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VIBRATION ANALYSIS OF MODERATE-THICK PLATES WITH SLOWLY VARYING THICKNESS

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Abstract

In this paper, the flexural vibration analysis of moderate-thick rectangular plates with slowly varying thickness using perturbation method is described, and the explict expressions of free vibration frequencies for arbitrary thickness functions are derived. Finally, several numerical examples have been given and comparisons have been made with other proposed solution techniques. This comparison shows that the method yields very good results, so that this method may be regarded as an alternative effective method for the vibration and buckling analysis of plates and shells.

I. Introduction

It is well-known that computational methods for the analysis of the natural frequencies of flexural vibration of plates based on the classical thin plate theory cannot present their accurate values for the higher modes, and thus have limited application. Timosheko[14](1921) was the first to include the effects of both transverse deformation and rotary inertia in the study of the flexural vibration of elastic beams. Later, Mindlin[2](1951) developed a theory for flexural vibration of elastic plates which included the influence of transverse shear deformation and rotary inertia. Since then, the application of Mindlin's theory to plate problems has been considered by various authors and a variety of methods have been developed to obtain approximate solutions, e.g., the Rayleigh-Ritz method[8], the finite element method[9-11], the finite strip method[12-13]. But there are relatively few analytical solutions[5-7], and only plates of uniform thickness have been considered[5].

For the vibration of moderate-thick plate with variable thickness, the solutions of the problem can be obtained only by approximate or numerical methods[15] owing to the mathematical difficulty of solving the differential equations with variable coefficients. In this paper, the difficulty is overcome by means of expanding the thickness function in power series in the thickness parameter and using perturbation method and the generalized Hale Law[4], and the explicit asymptotic expressions of free vibration frequency parameters for thickness functions are derived.

To demonstrate the applicability of the method presented herein, the frequency analysis of plates with linearly varying thickness and edges simply supported is made,and the results are compared with those obtained by other techniques. This comparison shows that the method yields very good results, and estimated values for the higher modes can be obtained without any difficulties. Finally, this method is applied to plates with bilinearly varying thickness, and some interesting results are presented.

II. Equations of Motion

In Mindlin's theory[2], the displacements are assumed to be

$$\left.\begin{aligned} u(x,y,t)&=-z\psi_x(x,y,t)\\ v(x,y,t)&=-z\psi_y(x,y,t)\\ w(x,y,t)&=w(x,y,t)\end{aligned}\right\}\tag{2.1}$$

The equations of motion of the plate are written as

$$\left.\begin{aligned} &M_{x,x}+M_{xy,y}-Q_x+\rho h^3\psi_{x,tt}1/12=0\\ &M_{xy,x}+M_{y,y}-Q_y+\rho h^3\psi_{y,tt}1/12=0\\ &Q_{x,x}+Q_{y,y}-\rho hw_{,tt}=0\end{aligned}\right\}\tag{2.2}$$

The constitutive relations of the plate are written as

$$\left.\begin{aligned} &M_x=-D(\psi_{x,x}+\nu\psi_{y,y})\\ &M_y=-D(\psi_{y,y}+\nu\psi_{x,x})\\ &M_{xy}=-D\cdot(\psi_{x,y}+\psi_{y,x})(1-\nu)/2\\ &Q_x=kGh(w_{,x}-\psi_x)\\ &Q_y=kGh(w_{,y}-\psi_y)\end{aligned}\right\}\tag{2.3}$$

where u, v, and w are displacements in the directions of x, y, and z. ψ_x and ψ_y represent respectively rotations in the xz and yz planes due to bending only. M_x, M_y, and M_{xy} are the bending and twisting moments per unit length. Q_x and Q_y are the transverse shear forces per unit length. ρ is the material mass density and thickness h is a function of x and y. D is the flexural rigidity of the plate. ν Poisson's ratio, G the shear modulus and K the shear coefficient.

