Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 26 (1—2), (1975), 41—52.

FREE INVERSE SEMIGROUPS ARE NOT FINITELY PRESENTABLE

By

B. M. SCHEIN (Saratov)

In the memory of Professor A. Kertész

Free inverse semigroups became a subject of intense studies in the last few years. Their existence was proved long ago: as algebras with two operations (binary multiplication and unary involution) inverse semigroups may be characterized by a finite system of identities, i.e. they form a variety of algebras [10]. Therefore, free inverse semigroups do exist.

A construction of a free algebra in a variety of algebras (as a quotient algebra of an absolutely free word algebra) is well known. Free inverse semigroups in such a form were considered by V. V. VAGNER [14] who found certain properties of such semigroups. A monogenic free inverse semigroup (i.e. a free inverse semigroup with one generator) was described by L. M. GLUSKIN [2]. Later this semigroup was described by H. E. SCHEIBLICH in a slightly different form [8]. The most essential progress in this direction was made in a paper [9] by H. E. SCHEIBLICH who described arbitrary free inverse semigroups. A relevant paper [1] by C. EBERHART and J. SELDEN should be mentioned. There are papers on some special properties of free inverse semigroups. N. R. REILLY described free inverse subsemigroups of free inverse semigroups [7], results in this direction were obtained also by W. D. MUNN and L. O'CARROLL.

Let $\mathcal{F}\mathcal{I}_X$ denote the free inverse semigroup with the set X of free generators. A monogenic free inverse semigroup will be denoted $\mathcal{F}\mathcal{I}_1$. Time and then we will write $\mathcal{F}\mathcal{I}$ instead of $\mathcal{F}\mathcal{I}_X$. We do not consider $\mathcal{F}\mathcal{I}_\varnothing$, a one-element inverse semigroup.

This paper contains two main results. The first one coincides with the title, the second consists in a description of free inverse semigroups (if a free inverse semigroup is presented as a quotient algebra of a free involuted semigroup, then each element of $\mathcal{F}\mathcal{I}$ is a class of equivalent words, we give a canonical form of the words). Certain corollaries with properties of free inverse semigroups follow.

All results of the paper were reported by the author at a meeting of the seminar "Semigroups" in the Saratov State University on October 21, 1971.

THEOREM 1. *Free inverse semigroups are not finitely presentable either as semigroups or as involuted semigroups.*

The proof of the theorem is subdivided in a series of lemmas.

LEMMA 1. *A semigroup F generated by two elements u and v satisfying the infinite list of defining relations:* I) $uvu=u$, $vuv=v$; $A_{m,n}$) $u^m v^{m+n} u^n = v^n u^{m+n} v^m$ *for all natural m and n, is a free inverse semigroup.*

PROOF. F is inverse by a lemma from [12]. Since the defining relations $A_{m,n}$ are valid in any inverse semigroup generated by two mutually inverse elements u and v, F is free. F is a monogenic free inverse semigroup generated by u in the variety of all inverse semigroups considered as involuted semigroups (i.e. as algebras with two operations).

LEMMA 2. *The set of defining relations of F given in Lemma 1 is not equivalent to any finite subset of these defining relations.*

PROOF. Consider two partial transformations u and v of a finite set $A=\{0, 1, 2, \dots, n\}$:

$$u=\begin{pmatrix}0 & 1 & 2\dots n-2 & n-1\\ 1 & 2 & 3\dots n-1 & n\end{pmatrix},\quad v=\begin{pmatrix}0 & 1 & 2\dots n-1 & n\\ 0 & 0 & 1\dots n-2 & n-1\end{pmatrix}.$$

