Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 26 (1—2), (1975), 123—134.

APPROXIMATION BY BERNSTEIN TYPE RATIONAL FUNCTIONS

By
KATALIN BALÁZS (Budapest)

1. The Bernstein polynomials belonging to a function $f(x)$ defined on $[0, 1]$ are, as well-known, the following

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \quad (n = 1, 2, \ldots). \tag{1.1}$$

It is also known that if $f(x)$ is continuous in the closed interval $[0, 1]$ these polynomials converge uniformly to $f(x)$. Bernstein polynomials play an important role in approximation theory and in other fields of mathematics. On account of this a number of mathematicians have dealt with several generalizations of Bernstein polynomials, see e.g. [6].

In the present paper we are going to define Bernstein type rational functions and prove convergence theorems for them. Moreover, we prove an asymptotic approximation theorem and show that the derivatives of Bernstein type rational functions also converge to the derivative of the function.

Let $f(x)$ be a real, single valued function defined in $[0, \infty)$. By Bernstein type rational functions belonging to $f(x)$ we mean the following:

$$R_n(f; x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_n x)^k \quad (n = 1, 2, \ldots), \tag{1.2}$$

where a_n and b_n are suitably chosen real numbers, independent of x.

To compare (1.1) and (1.2) set

$$q_k(x) = x^k(1-x)^{n-k}$$

and

$$r_k(x) = \frac{1}{(1+a_n x)^n}(a_n x)^k \quad (k = 0, 1, 2, \ldots, n)$$

then we have $r_k(x) = q_k(t)$, where $t = \dfrac{a_n x}{1+a_n x}$.

2. Let $R_n(f; x)$ be the functions defined by (1.2) with $a_n = \dfrac{b_n}{n}$, $b_n = n^{2/3}$ $(n = 1, 2, \ldots)$ and let $\omega_{2A}(\delta)$ be the modulus of continuity of the function $f(x)$ in $[0, 2A]$. We shall prove the following

THEOREM I. *Let $f(x)$ be a continuous function defined in $[0, \infty)$ such that $f(x) = O(e^{\alpha x})$ $(x \to \infty)$, for some real number α. Then in any interval $0 \leq x \leq A$ $(A > 0)$ the* *inequality*

$$|f(x)-R_n(f;\,x)| \leqq c_0\left\{\omega_{2A}\left(\frac{1}{n^{1/3}}\right)+\frac{1}{n^{2/3}}\right\} \tag{2.1}$$

is valid if n is sufficiently large, where c_0 is a constant depending on A and α only.

(In what follows $c_i=0, 1, 2, \ldots$ will denote constants independent of n.)

The inequality (2.1) shows that $R_n(f;x)\to f(x)$ when $x\geqq 0$ if $n\to\infty$, and this convergence is uniform in every finite interval $0\leqq x\leqq A$.

We remark that $R_n(f;\,x)$ tends to $f(x)$ with other choices of a_n, and b_n, too. In the theorem such a choice of a_n and b_n was motivated by the fact that these seemed the most suitable with respect to the rate of approximation.

To prove the theorem some lemmas are needed.

LEMMA 2.1. *If $x\geqq 0$, then the following identities hold:*

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k = 1 \quad (n=1,2,\ldots), \tag{2.2}$$

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}(k-b_nx)\binom{n}{k}(a_nx)^k = \frac{-a_nb_nx^2}{1+a_nx}, \tag{2.3}$$

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}(k-b_nx)^2\binom{n}{k}(a_nx)^k = \frac{a_n^2b_n^2x^4+b_nx}{(1+a_nx)^2} \tag{2.4}$$

where $a_n=\dfrac{b_n}{n}$ and $b_n>0$ is an arbitrary real number.

PROOF. (2.2) is evident from the obvious formula

$$\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k = (1+a_nx)^n. \tag{2.5}$$

Differentiating (2.5) by x, then multiplying both sides by x and using $a_n=\dfrac{b_n}{n}$ we have the equality

$$\sum_{k=0}^{n}k\binom{n}{k}(a_nx)^k = b_nx(1+a_nx)^{n-1}. \tag{2.6}$$

Dividing both sides by $(1+a_nx)^n$, subtracting b_nx and using (2.2) we get (2.3).