To solve the equations conveniently, firstly we introduce the nondimensional parameters as follows:

$$\begin{aligned} &\{M_x^*,\ M_y^*,\ M_{xy}^*\}=h_0D_0^{-1}\{M_x,\ M_y,\ M_{xy}\}\\ &\{Q_x^*,\ Q_y^*\}=h_0^2D_0^{-1}\{Q_x,\ Q_y\}\\ &\{\xi,\ \alpha\}=a^{-1}\{x,\ h_0\}\\ &\{\eta,\ \beta\}=b^{-1}\{y,\ h_0\}\\ &\{w^*,\ h^*\}=h_0^{-1}\{w,\ h\}\\ &\tau=\omega_0\lambda t,\ \ k_1=kGh_0^3D_0^{-1}\\ &\gamma=(\rho h_0^5\omega_0^2)(12D_0)^{-1}\end{aligned}$$

in which a and b are the length and width of the plate, and γ is a factor relating to the free vibration frequency.

Substituting the above equations into Eqs.(2.3) and (2.2) leads to the nondimensional governing equations (in which, for brevity, we omit the index * of nondimensional parameters:

$$\left.\begin{aligned} &\alpha M_{\xi,\xi}+\beta M_{\xi\eta,\eta}-Q_\xi+\gamma\lambda^2h^3\psi_{\xi,\tau\tau}=0\\ &\alpha M_{\xi\eta,\xi}+\beta M_{\eta,\eta}-Q_\eta+\gamma\lambda^2h^3\psi_{\eta,\tau\tau}=0\end{aligned}\right\}\tag{2.4}$$

$$\alpha Q_{\xi,\xi}+\beta Q_{\eta,\eta}-12\gamma\lambda^2 hw_{,\tau\tau}=0$$

$$\left.\begin{aligned}
M_\xi&=-h^3(\alpha\psi_{\xi,\xi}+\nu\beta\psi_{\eta,\eta})\\
M_\eta&=-h^3(\beta\psi_{\eta,\eta}+\nu\alpha\psi_{\xi,\xi})\\
M_{\xi\eta}&=-h^3\cdot(1-\nu)(\beta\psi_{\xi,\eta}+\alpha\psi_{\eta,\xi})/2\\
Q_\xi&=hk_1(\alpha w_{,\xi}-\psi_\xi)\\
Q_\eta&=hk_1(\beta w_{,\eta}-\psi_\eta)
\end{aligned}\right\}\qquad(2.5)$$

Now Eq.(2.5) are substituted into Eq.(2.4). We obtain the equations of motion in terms of the displacement variables

$$\begin{aligned}
&\alpha h^3(\alpha\psi_{\xi,\xi\xi}+\nu\beta\psi_{\eta,\eta\xi})+\beta h^3(\beta\psi_{\xi,\eta\eta}+\alpha\psi_{\eta,\xi\eta})(1-\nu)/2-hk_1(\alpha w_{,\xi}-\psi_\xi)\\
&\quad-\gamma\lambda^2h^3\psi_{\xi,\tau\tau}+3\alpha h^2h_{,\xi}(\alpha\psi_{\xi,\xi}+\nu\beta\psi_{\eta,\eta})+3\beta h^2h_{,\eta}(\beta\psi_{\xi,\eta}+\alpha\psi_{\eta,\xi})(1-\nu)/2=0\\
&\alpha h^3(\beta\psi_{\xi,\eta\xi}+\alpha\psi_{\eta,\xi\xi})(1-\nu)/2+\beta h^3(\beta\psi_{\eta,\eta\eta}+\nu\alpha\psi_{\xi,\xi\eta})-hk_1(\beta w_{,\eta}-\psi_\eta)\\
&\quad-\gamma\lambda^2h^3\psi_{\eta,\tau\tau}+\frac{1-\nu}{2}3\alpha h^2h_{,\xi}(\beta\psi_{\xi,\eta}+\alpha\psi_{\eta,\xi})+3\beta h^2h_{,\eta}(\beta\psi_{\eta,\eta}+\nu\alpha\psi_{\xi,\xi})=0\\
&\alpha hk_1(\alpha w_{,\xi\xi}-\psi_{\xi,\xi})+\beta hk_1(\beta w_{,\eta\eta}-\psi_{\eta,\eta})-12\gamma\lambda^2hw_{,\tau\tau}\\
&\quad+\alpha h_{,\xi}k_1(\alpha w_{,\xi}-\psi_\xi)+\beta h_{,\eta}k_1(\beta w_{,\eta}-\psi_\eta)=0
\end{aligned}\qquad(2.6)$$