Here v is defined on the whole set A and u is defined on every element of A except n. It is easy to verify that $uvu=u$ and $vuv=v$ (here xy denotes the partial transformation obtained when y acts after x). One can compute without difficulty that for $k\leqq n$ $u^k=\begin{pmatrix}0 & 1 & 2 & \dots & n-k\\ k & k+1 & k+2 & \dots & n\end{pmatrix}$ and for $k>n$ u^k is the empty partial transformation $\varnothing$. Analogously we may verify that for $k\leqq n$ $v^k=\begin{pmatrix}0 & 1 & \dots & k & k+1 & k+2 & \dots & n\\ 0 & 0 & \dots & 0 & 1 & 2 & \dots & n-k\end{pmatrix}$. If $i>n$ or $j>n$ then both sides of the defining relation $A_{i,j}$ contain $\varnothing$ as a factor, therefore, $A_{i,j}$ holds. Now let $i\leqq n$ and $j\leqq n$. Then

$$u^iv^i=\begin{pmatrix}0 & 1 & \dots n-i\\ i & i+1 & \dots\ n\end{pmatrix}\begin{pmatrix}0\dots i & i+1 & \dots & n\\ 0\dots 0 & 1 & \dots & n-i\end{pmatrix}=\begin{pmatrix}0 & 1\dots n-i\\ 0 & 1\dots n-i\end{pmatrix},$$

$$v^ju^j=\begin{pmatrix}0\dots j & j+1 & \dots & n\\ 0\dots 0 & 1 & \dots & n-j\end{pmatrix}\begin{pmatrix}0 & 1 & \dots n-j\\ j & j+1 & \dots\ n\end{pmatrix}=\begin{pmatrix}0 & 1\dots j & j+1\dots n\\ j & j\dots j & j+1\dots n\end{pmatrix}.$$

$A_{i,j}$ means that partial transformations u^iv^i and v^ju^j commute. We may compute now that

$$u^iv^{i+j}u^j=\begin{pmatrix}0\dots n-i\\ j\dots\ j\end{pmatrix}\quad\text{if}\quad n-i<j,$$

and

$$u^iv^{i+j}u^j=\begin{pmatrix}0\dots j & j+1\dots n-i\\ j\dots j & j+1\dots n-i\end{pmatrix}\quad\text{if}\quad n-i\geqq j.$$

Analogously, $v^ju^{i+j}v^i=\varnothing$ if $i+j>n$, and

$$v^ju^{i+j}v^i=\begin{pmatrix}0 & 1\dots j & j+1\dots n-i\\ j & j\dots j & j+1\dots n-i\end{pmatrix}\quad\text{if}\quad n-i\leqq j.$$

Therefore, $A_{i,j}$ is satisfied whenever $i+j\leqq n$ and is not satisfied otherwise.

Let S_n denote the semigroup of partial transformations of A generated by u and v. We have seen that the defining relation $A_{i,j}$ does not hold in S_n if and only if $i,j\leqq n<i+j$.

Suppose now that the defining relations given in Lemma 1 are equivalent to a finite subset B of these relations. Let $n=\max\{i+j\colon A_{i,j}\in B\}$ and if $\{i+j\colon A_{i,j}\in B\}=\varnothing$ let n be any natural number. If $A_{i,j}\in B$ then $i+j\leqq n$; it follows that $A_{i,j}$ holds in

S_n. Therefore, all defining relations from B hold in S_n. Therefore, all the relations given in Lemma 1 hold in S_n. However, $A_{1,n}$ does not hold in S_n. This contradiction completes the proof.

LEMMA 3. *The inverse semigroup F is not finitely presentable either as a semigroup or as an involuted semigroup.*

PROOF. 1. Consider F as a semigroup. Suppose F is finitely presentable over a set X of generators by means of defining relations R. We may replace X by $\{u, v\}$ and every relation from R is substituted by a relation resulting from replacement of all occurrences of elements of X by their expressions as products of u and v. Thus, F is definable over the alphabet $\{u, v\}$ by a finite set D of defining relations. Therefore, all the relations from D can be deduced from the defining relations given in Lemma 1. During such an inference one cannot use but a finite number of defining relations among those given in Lemma 1. Since the relations I and $A_{i,j}$, in their own turn, may be deduced from D, the defining relations from Lemma 1 are equivalent to their own finite subset which contradicts Lemma 2. Thus, the semigroup F is not finitely presentable.

