta)|^2 d\zeta \right)^{1/2} \tag{5.30}$$

and our next purpose is to estimate these integrals.

Choose

$$p_1 > p_0 , \quad p_1 > p_2 > 2 , \quad \frac{1}{p_2} = \frac{1-\theta_2}{p_1} + \frac{\theta_2}{2} . \tag{5.31}$$

The dual form of (5.10) yields

$$\left( \int_{\Lambda_{K,I}} |\widehat{w}_I(\zeta)|^2 d\zeta \right)^{1/2} < c\, K^{\frac{1}{2}(1-\theta_2)}\, M_1^{(\frac{d-1}{2} - \frac{d+1}{p_1})(1-\theta_2)} \parallel w_I \parallel_{p_2'} . \tag{5.32}$$

By (5.29) and Hölder's inequality

$$\|w_I\|_{p_2'} \leq \|\Delta_I u\|_{p_2} \left( \int |\Delta_{M_1} u \cdot \psi|^{\frac{p_2 p_2'}{p_2 - p_2'}} \right)^{\frac{p_2 - p_2'}{p_2 p_2'}} . \tag{5.33}$$

For the first factor of (5.33), write

$$\Delta_I u = \sum_{K_1 \text{ dyadic}} \left[ \int_{\Lambda_{K_1,I}} \widehat{u}(\zeta)\, e^{i(\langle x,\xi\rangle + \lambda t)}\, d\zeta \right] \tag{5.34}$$

(**) This is clear if $\alpha$=2. In general there are again some extra technicalities in which we do not want to sidetrack the reader.

and use triangle inequality and (5.10) to estimate

$$\|\Delta_I u\|_{p_2} \le c \sum_{K_1 \text{ dyadic}} K_1^{\frac{1}{2}(1-\theta_2)} M_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \left(\int_{\Lambda_{K_1,I}} |\widehat{u}(\zeta)|^2\right)^{1/2} \tag{5.35}$$

and using the definition of the norm $|||u|||$

$$\left(\sum_I \|\Delta_I u\|_{p_2}^2\right)^{1/2} \le$$

$$\le cM_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \left\{\sum_I \left\{\sum_{K_1 \text{ dyadic}} K_1^{-\frac{\theta_2}{2}} \left[K_1^{1/2} \cdot \left(\int_{\Lambda_{K_1,I}} |\widehat{u}(\zeta)|^2\right)^{1/2}\right]\right\}^2\right\}^{1/2}$$

$$\le cM_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \left\{\sum_{K_1 \text{ dyadic}} K_1^{-\frac{\theta_2}{2}} \left[K_1 \cdot \left(\int_{\Lambda_{K_1,M}} |\widehat{u}(\zeta)|^2\right)\right]\right\}^{1/2} \tag{5.36}$$

$$\le c\, M_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \cdot M^{-s} |||u||| \,. \tag{5.37}$$

Collecting estimates (5.32), (5.33), (5.35), (5.37), it follows that

$$(5.30) \lesssim K^{-\frac{\theta_2}{2}} \cdot M_1^{2(1-\theta_2)(\frac{d-1}{2}-\frac{d+1}{p_1})} \cdot \left(\frac{N}{M}\right)^s \cdot |||u||| \cdot \left(\int |\Delta_{M_1} u \cdot \psi|^{\frac{p_2 p_2'}{p_2 - p_2'}}\right)^{\frac{p_2 - p_2'}{p_2 p_2'}} . \tag{5.38}$$

We estimate the last factor in (5.38) again by Hölder's inequality.