In basic Eq.(2.6), the thickness is a function of ξ and η, so these differential equations of variable coefficients are difficult to be sloved directly by analytical method. In this paper, this difficulty is avoided by using the PLK method in perturbation theory[1]. Thickness h, displacements w, ψ_x, ψ_y and the frequency factor λ^2 are expanded in power series in the following form:

$$\left.\begin{aligned}
h(\xi,\eta)&=1+\sum_{m=1}\varepsilon^m h_m(\xi,\eta)\\
\lambda^2&=1+\sum_{m=1}\varepsilon^m\lambda_m^2\\
w(\xi,\eta,\tau)&=\sum_{m=0}\varepsilon^m w_m(\xi,\eta,\tau)\\
\psi_\xi(\xi,\eta,\tau)&=\sum_{m=0}\varepsilon^m\psi_{\xi m}(\xi,\eta,\tau)\\
\psi_\eta(\xi,\eta,\tau)&=\sum_{m=0}\varepsilon^m\psi_{\eta m}(\xi,\eta,\tau)
\end{aligned}\right\}\qquad(2.7)$$

where ε is a small parameter related to thickness, the second equation is introduced as a much more accurate relation for the frequency factor λ can be found by expanding λ^2, rather than λ, in a power series in ε to the same order[3].

Substituting Eq.(2.7) into Eq.(2.6) and equating like powers of ε lead to the following recurrence differential equations with constant coefficients.

Order ε^0

$$\left.\begin{array}{l}(\alpha^2\psi_{\xi0,\xi\xi}+\nu\alpha\beta\psi_{\eta0,\eta\xi})+\frac{1-\nu}{2}(\beta^2\psi_{\xi0,\eta\eta}+\alpha\beta\psi_{\eta0,\xi\eta})\\ \quad+k_1(\alpha w_{0,\xi}-\psi_{\xi0})-\gamma\psi_{\xi0,\tau\tau}=0\\ \frac{1-\nu}{2}(\alpha\beta\psi_{\xi0,\xi\eta}+\alpha^2\psi_{\eta0,\xi\xi})+(\beta^2\psi_{\eta0,\eta\eta}+\nu\alpha\beta\psi_{\xi0,\xi\eta})\\ \quad+k_1(\beta w_{0,\eta}-\psi_{\eta0})-\gamma\psi_{\eta0,\tau\tau}=0\\ \alpha k_1(\alpha w_{0,\xi\xi}-\psi_{\xi0,\xi})+\beta k_1(\beta w_{0,\eta\eta}-\psi_{\eta0,\eta})-12\gamma w_{0,\tau\tau}=0\end{array}\right\}\quad(2.8)$$