2. Now consider F as an involuted semigroup. Let S be a semigroup generated by two elements u and v satisfying the defining relations I and $A_{i,j}$ for $i+j\leqq n$. Since $A_{1,n}$ does not follow from these relations, $A_{1,n}$ does not hold in S. For every word α in the alphabet $\{u, v\}$ define a word α^{-1} inductively: $u^{-1}=v$, $v^{-1}=u$, if β^{-1} and γ^{-1} are defined then $(\beta\gamma)^{-1}=\gamma^{-1}\beta^{-1}$. E.g. $(uvvuv)^{-1}=uvuuv$. Clearly, $(\alpha^{-1})^{-1}=\alpha$ and $(\alpha\beta)^{-1}=\beta^{-1}\alpha^{-1}$ for all words α, β. Suppose the words α and β represent the same element of S. Then α^{-1} and β^{-1} also represent equal elements of S. In effect, all defining relations of S are invariant under the involution $^{-1}$: the relations from I are transformed one into the other, $A_{i,j}$ is transformed into $A_{j,i}$ if $^{-1}$ is applied to both parts of $A_{i,j}$. Since $A_{i,j}$ and $A_{j,i}$ are valid or not valid in S simultaneously, every chain of elementary transformations which transforms α into β turns into a chain of elementary transformations transforming α^{-1} onto β^{-1} if the involution $^{-1}$ is applied to all terms of the first chain. Thus, α^{-1} and β^{-1} represent the same element of S.

It follows that S may be considered as an involuted semigroup with one generator u satisfying the defining relations J: $uu^{-1}u=u$ and $B_{i,j}$: $u^iu^{-i-j}u^j=u^{-j}u^{i+j}u^{-i}$ for $i+j\leqq n$. Since S does not satisfy $A_{1,n}$, the relation $B_{1,n}$ does not follow from J and $B_{i,j}$ for $i+j\leqq n$ in the class of involuted semigroups. Therefore, defining relations J and $B_{i,j}$ for all i and j, which define a monogenic free inverse semigroup are not equivalent to a finite subset of these relations. To prove that F is not finitely presentable we proceed now along the same lines as in case 1 where F was considered as a semigroup.

Let $\mathcal{F}\mathcal{I}_X$ be a free inverse semigroup. Suppose it is finitely presentable with a finite set Y of generators by means of defining relations R. We may express each element of Y as a product of elements of X. Thus without loss of generality we may suppose $Y=X$. If X is infinite then some elements of X do not occur in the defining relations from R, therefore, $\mathcal{F}\mathcal{I}_X$ cannot be an inverse semigroup (if $x\in X$ does not occur in R then $xx^{-1}x=x$ does not hold in $\mathcal{F}\mathcal{I}_X$). Therefore, if $\mathcal{F}\mathcal{I}_X$ is finitely presentable, then X should be finite.

Now add to R a finite set of all defining relations of the form $x_i=x_j$ for all $x_i, x_j\in X$, $i\neq j$. Then we obtain an inverse semigroup F_0 which is a homomorphic image of $\mathcal{FI}_X$. Clearly, F_0 is a monogenic inverse semigroup (since all the generators X of $\mathcal{FI}_X$ are identified in F_0). Clearly, F_0 is a free inverse semigroup since $\mathcal{FI}_X$ is free. Thus, F_0 is a monogenic free inverse semigroup and F_0 is finitely presentable which contradicts Lemma 3. Thus, $\mathcal{FI}_X$ is not finitely presentable. This argument is valid both for semigroups and involuted semigroups.

Theorem 1 is proved.

REMARK. Defining relations given in Lemma 1 are not independent. E.g., the relations I, $A_{1,1}$, $A_{1,2}$, $A_{2,1}$ and $A_{3,1}$ imply $A_{2,2}$ and $A_{3,2}$:

$$u^2v^4u^2 = u^2v(vuv)(vuv)vu^2 = u^2v^2(uv^2u)v^2u^2 = u^2v^2(vu^2v)v^2u^2 =$$
$$= u(uv^3u^2)v^3u^2 = u(v^2u^3v)v^3u^2 = (uv^2u)u^2v^4u^2 = (vu^2v)u^2v^4u^2 = vu(uvu)uv^4u^2 =$$
$$= vuuuv^4u^2 = v(u^3v^4u)u = v(vu^4v^3)u = v^2u^2(u^2v^3u) = v^2u^2(vu^3v^2) =$$
$$= v^2u(uvu)u^2v^2 = v^2uuu^2v^2 = v^2u^4v^2.$$

The relation $A_{3,1}$ may be deduced analogously.