Again differentiating (2.6) and multiplying by x we obtain

$$\sum_{k=0}^{n}k^2\binom{n}{k}(a_nx)^k = (b_n^2x^2+b_nx)(1+a_nx)^{n-2}. \tag{2.7}$$

Multiplying both sides of (2.5), (2.6) and (2.7) by the factors $b_n^2x^2(1+a_nx)^{-n}$, $-2b_nx(1+a_nx)^{-n}$ and $(1+a_nx)^{-n}$ respectively, and summing up the three equalities, we get (2.4).

LEMMA 2.2. *If $x \geqq 0$ then the inequality*

$$A_n = \frac{1}{(1+a_n x)^n} \sum_{\left|\frac{k}{b_n}-x\right| \geqq \delta} e^{\gamma \frac{k}{b_n}} \binom{n}{k} (a_n x)^k \leqq c_4 \frac{a_n^2 x^4 + \frac{x}{b_n}}{(1+a_n x)^2} \tag{2.8}$$

holds for sufficiently large n where $\delta > 0$ and γ are arbitrary fixed real numbers. $a_n = \frac{b_n}{n} \to 0$, $b_n \to \infty$ if $n \to \infty$.

PROOF. By Lagrange's theorem

$$e^{\frac{\gamma}{b_n}} - 1 = \frac{\gamma}{b_n} e^{\vartheta \frac{\gamma}{b_n}} \leqq \frac{\gamma}{b_n} e^{\frac{\gamma}{b_n}} \leqq c_1 \frac{\gamma}{b_n}$$

for some $0 < \vartheta < 1$, if γ is fixed and $b_n \to \infty$. By this $\left(\text{as } a_n = \frac{b_n}{n}\right)$

$$\left(\frac{1+a_n x e^{\frac{\gamma}{b_n}}}{1+a_n x}\right)^n = \left[\frac{1+a_n x + a_n x\left(e^{\frac{\gamma}{b_n}} - 1\right)}{1+a_n x}\right]^n \leqq \left[1 + \frac{b_n x c_1 \frac{\gamma}{b_n}}{n(1+a_n x)}\right]^n \leqq e^{c_2 x}. \tag{2.9}$$

With the notation $t = x e^{\frac{\gamma}{b_n}}$ we have, if $\left|\frac{k}{b_n} - x\right| \geqq \delta$,

$$\left|\frac{k}{b_n} - t\right| = \left|\frac{k}{b_n} - x e^{\frac{\gamma}{b_n}}\right| = \left|\frac{k}{b_n} - x + x\left(1 - e^{\frac{\gamma}{b_n}}\right)\right| \geqq \tag{2.10}$$

$$\geqq \left|\frac{k}{b_n} - x\right| - |x| \left|1 - e^{\frac{\gamma}{b_n}}\right| \geqq \delta - |x| \cdot \left|1 - e^{\frac{\gamma}{b_n}}\right| \geqq \delta^*$$

for sufficienty large n, where $\delta^* > 0$ is constant. If $\left|\frac{k}{b_n} - x\right| \geqq \delta$, then (2.10) shows that

$$\frac{(k - b_n t)^2}{b_n^2 \delta^*} \geqq 1. \tag{2.11}$$

Using (2.9), (2.11) and summing for all k, the inequality

$$A_n = \frac{1}{(1+a_n x)^n} \sum_{\left|\frac{k}{b_n}-x\right| \geqq \delta} e^{\gamma \frac{k}{b_n}} \binom{n}{k} (a_n x)^k = \tag{2.12}$$

$$= \left(\frac{1+a_n x e^{\frac{\gamma}{b_n}}}{1+a_n x}\right)^n \frac{1}{\left(1+a^n x e^{\frac{\gamma}{b_n}}\right)^n} \sum_{\left|\frac{k}{b_n}-x\right| \geqq \delta} \binom{n}{k} \left(a_n x e^{\frac{\gamma}{b_n}}\right)^k \leqq$$

$$\leqq \frac{e^{c_2 x}}{b_n^2 \delta^* (1+a_n t)^n} \sum_{k=0}^{n} (k - b_n t)^2 \binom{n}{k} (a_n t)^k.$$

is true. By (2.12) and (2.4) applying $t=xe^{\frac{\gamma}{b_n}}$, where $\frac{\gamma}{\beta_n}\to 0$, if $n\to\infty$, we get

$$A_n \leqq c_3 \frac{a_n^2 t^4 + \frac{t}{b_n}}{(1+a_n t)^2} \leqq c_4 \frac{a_n^2 x^4 + \frac{x}{b_n}}{(1+a_n x)^2},$$

which proves the lemma.