Order ϵ^1

$$\left.\begin{array}{l}(\alpha^2\psi_{\xi1,\xi\xi}+\nu\alpha\beta\psi_{\eta1,\eta\xi})+\frac{1-\nu}{2}(\beta^2\psi_{\xi1,\eta\eta}+\alpha\beta\psi_{\eta1,\xi\eta})\\ \quad+k_1(\alpha w_{1,\xi}-\psi_{\xi1})-\gamma\psi_{\xi1,\tau\tau}=-3h_1(\alpha^2\psi_{\xi0,\xi\xi}+\nu\alpha\beta\psi_{\eta0,\xi\eta})\\ \quad-3h_1\frac{1-\nu}{2}(\beta^2\psi_{\xi0,\eta\eta}+\alpha\beta\psi_{\eta0,\xi\eta})-k_1h_1(\alpha w_{0,\xi}-\psi_{\xi0})+\gamma\lambda_1^2\psi_{\xi0,\tau\tau}\\ \quad+3h_1\gamma\psi_{\xi0,\tau\tau}-3h_{1,\xi}(\alpha^2\psi_{\xi0,\xi}+\nu\alpha\beta\psi_{\eta0,\eta})\\ \quad-\frac{1-\nu}{2}3h_{1,\eta}(\beta^2\psi_{\xi0,\eta}+\alpha\beta\psi_{\eta0,\xi})\\ \frac{1-\nu}{2}(\alpha\beta\psi_{\xi1,\xi\eta}+\alpha^2\psi_{\eta1,\xi\xi})+(\beta^2\psi_{\eta1,\eta\eta}+\nu\alpha\beta\psi_{\xi1,\xi\eta})\\ \quad+k_1(\beta w_{1,\eta}-\psi_{\eta1})-\gamma\psi_{\eta1,\tau\tau}=-3h_1(\alpha\beta\psi_{\xi0,\xi\eta}+\alpha^2\psi_{\eta0,\xi\xi})(1-\nu)/2\\ \quad-3h_1(\beta^2\psi_{\eta0,\eta\eta}+\nu\alpha\beta\psi_{\xi0,\xi\eta})-k_1h_1(\beta w_{0,\eta}-\psi_{\eta0})+\gamma\lambda_1^2\psi_{\eta0,\tau\tau}\\ \quad+3h_1\gamma\psi_{\eta0,\tau\tau}-3h_{1,\xi}(\alpha\beta\psi_{\xi0,\eta}+\alpha^2\psi_{\eta0,\xi})(1-\nu)/2-3h_{1,\eta}(\beta^2\psi_{\eta0,\eta}+\nu\alpha\beta\psi_{\xi0,\xi})\\ k_1(\alpha^2w_{1,\xi\xi}-\alpha\psi_{\xi1,\xi})+k_1(\beta^2w_{1,\eta\eta}-\beta\psi_{\eta1,\eta})-12\gamma w_{1,\tau\tau}\\ \quad=-k_1h_1(\alpha^2w_{0,\xi\xi}-\alpha\psi_{\xi0,\xi})-k_1h_1(\beta^2w_{0,\eta\eta}-\beta\psi_{\eta0,\eta})+12\gamma\lambda_1^2w_{0,\tau\tau}\\ \quad-k_1h_{1,\xi}(\alpha^2w_{0,\xi}-\alpha\psi_{\xi0})-k_1h_{1,\eta}(\beta^2w_{0,\eta}-\beta\psi_{\eta0})+12\gamma h_1w_{0,\tau\tau}\end{array}\right\}\quad(2.9)$$

III. Perturbation Solution

Let us consider the plate with edges simply supported. The displacements in Eq.(2.8) are described as

$$\left.\begin{array}{l}w_0(\xi,\eta,\tau)=\sum_{m=1}\sum_{n=1}w_{mn}\sin m\pi\xi\cdot\sin n\pi\eta\cdot e^{i\tau}\\ \psi_{\eta0}(\xi,\eta,\tau)=\sum_{m=1}\sum_{n=1}v_{mn}\sin m\pi\xi\cdot\cos n\pi\eta\cdot e^{i\tau}\\ \psi_{\xi0}(\xi,\eta,\tau)=\sum_{m=1}\sum_{n=1}u_{mn}\cos m\pi\xi\cdot\sin n\pi\eta\cdot e^{i\tau}\end{array}\right\}\quad(3.1)$$

where m and n are the numbers of half waves in the x and y directions. It is apparent that all the following boundary conditions are satisfied.

$$\left.\begin{array}{l}w_0(0,\eta,\tau)=\psi_{\eta0}(0,\eta,\tau)=M_{\xi0}(0,\eta,\tau)=0\\ w_0(1,\eta,\tau)=\psi_{\eta0}(1,\eta,\tau)=M_{\xi0}(1,\eta,\tau)=0\\ w_0(\xi,0,\tau)=\psi_{\xi0}(\xi,0,\tau)=M_{\eta0}(\xi,0,\tau)=0\\ w_0(\xi,1,\tau)=\psi_{\xi0}(\xi,1,\tau)=M_{\eta0}(\xi,1,\tau)=0\end{array}\right\}\quad(3.2)$$