Let

$$p_3 > p_0\ ,\ p_3 > p_4 > 2\ ,\ \frac{1}{p_4} = \frac{1-\theta_4}{p_3} + \frac{\theta_4}{2} \text{ and assume } 1 > \frac{2}{p_2} + \frac{1}{p_4}\ . \tag{5.39}$$

Then

$$\|\Delta_{M_1} u \cdot \psi\|_{\frac{p_2 p_2'}{p_2 - p_2'}} \le \|\Delta_{M_1} u\|_{p_4} \cdot \left\|\psi(u_{M_1}, u_{\frac{M_1}{2}}, u_{\frac{M_1}{4}})\right\|_{(1-\frac{2}{p_2}-\frac{1}{p_4})^{-1}} . \tag{5.40}$$

Similarly to (5.35)

$$\|\Delta_{M_1} u\|_{p_4} \le c \sum_{K_1 \text{dyadic}} K_1^{\frac{1}{2}(1-\theta_4)} M_1^{(\frac{d-1}{2}-\frac{d+1}{p_3})(1-\theta_4)} \left(\int_{\Lambda_{K_1,M_1}} |\widehat{u}(\zeta)|^2\right)^{1/2} \tag{5.41}$$

$$\le cM_1^{(\frac{d-1}{2}-\frac{d+1}{p_3})(1-\theta_4)-s}\ |||u||| \,. \tag{5.42}$$

Consider the second factor in (5.40), which amounts to estimate, by (5.26),

$$\|u_{M_1}\|^{\alpha-1}_{(\alpha-1)(1-\frac{2}{p_2}-\frac{1}{p_4})^{-1}} \cdot \tag{5.43}$$

Take $p_5$, $p_6$ such that

$$p_5 > p_0\ ,\ p_5 > p_6 > 2\ ,\ \frac{1}{p_6} = \frac{1-\theta_6}{p_5} + \frac{\theta_6}{2} \text{ and assume } \frac{\alpha-1}{p_6} \leq 1 - \frac{2}{p_2} - \frac{1}{p_4}\ . \tag{5.44}$$

Then

$$\begin{aligned}(5.43) &\leq \|u_{M_1}\|^{\alpha-1}_{p_6} \text{ and } \|u_{M_1}\|_{p_6} \leq \\ &\leq \sum_{\substack{M_2<M_1 \\ M_2 \text{ dyadic}}} \|\Delta_{M_2} u\|_{p_6} \leq c \sum_{M_2} M_2^{(\frac{d-1}{2}-\frac{d+1}{p_5})(1-\theta_6)-s} |||u||| \leq c|||u|||\end{aligned} \tag{5.45}$$

provided $s$ satisfies

$$s \geq \frac{d-1}{2} - \frac{d+1}{p_5}\ . \tag{5.46}$$

Collecting estimates (5.40), (5.42), (5.45)

$$\|\Delta_{M_1} u \cdot \psi\|_{\frac{p_2 p_2'}{p_2 - p_2'}} \leq c M_1^{(\frac{d-1}{2}-\frac{d+1}{p_3})(1-\theta_4)-s}\ |||u|||^{\alpha} \tag{5.47}$$

and with (5.38)

$$(5.30) \leq cK^{-\frac{\theta_2}{2}} \cdot M_1^{2(1-\theta_2)(\frac{d-1}{2}-\frac{d+1}{p_1})+(1-\theta_4)(\frac{d-1}{2}-\frac{d+1}{p_3})-s} \cdot \left(\frac{N}{M}\right)^{s} \cdot |||u|||^{1+\alpha}\ . \tag{5.48}$$

Assume $s$ also satisfies the condition

$$s > 2\left(\frac{d-1}{2} - \frac{d+1}{p_1}\right) + \left(\frac{d-1}{2} - \frac{d+1}{p_3}\right)\ . \tag{5.49}$$

One may then estimate

$$(5.17) \leq c\, K^{-\frac{\theta_2}{2}} \cdot |||u|||^{1+\alpha} \tag{5.50}$$

performing the summations over $M_1$ and $M > N$ ($s > 0$).

It remains to analyze the conditions on $s$, $\alpha$ and the various exponents introduced above. Consider the conditions (5.39), (5.44), (5.46), (5.49).