COROLLARY. $\lim\limits_{n\to\infty} A_n = 0.$

It is well-known (see e.g. [6]) that if λ and δ are arbitrary positive values, then

$$\omega_{2A}(\lambda\delta) \leqq \omega_{2A}(\delta)(\lambda+1). \tag{2.13}$$

Now we prove the convergence theorem. By (1.2) and (2.2)

$$\Delta_n(f;x) = |f(x)-R_n(f;x)| \leqq \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \left|f(x)-f\left(\frac{k}{b_n}\right)\right| \binom{n}{k} (a_n x)^k \leqq \tag{2.14}$$

$$\leqq \frac{1}{(1+a_n x)^n}\Big(\sum_{\frac{k}{b_n}\leqq 2A} + \sum_{\frac{k}{b_n}>2A}\Big) = S_1+S_2.$$

We obtain by (2.13)

$$\left|f(x)-f\left(\frac{k}{b_n}\right)\right| \leqq \omega_{2A}\left(\left|x-\frac{k}{b_n}\right|\right) = \omega_{2A}\left(\frac{1}{n^\beta}\cdot n^\beta\left|x-\frac{k}{b_n}\right|\right) \leqq \tag{2.15}$$

$$\leqq \omega_{2A}\left(\frac{1}{n^\beta}\right)\left(n^\beta\left|x-\frac{k}{b_n}\right|+1\right).$$

We shall choose the number $\beta>0$ suitably later on. By (2.14), (2.15) and (2.2)

(2.16)
$$S_1 \leqq \omega_{2A}\left(\frac{1}{n^\beta}\right)\frac{n^\beta}{(1+a_n x)^n}\sum_{\frac{k}{b_n}\leqq 2A}\left|x-\frac{k}{b_n}\right|\binom{n}{k}(a_n x)^k + \omega_{2A}\left(\frac{1}{n^\beta}\right) = S_1' + \omega_{2A}\left(\frac{1}{n^\beta}\right).$$

Using the Schwarz inequality, then considering (2.4) and (2.2) we obtain

$$S_1' \leqq \omega_{2A}\left(\frac{1}{n^\beta}\right)\frac{n^\beta}{b_n}\left\{\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}(b_n x-k)^2\binom{n}{k}(a_n x)^k \times\right. \tag{2.17}$$

$$\left.\times\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k\right\}^{1/2} \leqq \omega_{2A}\left(\frac{1}{n^\beta}\right)\frac{n^\beta}{b_n}\left\{\frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2}\right\}^{1/2}.$$

Assuming $\beta=\frac{1}{3}$ and $b_n=n^{2/3}$, in this case $a_n=\frac{b_n}{n}=n^{-1/3}$, then by (2.16) and (2.17) we have

$$S_1 \leqq \omega_{2A}\left(\frac{1}{n^{1/3}}\right)[(x^4+x)^{1/2}+1]. \tag{2.18}$$

Since $f(x)=O(e^{\alpha x})$ $(x\to\infty,\ \alpha$ fixed), the estimation of S_2 is an easy consequence of Lemma 2.2, if δ was chosen small enough:

$$(2.19)\qquad S_2=\frac{1}{(1+a_nx)^n}\sum_{\frac{k}{b_n}>2A} c_5 e^{\alpha\frac{k}{b_n}}\binom{n}{k}(a_nx)^k\leqq$$

$$\leqq\frac{1}{(1+a_nx)^n}\sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta} c_6 e^{\alpha\frac{k}{b_n}}\binom{n}{k}(a_nx)^k\leqq c_7\left\{a_n^2x^4+\frac{x}{b_n}\right\}\leqq\frac{c_7}{n^{2/3}}(x^4+x).$$

Now, on the basis of (2.18) and (2.19) the inequality (2.14) may be written in the following way

$$(2.20)\qquad \Delta_n(f;\,x)\leqq c_0\left\{\omega_{2A}\left(\frac{1}{n^{1/3}}\right)+\frac{1}{n^{2/3}}\right\}\quad(0\leqq x\leqq A).$$

This establishes the proof of Theorem I.

3. E. V. Voronovskaya proved in [9] for the Bernstein polynomials that

$$(3.1)\qquad B_n(f;\,x)=f(x)+\frac{f''(x)}{2n}x(1-x)+\frac{\varrho_n}{n},$$

if $f(x)$ is bounded in $[0, 1]$, and has a finite second derivative at the point x. In (3.1) ϱ_n tends to zero with $n\to\infty$.