Substituting the displacements Exp.(3.1) into the governing Eq.(2.8), the resulting expressions have the following forms:

$$[c_{ij}][f]=0 \tag{3.3}$$

where

$$[f]=\{u_{mn},\ v_{mn},\ w_{mn}\}^T$$

$$c_{11}=\gamma-k_1-\frac{1-\nu}{2}\beta^2n^2\pi^2-\alpha^2m^2\pi^2$$

$$c_{12}=c_{21}=-\frac{1+\nu}{2}\alpha\beta mn\pi^2$$

$$c_{13}=c_{31}=\alpha m\pi k_1$$

$$c_{22}=\gamma-k_1-\frac{1-\nu}{2}\alpha^2m^2\pi^2-\beta^2n^2\pi^2$$

$$c_{23}=c_{32}=\beta n\pi k_1$$

$$c_{33}=12\gamma-k_1\alpha^2m^2\pi^2-k_1\beta^2n^2\pi^2$$

The frequency factor γ in the zeroth order approximation is obtained from

$$[c_{ij}]=0 \tag{3.4}$$

The solution of Eq.(3.4) yields the estimated values for the three natural frequencies and the corresponding eigenmodes.

Before solving Eq.(2.9), we change Eq.(2.9) into the following form

$$\mathsf{Lu}=\mathsf{f} \tag{3.5}$$

in which

$$\mathsf{u}=\{\psi_{\xi 1},\ \psi_{\eta 1},\ w_1\}^T$$
$$\mathsf{f}=\{P_1,\ P_2,\ P_3\}^T$$

$$\mathsf{L}=\begin{bmatrix} \alpha^2\frac{\partial^2}{\partial\xi^2}+\frac{1-\nu}{2}\beta^2\frac{\partial^2}{\partial\eta^2}-k_1-\gamma\frac{\partial^2}{\partial\tau^2}, & \alpha\beta\frac{1+\nu}{2}\frac{\partial^2}{\partial\xi\partial\eta}, & \alpha k_1\frac{\partial}{\partial\xi} \\ \alpha\beta\frac{1+\nu}{2}\frac{\partial^2}{\partial\xi\partial\eta}, & \frac{1-\nu}{2}\alpha^2\frac{\partial^2}{\partial\xi^2}+\beta^2\frac{\partial^2}{\partial\eta^2}-k_1-\gamma\frac{\partial^2}{\partial\tau^2}, & \beta k_1\frac{\partial}{\partial\eta} \\ -\alpha k_1\frac{\partial}{\partial\xi}, & -\beta k_1\frac{\partial}{\partial\eta}, & \alpha^2k_1\frac{\partial^2}{\partial\xi^2}+\beta^2k_1\frac{\partial^2}{\partial\eta^2}-\gamma\frac{\partial^2}{\partial\tau^2} \end{bmatrix}$$

$$\begin{aligned} P_1=&2h_1k_1(\alpha w_{0,\xi}-\psi_{\xi 0})+\gamma\lambda_1^2\psi_{\xi 0,\tau\tau}-3h_{1,\xi}(\alpha^2\psi_{\xi 0,\xi}\\ &+\nu\alpha\beta\psi_{\eta 0,\eta})-3h_{1,\eta}(\beta^2\psi_{\xi 0,\eta}+\alpha\beta\psi_{\eta 0,\xi})(1-\nu)/2\\ P_2=&2h_1k_1(\beta w_{0,\eta}-\psi_{\eta 0})+\gamma\lambda_1^2\psi_{\eta 0,\tau\tau}-3h_{1,\xi}(\alpha\beta\psi_{\xi 0,\eta}\\ &+\alpha^2\psi_{\eta 0,\xi})(1-\nu)/2-3h_{1,\eta}(\beta^2\psi_{\eta 0,\eta}+\nu\alpha\beta\psi_{\xi 0,\xi})\\ P_3=&12\gamma\lambda_1^2w_{0,\tau\tau}-k_1h_{1,\xi}(\alpha^2w_{0,\xi}-\alpha\psi_{\xi 0})-k_1h_{1,\eta}(\beta^2w_{0,\eta}-\beta\psi_{\eta 0}) \end{aligned}$$