It would be interesting to study interdependence of the defining relations given in Lemma 1 and, if possible, to find a set of independent defining relations for a monogenic free inverse semigroup.

Now we give a construction for $\mathcal{FI}_X$. Let $X^{-1}=\{x^{-1}: x\in X\}$ and suppose the alphabets X and X^{-1} are disjoint. Let $Y=X\cup X^{-1}$ and $\mathcal{FS}_Y$ be a free semigroup over Y. The elements of $\mathcal{FS}_Y$ are all non-empty words over Y. Clearly, $\mathcal{FS}_Y$ admits an involution defined inductively: $(x)^{-1}=x^{-1}$, $(x^{-1})^{-1}=x$, $(\alpha\beta)^{-1}=\beta^{-1}\alpha^{-1}$ for all $\alpha, \beta\in\mathcal{FS}_Y$. Together with this involution, $\mathcal{FS}_Y$ is an involuted semigroup: a free involuted semigroup $\mathcal{FIn}_X$ with the set X of generators. We visualize $\mathcal{FI}_X$ as a quotient semigroup of $\mathcal{FIn}_X$. Thus, the elements of $\mathcal{FI}_X$ are classes of equivalent words over Y, we say that equivalent words *represent* the same element of $\mathcal{FI}_X$.

A word α over Y is called *reduced* if it is empty or if it does not contain occurrences xx^{-1} and $x^{-1}x$ for $x\in X$. A word $w\in\mathcal{FIn}_X$ is called *left canonical* if $w=(a_1a_1^{-1})\dots(a_na_n^{-1})a$ where $a, a_1, \dots, a_n$ are reduced words, for every i the word a_i is not a beginning of the word a or of the word a_j for $j\neq i$. In particular, the words $a_1, \dots, a_n$ are nonempty, a may be empty, n is any nonnegative integer (if $n=0$, then $w=a$, in this case a cannot be empty). Speaking of left canonical words, we will omit "left" since no other types of canonical words occur in this paper.

Note that for words $v, w\in\mathcal{FIn}_X$, $v=w$ means that v and w are the same word; $v\equiv w$ means that v and w represent the same element of $\mathcal{FI}_X$.

If $w=(a_1a_1^{-1})\dots(a_na_n^{-1})a$ is a canonical word then the *prefix* of w is the word $\Pr(w)=(a_1a_1^{-1})\dots(a_na_n^{-1})$, the words $a_1, \dots, a_n$ are called *components* of $\Pr(w)$, the *root* of w is the word $R(w)=a$.

Let $\mathcal{FG}_X$ be a free group with the set X of generators. The elements of $\mathcal{FG}_X$ are all reduced words from $\mathcal{FIn}_X$ and the empty word, the operations of multiplication and involution in $\mathcal{FG}_X$ are usual [3]. Then $R: w\to R(w)$ is a mapping of the set of all canonical words onto $\mathcal{FG}_X$.

Now we give an algorithm which transforms every word from $\mathcal{FIn}_X$ into a canonical word. Let $w_0\in\mathcal{FIn}_X$.

Algorithm. 1. Read w_0 from left to right. If w_0 is reduced, stop. Otherwise, pass on to 2.

2. Let $w=byy^{-1}c$ for some $y\in Y$ and this is the first occurrence of yy^{-1} for some $y\in Y$ in w_0. Find the longest beginning d of c such that d^{-1} is an end of b. Then $b=ed^{-1}$ and $c=df$ for some words e and f. Let $a_1=by$, $w_1=ef$. Now pass on to 3.