In this part of the paper we prove an asymptotic approximation theorem similar to (3.1) for Bernstein type rational functions defined in (1.2).

Theorem II. *Let $f(t)$ be a function defined in $[0,\infty)$, for which $f(t)=O(e^{\alpha t})$ $(t\to\infty$, α is a fixed real number), then at each point $t=x$, in which $f''(t)$ exists and is finite*

$$(3.2)\qquad R_n(f;\,x)=f(x)+a_nf'(x)g_1(x)+a_nf''(x)g_2(x)+a_n\varrho_n,$$

where $\varrho_n\to 0$, $a_n=\dfrac{b_n}{n}\to 0$ *and* $\dfrac{n^{1/2}}{b_n}\to 0$, *if* $n\to\infty$, *moreover*

$$g_1(x)=\frac{-x^2}{1+a_nx},\quad g_2(x)=\frac{a_nb_nx^4+\dfrac{x}{a_n}}{2b_n(1+a_nx)^2}.$$

We remark that satisfying the conditions concerning a_n and b_n, $g_1(x)$ and $g_2(x)$ remain under a limit depending only on x, so Theorem II is indeed an asymptotic approximation theorem.

It is immediately seen that $R_n(f;x)$, similarly to $B_n(f;x)$, is a linear operator. For certain linear operators asymptotic approximation theorems similar to our Theorem II were proved by a number of mathematicians, see e.g. O. Szász [8], J. Gróf [2], R. G. Mamedov [4], M. W. Müller [5] and F. Schurer [7].

PROOF. By the conditions of the theorem, $f''(x)$ is finite, thus we may write

$$f(t)=f(x)+f'(x)(t-x)+\left[\frac{f''(x)}{2}+\lambda(t)(t-x)^2\right], \tag{3.3}$$

where $\lambda(t)\to 0$, if $t\to x$. By reason of this

$$f\left(\frac{k}{b_n}\right)=f(x)+f'(x)\left(\frac{k}{b_n}-x\right)+\left[\frac{f''(x)}{2}+\lambda\left(\frac{k}{b_n}\right)\right]\left(\frac{k}{b_n}-x\right)^2. \tag{3.4}$$

Substituting this expression in $R_n(f; x)$ and taking into account the identities (2.2), (2.3) and (2.4) we get

$$\begin{aligned} R_n(f; x) &= \frac{f(x)}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k+\frac{f'(x)}{b_n(1+a_nx)^n}\sum_{k=0}^{n}(k-b_nx)\binom{n}{k}(a_nx)^k+ \\ &+\frac{f''(x)}{2b_n^2(1+a_nx)^n}\sum_{k=0}^{n}(k-b_nx)^2\binom{n}{k}(a_nx)^k+ \\ &+\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\lambda\left(\frac{k}{b_n}\right)\left(\frac{k}{b_n}-x\right)^2\binom{n}{k}(a_nx)^k= \\ &=f(x)+f'(x)\frac{-a_nx^2}{1+a_nx}+f''(x)\frac{a_n^2b_nx^4+x}{2b_n(1+a_nx)^2}+r_n, \end{aligned} \tag{3.5}$$

where

$$r_n=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\lambda\left(\frac{k}{b_n}\right)\left(\frac{k}{b_n}-x\right)^2\binom{n}{k}(a_nx)^k. \tag{3.6}$$

Now given an arbitrary small number $\varepsilon>0$, let us choose $\delta>0$ so small, that assuming $|t-x|<\delta$, then $|\lambda(t)|<\varepsilon$ be satisfied. With such a δ, decompose the sum (3.6) into two parts:

$$r_n=\sum\nolimits_1+\sum\nolimits_2 \tag{3.7}$$

where Σ_1 contains the members where $\left|\frac{k}{b_n}-x\right|<\delta$, and Σ_2 the ones where $\left|\frac{k}{b_n}-x\right|\geqq\delta$. By the property of $\lambda(t)$ and by (2.4) we obtain

$$\left|\sum\nolimits_1\right|<\varepsilon\frac{a_n^2x^4+\dfrac{x}{b_n}}{(1+a_nx)^2}. \tag{3.8}$$

Now we give an upper estimation for $|\Sigma_2|$. (Henceforth c_i, $i=8, 9, \ldots$ are positive numbers depending only on x and α.) By $f(t)=O(e^{\alpha t})$ ($t\to\infty$, α fixed) it follows from