Eq.(3.5) is formally the same as the equation of forced vibration. Owing to the properties of eigenvalue γ, in general, there are no solutions in Eq.(3.5) except that λ is a characteristic value. Based on the solvability condition, there must be

$$\int_0^t \int_\Omega \mathbf{u}^{*T} \mathbf{f} \, d\Omega d\tau = 0 \tag{3.6}$$

where $\mathbf{u}^*$ is the solution of equation $\mathbf{L}^T \mathbf{u}^* = 0$, t is the period of vibration, L^T is the transposed operator of the differential operator L. In fact, Eq.(3.6) is the generalized form of Hale Law[4]. Here, it is apparent that

$$\mathbf{u}^* = \{\psi_{\xi 0}, \psi_{\eta 0}, w_0\}^T \tag{3.7}$$

Substituting Eq.(3.7) into Eq.(3.6) and introducing the following parameters

$$A_1 = \int_\Omega \{2k_1 h_1 (\alpha m\pi w_{mn} - u_{mn}) u_{mn} \cos^2 m\pi\xi \sin^2 n\pi\eta - 3h_{1,\xi}(\alpha^2 m\pi u_{mn}$$
$$+ \nu\alpha\beta n\pi v_{mn}) u_{mn} \sin m\pi\xi \cos m\pi\xi \sin^2 n\pi\eta - 3h_{1,\eta}(\beta^2 n\pi u_{mn}$$
$$+ \alpha\beta m\pi v_{mn}) u_{mn} \cos^2 m\pi\xi \sin n\pi\eta \cos n\pi\eta \cdot (1-\nu)/2\} d\xi d\eta$$

$$A_2 = \int_\Omega \{2k_1 h_1 (\beta n\pi w_{mn} - v_{mn}) v_{mn} \sin^2 m\pi\xi \cos^2 n\pi\eta - 3h_{1,\xi}(\alpha\beta n\pi u_{mn}$$
$$+ \alpha^2 m\pi v_{mn}) v_{mn} \cos m\pi\xi \sin m\pi\xi \cos^2 n\pi\eta \cdot (1-\nu)/2 + 3h_{1,\eta}(\beta^2 n\pi v_{mn}$$
$$+ \nu\alpha\beta m\pi u_{mn}) v_{mn} \sin^2 m\pi\xi \cos n\pi\eta \sin n\pi\eta\} d\xi d\eta$$

$$A_3 = -\int_\Omega \{k_1 h_{1,\xi}(\alpha^2 m\pi w_{mn} - \alpha u_{mn}) w_{mn} \sin m\pi\xi \cos m\pi\xi \sin^2 n\pi\eta$$
$$+ k_1 h_{1,\eta}(\beta^2 n\pi w_{mn} - \beta v_{mn}) w_{mn} \sin^2 m\pi\xi \sin n\pi\eta \cos n\pi\eta\} d\xi d\eta$$

$$A_0 = \int_\Omega \gamma \{u_{mn}^2 \cos^2 m\pi\xi \sin^2 n\pi\eta + v_{mn}^2 \sin^2 m\pi\xi \cos^2 n\pi\eta + 12 w_{mn}^2 \sin^2 m\pi\xi \sin^2 n\pi\eta\} d\xi d\eta$$

We have the resulting expression in the following form:

$$\lambda_1^2 = \sum_{k=1}^{3} A_k / A_0 \tag{3.8}$$

For various plates with varying thickness, we can obtain the frequency factor λ_1 in the first order approximation by calculating integral Exp.(3.8).