Since $p_0$, $p_1$, $p_2$ (resp. $p_3$, $p_4$ and $p_5$, $p_6$) may be choosen arbitrarily close, these conditions may be replaced by

$$1 > \frac{2}{p_0} + \frac{1}{p_3} \tag{5.51}$$

$$\frac{\alpha - 1}{p_5} < 1 - \frac{2}{p_0} - \frac{1}{p_3} \tag{5.52}$$

$$s > \frac{d-1}{2} - \frac{d+1}{p_5} \tag{5.53}$$

$$s > \frac{3}{2}\,(d-1) - 2\,\frac{d+1}{p_0} - \frac{d+1}{p_3} \tag{5.54}$$

where

$$p_3\ ,\ p_5 \geq p_0\ . \tag{5.55}$$

Choose $p_5$ with approximative equality in (5.53), assuming

$$s > \frac{d-1}{2} - \frac{d+1}{p_0} \tag{5.56}$$

to ensure $p_5 \geq p_0$. (5.52) becomes

$$\frac{1}{p_3} < 1 - \frac{2}{p_0} - (\alpha - 1)\,\frac{d-1-2s}{2(d+1)}\ . \tag{5.57}$$

Rewrite (5.54), (5.55) as

$$\frac{d+1}{p_3} > \frac{3}{2}\,(d-1) - 2\,\frac{d+1}{p_0} - s \tag{5.58}$$

$$\frac{1}{p_3} \leq \frac{1}{p_0} \tag{5.59}$$

and verify the existence of $p_3$ fulfilling (5.57), (5.58), (5.59).

Their compatibility requires

$$\begin{cases} (d+1)\left(1 - \frac{2}{p_0} - (\alpha-1)\,\frac{d-1-2s}{2(d+1)}\right) > \frac{3}{2}\,(d-1) - 2\,\frac{d+1}{p_0} - s & (5.60)\\ \frac{d+1}{p_0} > \frac{3}{2}\,(d-1) - 2\,\frac{d+1}{p_0} - s & (5.61)\end{cases}$$

and the resulting conditions become

$$\alpha < \frac{4}{d-1-2s} \qquad \left(s < \frac{d-1}{2}\right) \tag{5.62}$$

$$p_0 < \frac{2(d+1)}{d-1-\frac{2}{3}s} \tag{5.63}$$

(5.63) also implies (5.56).

Hence we proved

LEMMA 5.64. *If* (5.62) *and* (5.63) *hold, then so does* (5.50), *i.e.*

$$K^{-1/2}\ N^s\Big(\int_{\Lambda_{K,N}} \big|\widehat{w}(\zeta)\big|^2 d\zeta\Big)^{1/2} < c\ K^{-\theta}\ |||u|||^{1+\alpha} \tag{5.65}$$

*for some* $\theta > 0$.

Now come back to the estimate in $|||\ \ |||$-norm of (4.14), (4.15), (4.16).

**(4.14).** Observe that $\big|t^k\widehat{\psi_1(t)}\ (\lambda)\big| < c\ k\ \delta^k\ |\lambda|^{-1}$. Hence by (4.9)

$$\begin{aligned} |||(4.14)||| &\leq c\ \delta N^s\Big(\sum_{|\xi|\sim N}\Big(\int_{|\lambda-|\xi|^2|\leq B}\big|\widehat{w}(\xi,\lambda)\big|d\lambda\Big)^2\Big)^{1/2} \\ &\leq c\ \delta B^{1/2}N^s\Big(\sum_{K\leq B}\int_{\Lambda_{K,N}}\big|\widehat{w}(\zeta)\big|^2 d\zeta\Big)^{1/2} \\ &\leq c\ \delta B\ |||u|||^{1+\alpha}\ . \end{aligned} \tag{5.66}$$

**(4.15).** One gets

$$K^{1/2}\ N^s\left(\int_{\Lambda_{K,N}}\left|\frac{(1-\psi_2)(\lambda-|\xi|^2)}{\lambda-|\xi|^2}\ \widehat{w}(\zeta)\right|^2 d\zeta\right)^{1/2} \tag{5.67}$$