(3.4) for some c_8

(3.9) $$\left|\lambda\left(\frac{k}{b_n}\right)\left(\frac{k}{b_n}-x\right)^2\right| = \left|f\left(\frac{k}{b_n}\right)-f(x)-f'(x)\left(\frac{k}{b_n}-x\right)-\right.$$
$$\left.-\frac{f''(x)}{2}\left(\frac{k}{b_n}-x\right)^2\right| < c_8 e^{\alpha\frac{k}{b_n}} \quad (k = 0, 1, 2, \dots, n).$$

Using (3.6), (3.7), (3.9) and (2.8) we get

(3.10) $$\left|\sum\nolimits_2\right| < c_9 \frac{a_n^2 x^4 + \dfrac{x}{b_n}}{(1+a_n x)^2}.$$

Let now

(3.11) $$\varrho_n \overset{\text{def}}{=} \frac{r_n}{a_n}.$$

By (3.11), (3.7), (3.8) and (3.10) the relation

(3.12) $$|\varrho_n| < \varepsilon \frac{a_n^2 x^4 + \dfrac{x}{b_n}}{a_n(1+a_n x)^2} + c_9 \frac{a_n^2 x^4 + \dfrac{x}{b_n}}{a_n(1+a_n x)^2} = c_{10}\left(a_n x^4 + \frac{x}{a_n b_n}\right) \to 0 \quad (n \to \infty)$$

holds, because $a_n = \dfrac{b_n}{n} \to 0$ and $\dfrac{n^{1/2}}{b_n} \to 0$, if $n \to \infty$. (3.5), (3.6), (3.11) and (3.12) give the proof of Theorem II.

4. In this part we prove a convergence theorem concerning the derivative of $R_n(f; x)$. The derivative by x of the rational function defined in (1.2) belonging to $f(x)$ is denoted by $R_n'(f; x)$.

THEOREM III. *Let $f(t)$ be a function defined in $[0, \infty)$, for which $f(t) = O(e^{\alpha t})$ ($t \to \infty$, α is a fixed, real number). If $f'(t)$ exists at the point $t = x$, then*

(4.1) $$R_n'(f; x) \to f'(x) \quad \textit{if} \quad n \to \infty,$$

where $a_n = \dfrac{b_n}{n} \to 0$, *and* $b_n = n^{2/3}$.

To prove Theorem III we need some lemmas.

LEMMA 4.1. *In the case $x \geqq 0$, for the rational functions*

(4.2) $$S_m(x) \overset{\text{def}}{=} \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} (k - b_n x)^m \binom{n}{k} (a_n x)^k \quad (m = 0, 1, 2, \dots),$$

the recurrent formula

(4.3) $$S_{m+1}(x) = x\left[S_m'(x) + m b_n S_{m-1}(x) - \frac{a_n b_n x}{1+a_n x} S_m(x)\right] \quad (m = 1, 2, \dots),$$

holds, if $a_n = \dfrac{b_n}{n}$.

PROOF. Differentiating $S_m(x)$, we get

$$S_m'(x) = \frac{-1}{(1+a_nx)^n} \sum_{k=0}^{n} (k-b_nx)^{m-1} \binom{n}{k} (a_nx)^k mb_n +$$

$$+\frac{1}{(1+a_nx)^n} \sum_{k=0}^{n} (k-b_nx)^{m-1} \binom{n}{k} (a_nx)^{k-1}(k-b_nx)\times$$

$$\times\left(a_nk - a_nb_nx + a_nb_nx - \frac{a_nb_nx}{1+a_nx}\right).$$

From this by appropriate transcription we have

$$S_m'(x) = -mb_nS_{m-1}(x) + \frac{1}{x} S_{m+1}(x) + \left(b_n - \frac{b_n}{(1+a_nx)}\right) S_m(x),$$

which gives (4.3).

LEMMA 4.2. *The rational function $S_m(x)$ defined in* (4.2) *is identical with*

$$S_m(x) = \frac{1}{(1+a_nx)^m} \sum_{i=0}^{m} A_{m,i}(x) b_n^i \quad (m = 0, 1, 2, \ldots), \tag{4.4}$$

where the polynomials $A_{m,i}(x)$ are independent of b_n, and their coefficients are polynomials of a_n.