3. Apply 1 and 2 to w_1.

Applying 1—3 to w_0 we obtain successively the words a_1, w_1; then the words $a_2, w_2; \dots$, after a finite number of steps we obtain the words a_n, w_n such that w_n is a reduced word. This follows from the fact that $|w_0|>|w_1|$ where $|w|$ denotes the length of the word w. Notice that the words $a_1, \dots, a_n$ are reduced and non-empty.

In the list $\{a_1, \dots, a_n\}$ check every word: if a_1 is a beginning of some of the words $\{a_2, \dots, a_n, w_n\}$, omit a_1; otherwise, retain it. Pass on to a_2 in the new list (with omitted or retained a_1). After a finite number of steps one obtains a list of words where no word is a beginning of another word or of w_n.

Suppose $\{b_1, \dots, b_m\}$ is such a list. Then the final canonical word is $C(w_0)=$ $=(b_1b_1^{-1})\dots(b_mb_m^{-1})w_n$. Clearly, $C(w_0)$ is a canonical word. If $m=0$ then $C(w_0)=w_n$.

EXAMPLE. Let $w_0=x_1x_2^{-1}x_2x_2^{-1}x_3x_3^{-1}$. Then $a_1=x_1x_2^{-1}$, $w_1=x_1x_2^{-1}x_2x_3^{-1}$; $a_2=x_1x_2^{-1}x_3$, $w_2=x_1x_2^{-1}$. Now w_2 is a reduced word. The word a_1 is a beginning of a_2 and of w_2 and should be omitted. Now $C(w_0)=(x_1x_2^{-1}x_3)(x_3^{-1}x_2x_1^{-1})x_1x_2^{-1}$.

Now we prove that $w_0\equiv C(w_0)$ for every $w_0\in\mathscr{F}\mathscr{I}n_X$. Clearly,

$$w_0 = byy^{-1}c = ed^{-1}yy^{-1}df \equiv ee^{-1}ed^{-1}yy^{-1}df \equiv ed^{-1}yy^{-1}de^{-1}ef =$$
$$= byy^{-1}b^{-1}ef = a_1a_1^{-1}w_1.$$

Analogously, $w_2\equiv a_2a_2^{-1}w_2$, whence, $w_0\equiv(a_1a_1^{-1})(a_2a_2^{-1})w_2$. After a finite number of steps we obtain $w_0\equiv(a_1a_1^{-1})\dots(a_na_n^{-1})w_n$. Suppose a_i is a beginning of a_j for $j\neq i$. Then $a_j=a_ig$ for some (possibly, empty) word g. Now

$$(a_1a_1^{-1})\dots(a_na_n^{-1})w_n \equiv (a_ia_i^{-1})(a_ja_j^{-1})(a_1a_1^{-1})\dots(a_na_n^{-1})w_n.$$

Here we have written $(a_ia_i^{-1})$ and $(a_ja_j^{-1})$ at the very beginning. Now

$$a_ia_i^{-1}a_ja_j^{-1} = a_ia_i^{-1}a_igg^{-1}a_i^{-1} \equiv a_igg^{-1}a_i^{-1} = (a_ja_j^{-1}),$$

therefore,

$$(a_1a_1^{-1})\dots(a_na_n^{-1})w_n \equiv (a_1a_1^{-1})\dots(a_{i-1}a_{i-1}^{-1})(a_{i+1}a_{i+1}^{-1})\dots(a_na_n^{-1})w_n,$$

i.e. the factor $(a_ia_i^{-1})$ may be omitted. Analogously, $(a_ia_i^{-1})$ may be omitted in case when a_i is a beginning of w_n. Omitting all factors $(a_ia_i^{-1})$ where a_i is a beginning of some other word a_j or w_n, we obtain $C(w_0)$. Thus, $w_0\equiv C(w_0)$.

LEMMA 4. *Every element of $\mathscr{F}\mathscr{I}_X$ may be represented by a canonical word.*

PROOF. Every element of $\mathscr{F}\mathscr{I}_X$ may be represented by a word $w\in\mathscr{F}\mathscr{I}n_X$. Since $w\equiv C(w)$, the element of $\mathscr{F}\mathscr{I}_X$ may be represented by $C(w)$, the latter word having the canonical form.