(5.67) vanishes, unless $K \geq B$. One then has by (5.64) the estimate

$$K^{-1/2}\cdot N^s\cdot\Big(\int_{\Lambda_{K,N}}\big|\widehat{w}(\zeta)\big|^2 d\zeta\Big)^{1/2} \leq cB^{-\theta}|||u|||^{1+\alpha}\ . \tag{5.68}$$

**(4.16).** Contribution to $|||\ |||$ appears mainly for bounded $K$ and amounts to

$$\begin{aligned} &N^s\Big(\sum_{|\xi|\sim N}\Big|\int\frac{(1-\psi_2)(\lambda-|\xi|^2)}{\lambda-|\xi|^2}\ \widehat{w}(\xi,\lambda)\ d\lambda\Big|^2\Big)^{1/2} \leq \\ &\leq c\ N^s\Big(\sum_{\substack{K>B\\ K\ \text{dyadic}}} K^{-(1-\varepsilon)}\cdot\int_{\Lambda_{K,N}}\big|\widehat{w}(\zeta)\big|^2 d\zeta\Big)^{1/2} \leq \\ &\leq c\Big(\sum_{\substack{K>B\\ K\ \text{dyadic}}} K^{-2\theta+\varepsilon}\Big)^{1/2}|||u|||^{1+\alpha} \\ &< c\ B^{-\theta+\varepsilon}\ |||u|||^{1+\alpha} \end{aligned} \tag{5.69}$$

again invoking (5.65).

Collecting estimates (5.66), (5.68), (5.69), it follows that the terms (4.14), (4.15), (4.16) contribute in $|||\ \ |||$-norm for

$$c(\delta \cdot B + B^{-\theta/2})|||u|||^{1+\alpha} . \tag{5.70}$$

At this point, one finds that the transformation associated to (4.4) maps the ball $\{u \mid |||u||| \leq M\}$ into itself, for sufficiently large $M$. If $\alpha \geq 2$, one may perfom one more differentiation of the function $z|z|^\alpha$ and obtain an estimate

$$|||Tu - Tv||| \leq c(\delta \cdot B + B^{-\theta/2})(|||u|||^\alpha + |||v|||^\alpha)|||u - v||| \tag{5.71}$$

where

$$Tu = \psi_1(t)U(t)\phi + i\ \psi_1(t) \int_0^t U(t-\tau)\ w(\cdot, \tau)\ d\tau \tag{5.72}$$

instead of (5.70). Details are rather straightforward adjustments of above arguments and we leave them to the reader. Choosing $\delta$, $B$ in a suitable way, $T$ may be given Lipschitz constant $< 1$ so that Picard's theorem may again be invoked to get a local solution to (4.3). This solution is unique and persistent. Hence we established

PROPOSITION 5.73. *Consider the generalized period NLSE* (4.1) *with* $\alpha \geq 2$ *and initial data* $\phi \in H^s(\mathbf{T}^n)$, $s > 0$ $(n = d - 1)$ *with* $s, \alpha$ *satisfying the condition*

$$\alpha < \frac{4}{n - 2s} . \tag{5.74}$$

*Assume moreover we dispose of an admissible exponent*

$$p < \frac{2(n+2)}{n - \frac{2}{3}s} . \tag{5.75}$$

*Then the problem is locally well-posed in the space*

$$|||u||| = \sup_{K,N} (K+1)^{1/2}\ (N+1)^s \Big(\int_{\substack{|\xi| \sim N \\ |\lambda - |\xi|^2| \sim K}} \big|\widehat{u}(\xi, \lambda)\big|^2 d\xi d\lambda\Big)^{1/2} \tag{5.76}$$

*(understood as "restriction" norm wrt time variable").*

*Remarks*: (i) Condition (5.74) corresponds to the subcritical case in the non-periodic setting, the $H^s$-critical exponent being $\frac{4}{n-2s}$. Thus this relation between $n$, $s$, $\alpha$ is not unexpected. In view of the failure of the $L^{\frac{2(d+1)}{d+3}}(\mathbf{T}^d) - L^2(\mathbf{Z}^{d-1})$ restriction theorem in the periodic case (i.e. for sets $\{(\xi, |\xi|^2) \in \mathbf{Z}^n \times \mathbf{Z}\}$), it is unlikely one may include the critical exponent, at least for $s = 0$ (cf. [CW]).