PROOF. The proof is carried out by induction concerning m. $S_0(x)\equiv 1$, $S_1(x)$ is identical with (2.3):

$$S_1(x) = \frac{1}{(1+a_nx)^n} \sum_{k=0}^{n} (k-b_nx)\cdot \binom{n}{k} (a_nx)^k = \frac{-a_nb_nx^2}{1+a_nx} = \frac{A_{1,0}(x) + A_{1,1}(x)b_n}{1+a_nx},$$

where $A_{1,0}(x) = 0$, $A_{1,1}(x) = -a_nx^2$.

Suppose now that (4.4) is true for m, and prove it also for $m+1$. By the inductive assumption and by (4.3)

$$S_{m+1}(x) = x\left[\frac{\sum_{i=0}^{m} A_{m,i}'(x) b_n^i (1+a_nx)^m - \sum_{i=0}^{m} A_{m,i}(x) b_n^i m a_n (1+a_nx)^{m-1}}{(1+a_nx)^{2m}} + \right.$$

$$\left. + \frac{mb_n}{(1+a_nx)^{m-1}} \sum_{i=0}^{m-1} A_{m-1,i}(x) b_n^i - \frac{a_nb_nx}{(1+a_nx)^{m+1}} \sum_{i=0}^{m} A_{m,i}(x) b_n^i \right],$$

and hence by rearrangement we obtain

$$S_{m+1}(x) = \frac{1}{(1+a_nx)^{m+1}} \sum_{i=0}^{m+1} A_{m+1,i}(x) b_n^i,$$

where the polynomials $A_{m+1,i}(x)$ obviously satisfy the statement of the lemma.

LEMMA 4.3. *In every interval* $0 \leqq x \leqq A < \infty$, *the inequality*

$$(4.5)\quad |S_m(x)| = \frac{1}{(1+a_n x)^n}\left|\sum_{k=0}^{n}(k-b_n x)^m \binom{n}{k}(a_n x)^k\right| \leqq K_m(A)\, a_n^m b_n^m \quad (m = 0, 1, \ldots)$$

holds for sufficiently large n, where $K_m(A)$ *is a number depending only on* A, $a_n = \frac{b_n}{n}$, $b_n = n^{2/3}$.

PROOF. By (4.4)

$$(4.6)\qquad S_m(x) = \frac{g_m(x)}{(1+a_n x)^m} \quad (m = 0, 1, 2, \ldots),$$

where $g_m(x) = \sum_{i=0}^{m} A_{m,i}(x) b_n^i$. We show that $g_m(x)$ is a polynomial of x of degree $2m$. By (4.5)

$$(4.7)\qquad S_m(x) = \frac{P_{n+m}(x)}{(1+a_n x)^n} \quad (m = 0, 1, 2, \ldots),$$

where the degree of $P_{n+m}(x)$ is $n+m$ exactly. We have got (4.7) by multiplying the numerator and the denominator of (4.6) by $(1+a_n x)^{n-m}$, and this is possible only when the degree of $g_m(x)$ is $2m$ exactly. After these we show (4.5) by induction on m. In the case $m=0$ $S_0(x) \equiv 1$, thus (4.5) is trivially fulfilled. In the case $m=1$ in the sense of (2.3)

$$|S_1(x)| = \frac{a_n b_n x^2}{1+a_n x} \leqq A^2 a_n b_n = K_1(A) a_n b_n.$$

Now suppose that (4.5) is true for a natural number m, and prove it for $m+1$. By (4.6) and (4.3)

$$(4.8)\qquad S_{m+1}(x) = x\left[\frac{g_m'(x)(1+a_n x)^m + g_m(x)(1+a_n x)^{m-1} m a_n}{(1+a_n x)^{2m}} + \right.$$

$$\left. + \frac{m b_n g_{m-1}(x)}{(1+a_n x)^{m-1}} - \frac{a_n b_n x g_m(x)}{(1+a_n x)^{m+1}}\right] =$$

$$= \frac{g_m'(x) x}{(1+a_n x)^m} + \frac{g_m(x)(x m a_n - a_n b_n x^2)}{(1+a_n x)^{m+1}} + \frac{g_{m-1}(x) m b_n x}{(1+a_n x)^{m-1}}.$$