As an example, the free vibration frequencies of the Mindlin plate with edges simply supported (SSSS) and linear ($h_1 = \xi$) and bilinear ($h_1 = \xi\eta$) thickness distributions are discussed and the results are shown in Table 1 to Table 4. In order to compare them with the existing solutions based on the thin plate theory[16] and Mindlin plate theory[15], we consider the SSSS plate with a taper ratio $\varepsilon = 0.6$, the side thickness ratios of the plate varies from 10 to 500. The numerical results for the fundamental frequency are shown in Fig.1.

IV. Conclusions

It can be seen from the example, the results of the first-order perturbation solution agree very well with the results of [15], and therefore, for Mindlin plates with arbitrary variable thickness, it is not difficult to calculate the free vibration frequencies by means of the method proposed herein.

As can be seen from Fig. 1, the results obtained by the present method are close to the existing solutions, which is obtained by using the collocation method[15], for the case of $a / h_0 > 100$, it may

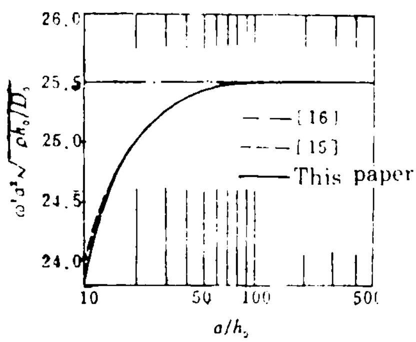

Fig.1 Variation of fundamental frequency with (a/ho) for SSSS plate (b/a 1, [illegible] 0.6)

Table 1 $m=n=1$ $\omega=[D/(\rho a^4h_0)]^{1/2}\cdot a\cdot[1+\beta(h_1)\varepsilon]^{1/2}$ K [illegible]

a/h_0	10	20	30	40	50	60	80	100
a	19.06	19.56	19.66	19.62	19.73	19.74	19.84	20.04
$\alpha(\xi)$	0.933	0.982	0.992	1.00	0.995	0.996	0.988	0.968
$\beta(\xi\eta)$	0.466	0.491	0.496	0.500	0.498	0.498	0.494	0.484

Table 2. $m=n=2$ $\omega=[D/(\rho a^4h_0)]^{1/2}\cdot a\cdot[1+\beta(h_1)\varepsilon]^{1/2}$

a/h_0	10	20	30	40	50	60	80	100
a	69.71	76.23	77.70	78.25	78.50	78.65	77.05	78.92
$\beta(\xi)$	0.786	0.933	0.933	0.982	0.988	0.992	1.04	0.995
$\beta(\xi\eta)$	0.398	0.466	0.466	0.491	0.494	0.496	0.52	0.496

Table 3 $m=n=3$ $\omega=[D/(\rho a^4h_0)]^{1/2}\cdot a\cdot[1+\beta(h_1)\varepsilon]^{1/2}$

a/h_0	10	20	30	40	50	60	80	100
a	139.8	164.8	171.5	174.1	175.4	176.1	176.9	177.9
$\beta(\xi)$	0.637	0.863	0.933	0.960	0.974	0.982	0.992	0.984
$\beta(\xi\eta)$	0.319	0.432	0.466	0.480	0.487	0.491	0.496	0.492

Table 4. $m=n=4$ $\omega=[D/(\rho a^4h_0)]^{1/2}\cdot a\cdot[1+\beta(h_1)\varepsilon]^{1/2}$

a/h_0	10	20	30	40	50	60	80	100
a	219.9	278.8	297.3	304.9	308.7	310.8	313.6	314.0
$\beta(\xi)$	0.514	0.786	0.888	0.932	0.955	0.968	0.986	0.986
$\beta(\xi\eta)$	0.257	0.393	0.444	0.466	0.478	0.484	0.493	0.494

be observed that the present results are almost coincide with the solutions obtained from classical thin plate theroy[16], so this method can avoid the "lock" phenomenon which cannot be overcome by the finite element method[11].

Although the analysis is limited to plates with edges simply supported, the proposed method may be extended, without special treatment, to the stability and vibration problems of Mindlin's plate under initial stress and with other boundary conditions.

In the present paper, since the explicit expressions of the free vibration frequencies are presented relating to the thickness function, it is very convenient to optimize further the thickness for dynamic problems.

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