(ii) One may replace $|||\ \ |||$ by a Hilbert-space norm, for instance

$$|||u|||_2 = \left\{ \int_{\mathbf{Z}^n \times \mathbf{R}} (1 + |\xi|)^{2s} (1 + |\lambda - |\xi|^2|) |\widehat{u}(\xi, \lambda)|^2 d\zeta \right\}^{1/2} \tag{5.77}$$

and conclude for $\alpha \geq 1$ under the assumptions (5.74), (5.75) to a local solution, invoking Schauder's fixpoint theorem. We loose uniqueness here however, which is a major consideration in these problems.

(iii) Observe that in particular the solution $u$ satisfies

$$u \in L^p(\mathbf{T}^n \times [-\delta, \delta]) \quad \text{for} \quad p < \frac{2(n+2)}{n-2s} \ . \tag{5.78}$$

(iv) The regularization method discussed in section 4 applies equally well here, considering norms of the form

$$|||u||| + \sum_{(\tau)} \rho_{|\tau|}\ |||D_x^{(\tau)} u||| \tag{5.79}$$

$(\tau = (\tau_1, \dots, \tau_n),\ |\tau| = |\tau_1| + \cdots + |\tau_n|)$.

(v) It follows from (5.73) and (2.34) ($p = 6$ is admissible) that for $n = 1$ the periodic NLSE is locally well-posed in the space with norm (5.76). This specifies Theorem 1.

## 6. Consequences and Global Results

Recall the two conserved quantities for the NLSE

$$\int_{\mathbf{T}^n} |\psi(x)|^2 dx \quad (L^2 - \text{norm}) \tag{6.1}$$

$$H(\psi) = \frac{1}{2} \int_{\mathbf{T}^n} |\nabla \psi|^2 dx - \frac{1}{p} \int_{\mathbf{T}^n} |\psi(x)|^p dx \quad \text{(Hamiltonian)}\ . \tag{6.2}$$

Global results for $H^1$-data will be derived from

- the local result (given by (5.73))
- the conservation of $H(u(\cdot,t))$ and (6.1)

(+ a standard regularization process in order to justify the conservation law for the generalized equation we are dealing with here).

The negative sign of $\int|\psi|^p$ in (6.2) leads to some problems when estimating the $H^1$-norm from the Hamiltonian. In order to interpolate the $L^p$-norm between $L^2$ and $H^1$, one needs the condition

$$\theta \equiv n\left(\frac{1}{2}-\frac{1}{p}\right) < 1 \tag{6.3}$$

in which case

$$\|f\|_{L^p(\mathbf{T}^n)} \leq c\|f\|_2^{1-\theta}\ \|f\|_{H^1}^{\theta}\ . \tag{6.4}$$

Let $u = u(x,t)$ be a solution of the periodic NLSE $i\partial_t u + \Delta u + u|u|^{p-2} = 0$ with initial data $\phi$. Assuming sufficient smoothness (which may be achieved using a regularization), one gets from (6.1), (6.2), (6.4) (assuming (6.3) valid)

$$\frac{1}{2}\|\phi\|_{H^1}^2 - \frac{1}{p}\|\phi\|_p^p = \frac{1}{2}\big\|u(\cdot,t)\big\|_{H^1}^2 - \frac{1}{p}\|u(\cdot,t)\|_p^p \tag{6.5}$$

$$\begin{aligned}\frac{1}{2}\big\|u(\cdot,t)\big\|_{H^1}^2 &\leq \frac{1}{2}\|\phi\|_{H^1}^2 + c\big\|u(\cdot,t)\big\|_2^{p(1-\theta)}\|u(\cdot,t)\|_{H^1}^{p\theta} = \\ &= \frac{1}{2}\|\phi\|_{H^1}^2 + c\|\phi\|_2^{p(1-\theta)}\big\|u(\cdot,t)\big\|_{H^1}^{p\theta}\ .\end{aligned} \tag{6.6}$$

We distinguish 3 cases

(I) $\underline{p\theta < 2}$ : Then (6.6) yields an a priori bound on $\|u(\cdot,t)\|_{H^1}$.