The Markov inequality concerning polynomials states that if a polynomial $Q(x)$ of degree k remains between $-C$ and C in an interval $[a, b]$, then

$$|Q'(x)| \leqq \frac{2Ck^2}{b-a} \quad \text{if} \quad a \leqq x \leqq b.$$

We apply the Markov inequality for $g_m(x)$. By the inductive assumption

$$(4.9)\qquad |g_m(x)| \leqq K_m(A) a_n^m b_n^m (1+a_n x)^m,$$

thus

$$|g_m'(x)| \leqq \frac{8m^2 K_m(A) a_n^m b_n^m}{A} (1+a_n x)^m \leqq K_m'(A) a_n^m b_n^m, \tag{4.10}$$

if $0 \leqq x \leqq A$. Using (4.8), (4.9) and (4.10) and taking out $a_n^{m+1} b_n^{m+1}$ it follows

$$|S_{m+1}(x)| \leqq a_n^{m+1} b_n^{m+1} \left[\frac{K_m'(A)}{a_n b_n (1+a_n x)^m} + \frac{K_m(A)\left(\dfrac{xm}{b_n} - x^2\right)}{(1+a_n x)^{m+1}} + \right.$$

$$\left. + \frac{K_{m-1}(A) m x}{a_n^2 b_n (1+a_n x)^{m-1}} \right] \leqq K_{m+1}(A) a_n^{m+1} b_n^{m+1}.$$

PROOF OF THEOREM III. Consider first the case when $x > 0$. By (1.2) and (2.2)

$$R_n'(f;\, x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) \binom{n}{k} a_n (a_n x)^{k-1} \left(k - \frac{b_n x}{1+a_n x}\right). \tag{4.11}$$

From this we have

$$R_n'(f;\, x) = \frac{1}{x(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n x)^k (k - b_n x) + \tag{4.12}$$

$$+ \frac{a_n b_n x}{(1+a_n x)^{n+1}} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n x)^k.$$

Since $f'(x)$ exists and is finite, so

$$f\left(\frac{k}{b_n}\right) = f(x) + \left[f'(x) + \lambda\left(\frac{k}{b_n}\right)\right] \left(\frac{k}{b_n} - x\right), \tag{4.13}$$

where $\lambda(t) \to 0$, if $t \to x$. Taking into consideration (4.12), (4.13), (2.4) and (2.3) it follows by simple modification that

$$R_n'(f;\, x) = f'(x) \frac{1}{(1+a_n x)^2} + \Delta_n, \tag{4.14}$$

where

$$\Delta_n = \frac{b_n}{x(1+a_n x)^n} \sum_{k=0}^{n} \lambda\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n x)^k \left(\frac{k}{b_n} - x\right)^2 + \tag{4.15}$$

$$+ \frac{a_n b_n x}{(1+a_n x)^{n+1}} \sum_{k=0}^{n} \lambda\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n x)^k \left(\frac{k}{b_n} - x\right) =$$

$$= \frac{b_n}{x(1+a_n x)^n} \Big\{ \sum_{\left|\frac{k}{b_n} - x\right| < \delta} + \sum_{\left|\frac{k}{b_n} - x\right| \geqq \delta} \Big\} + \frac{a_n b_n x}{(1+a_n x)^{n+1}} \Big\{ \sum_{\left|\frac{k}{b_n} - x\right| < \delta} + \sum_{\left|\frac{k}{b_n} - x\right| \geqq \delta} \Big\} =$$

$$= A_1 + A_2 + A_3 + A_4.$$

Let $\varepsilon>0$ be an arbitrary but fixed number, then by $\lambda(t)\to 0$ $(t\to x)$ there exists a number $\delta>0$ for which $|\lambda(t)|<\varepsilon$ is valid, if $|t-x|<\delta$, and so by (4.15) and (2.4)

$$|A_1| \leqq \varepsilon \frac{a_n^2 b_n x^3+1}{(1+a_n x)^2} < c_{17}\varepsilon \tag{4.16}$$

for sufficiently large n. Similarly, in sense of (4.15) and (2.3)

$$|A_3| \leqq \varepsilon \frac{a_n^2 b_n x^3}{(1+a_n x)^2} < c_{18}\varepsilon. \tag{4.17}$$

Since $f(t)=O(e^{\alpha t})$ $(t\to\infty)$, thus by (4.13)

$$\left|\lambda\left(\frac{k}{b_n}\right)\right| < c_{11} e^{\alpha\frac{k}{b_n}} \quad \text{if} \quad \left|\frac{k}{b_n}-x\right| > \delta. \tag{4.18}$$

We get from (4.15) and (4.18)

$$|A_2| \leqq \frac{c_{11}}{b_n x(1+a_n x)^n} \sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta} e^{\alpha\frac{k}{b_n}} \binom{n}{k} (a_n x)^k (k-b_n x)^2.$$

We apply the Cauchy—Schwarz inequality:

$$|A_2| \leqq \frac{c_{11}}{b_n x} \sqrt{\frac{1}{(1+a_n x)^n} \sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta} e^{2\alpha\frac{k}{b_n}} \binom{n}{k} (a_n x)^k} \times$$

$$\times \sqrt{\frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k (k-b_n x)^4}.$$

Using (2.8) if $\gamma=2\alpha$ and (4.5) we have

$$|A_2| \leqq \frac{c_{12}}{b_n x} \sqrt{\frac{a_n^2 x^4+\frac{x}{b_n}}{(1+a_n x)^2}} \sqrt{K_4(A) a_n^4 b_n^4} < c_{13}(a_n^3 b_n + a_n^2 b_n^{1/2}) \to 0 \quad (n\to\infty). \tag{4.19}$$

It follows from (4.13), that

$$\left|\lambda\left(\frac{k}{b_n}\right)\left(\frac{k}{b_n}-x\right)\right| < c_{14} e^{\alpha\frac{k}{b_n}} \quad \text{if} \quad \left|\frac{k}{b_n}-x\right| \geqq \delta. \tag{4.20}$$

We can estimate $|A_4|$ using (4.20) and (2.8):

$$|A_4| = \left|\frac{a_n b_n x}{(1+a_n x)^{n+1}} \sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta} \lambda\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n x)^k \left(\frac{k}{b_n}-x\right)\right| \leqq \tag{4.21}$$

$$\leqq \frac{a_n b_n x c_{14} c_4 \left(a_n^2 x^4 + \frac{x}{b_n}\right)}{(1+a_n x)^3} \leqq c_{15}(a_n^3 b_n + a_n) \to 0 \quad (n\to\infty).$$

We can see from (4.16), (4.17), (4.19) and (4.21)

$$|\Delta_n| \leqq \sum_{i=1}^{4} |A_i| \leqq c_{16}\varepsilon$$

for sufficiently large n, thus it follows from (4.14)

$$R_n'(f;\, x) \to f'(x) \quad \text{if} \quad n \to \infty \quad \text{and} \quad x > 0.$$

Let now $x=0$, then by the condition of Theorem III $f(+0)$ exists and is finite, and so by the definition of $R_n(f;\, x)$ in (1.2) we have

$$R'(f;\, x)|_{x=0} = \left\{ \frac{1}{(1+a_n x)^n} f(0) + nf\left(\frac{1}{b_n}\right) \frac{a_n x}{(1+a_n x)^n} \right\}'\Bigg|_{x=0} =$$

$$= b_n\left[-f(0) + f\left(\frac{1}{b_n}\right)\right] \to f'(0),$$

as $b_n \to \infty$, if $n \to \infty$. This completes the proof of Theorem III.

References

[1] K. BALÁZS, Függvényapproximáció Bernstein-típusú racionális törtfüggvényekkel és valószínűségszámítási vonatkozásai, *MTA III. Oszt. Közl.* (To appear in Hungarian.)

[2] J. GRÓF, A Szász Ottó féle operátor approximációs tulajdonságairól, *MTA III. Oszt. Közl.*, **20** (1971), 35—44 (in Hungarian).

[3] G. G. LORENTZ, *Bernstein polynomials* (Toronto, 1953).

[4] Р. Г. Мамедов, Асимптотические значение приближения дифференцируемых функций линейными положительными операторами, *Докл. Акад. Наук СССР*, **128** (1953), 471—474.

[5] M. W. MÜLLER, On asymptotic approximation theorems for sequences of linear positive operators, *Approximation Theory, Proceedings of a Symposium held at Lancaster*, July 1969 (London, 1970).

[6] И. П. Натансон, *Конструктивная теория функций* (Москва—Ленинград, 1949).

[7] F. SCHURER, *On linear positive operators in approximation theory*, Thesis, Techn. Univ. Delft, 1965.

[8] O. SZÁSZ, Generalizations of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bureau of Standards*, **45** (1950).

[9] Е. В. Вороновская, Определение асимптотического вида приближения функций полиномами С. Н. Бернштейна, *Докл. Акад. Наук* (A), (1932), 79—85.

(Received November 29, 1973)

KARL MARX UNIVERSITY
DEPARTMENT FOR COMPUTER SCIENCE
1093 BUDAPEST
DIMITROV TÉR 8
HUNGARY