(II) $\underline{p\theta = 2}$ : (6.6) implies a bound on $\|u(\cdot,t)\|_{H^1}$, provided the initial data is sufficiently small in $L^2$. One has

$$\big\|u(\cdot,t)\big\|_{H^1}^2 \leq \frac{\|\phi\|_{H^1}^2}{1-2c\|\phi\|_2^{p-2}}\ . \tag{6.7}$$

(III) $\underline{p > p\theta > 2}$ : If $\big\|u(\cdot,t)\big\|_{H^1}$ is sufficiently small, one may write

$$\big\|u(\cdot,t)\big\|_{H^1}^2 \leq \frac{\|\phi\|_{H^1}^2}{1-2c\|\phi\|_2^{p(1-\theta)}\big\|u(\cdot,t)\big\|_{H^1}^{p\theta-2}}\ . \tag{6.8}$$

In particular, if $\|\ u(\cdot,t)\ \|_{H^1} \leq 1$ and $\|\ \phi\ \|_2$ sufficiently small, (6.8) yields $\|u(\cdot,t)\|_{H^1} \leq 2\|\phi\|_{H^1}$. So for sufficiently small $L^2$ and $H^1$-data, one gets again an apriori control on $\|u(\cdot,t)\|_{H^1}$.

<u>$n=1$</u> : For $p \le 4$, see Theorem 4.45.

Propostion 5.73 yields a local well-posedness result provided

$$s > 0\ ,\ 4 \le p < \frac{4}{1-2s} + 2 \quad \left(s \le \frac{1}{2}\right)\ . \tag{6.9}$$

Global results for $H^1$ initial data are discussed in the paper [LeRSp].

<u>$n=2$</u> : Since 4 is an admissible exponent, (5.73) yields local well-posedness for $\phi \in H^s(\mathbf{T}^2)$ provided

$$4 \le p < \frac{4}{2(1-s)} + 2 \quad (0 < s \le 1)\ . \tag{6.10}$$

From the preceding, one gets global well-posedness if $p=4$ and $\phi \in H^1$ has sufficiently small $L^2$-norm and for arbitrary $p \ge 4$ assuming the initial data sufficiently small in $H^1$. The function space is given by (5.76) with $s=1$ and $u$ replaced by $u_T(x,t) = u(x,t)\ \chi_{[-T,T]}\ (t)$, for arbitrarily choosen finite time restriction $T$.

This proves in particular Theorem 2 from the introduction.

<u>$n=3$</u> : The smallest admissible exponent we know off is $q=4$ (see Proposition 3.6). Hence (5.73) implies local well-posedness in the space (5.76), in the parameter range

$$4 \le p < \frac{4}{3-2s} + 2\ ,\ \frac{3}{4} < s \le \frac{3}{2}\ . \tag{6.11}$$

For $4 \le p < 6$ one gets global well-posedness for sufficiently small $H^1$ data (this is case III above). This proves Theorem 3.

<u>$n \ge 4$</u> : The smallest critical exponent at our disposal is $q = \frac{2(n+4)}{n}$ (see Proposition 3.110). Hence (5.73) implies local well-posedness in the space (5.76), in the parameter range

$$4 \le p < \frac{4}{n-2s} + 2\ ,\ \frac{3n}{n+4} < s \le \frac{n}{2}\ . \tag{6.12}$$

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J. Bourgain
IHES
35 route de Chartres
Bures sur Yvette 91440
France


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