LEMMA 5. *Let $w=(a_1a_1^{-1})\dots(a_ma_m^{-1})a$ and $v=(b_1b_1^{-1})\dots(b_nb_n^{-1})b$ be canonical words. Then $w\equiv v$ if and only if $\{a_1, \dots, a_m\}=\{b_1, \dots, b_n\}$ and $a=b$, i.e. two canonical* *words represent the same element of $\mathscr{F}\mathscr{I}_X$ if and only if the components of prefixes of these words coincide and the roots of these words coincide.*

PROOF. The "if" part is trivial. Now suppose $w\equiv v$. Let $\bar{w}$ be the element of $\mathscr{F}\mathscr{I}_X$ represented by w. Then $\bar{w}=\bar{v}$. Let Δ_X be the identical mapping of X onto the set X of generators of $\mathscr{F}\mathscr{G}_X$. This mapping can be extended to a uniquely defined homomorphism $f\colon \mathscr{F}\mathscr{I}_X\to\mathscr{F}\mathscr{G}_X$. Clearly, f is surjective. Δ_X can be extended to a homomorphism $g\colon \mathscr{F}\mathscr{I}n_X\to\mathscr{F}\mathscr{G}_X$. Obviously, $g(w)$ is a reduced form of a word w (i.e. $g(w)$ may be obtained from w after all occurences of yy^{-1} for $y\in Y$ are omitted from w and from all words obtained from w in this way). If $h\colon \mathscr{F}\mathscr{I}n_X\to\mathscr{F}\mathscr{I}_X$ is the natural homomorphism, then $f\circ h=g$. It follows that $g(w)=g(v)\leftrightarrow f(h(w))=f(h(v))\leftrightarrow$ $\leftrightarrow f(\bar{w})=f(\bar{v})$ for all $w, v\in\mathscr{F}\mathscr{I}n_X$. Clearly, $R(C(w))$ is a reduced form of w. Since every word has a uniquely determined reduced form [3], $g(w)=R(C(w))$, and since w and v are canonical words, $C(w)=w$ and $C(v)=v$. Therefore, $g(w)=g(v)$, i.e., $R(w)=R(v)$.

Let R be the set of all nonempty reduced words. A nonempty finite subset $A\subset R$ is called *closed* [9] if for every $w\in A$ and every nonempty beginning v of w $v\in A$. Let E be the set of all closed subsets of R. Since the union of two closed subsets is closed, E is a semilattice with multiplication $\cup$. Let T_E denote the inverse semigroup of all isomorphisms between principal ideals of E [5]. For every $x\in X$, $\{x\}\in E$. Let $(\{x\})$ denote the principal ideal of E generated by $\{x\}$. Then $(\{x\})=$ $=\{A\in E\colon x\in A\}$. Define an isomorphism f_x of $(\{x\})$ onto $(\{x^{-1}\})$: let $A\in(\{x\})$, i.e. $x\in A$; then $f_x(A)$ consists of the word x^{-1} and all words of the form $g(x^{-1}w)$ for $w\in A$, $w\neq x$. It is a matter of straightforward computation to check that $f_x(A)\in(\{x^{-1}\})$ and f_a is an isomorphism. Thus, $f_x\in T_E$. Let $\bar{f}$ denote the mapping of X into T_E such that $\bar{f}(x)=f_x$. Then $\bar{f}$ may be extended in a unique way up to a homomorphism $\bar{f}\colon \mathscr{F}\mathscr{I}_X\to T_E$ (we denote this homomorphism by the same letter $\bar{f}$ as the mapping $X\to T_E$).

Let $u=(a_1a_1^{-1})\dots(a_na_n^{-1})$ be a canonical word with an empty root. Let I_u denote the set of all $A\in E$ such that $\{a_1, \dots, a_n\}\subset A$. Then I_u is an ideal of E. Let Δ_{I_u} denote the identical automorphism of this ideal. Then $\Delta_{I_u}\in T_E$. It can be computed that $f(\bar{u})=\Delta_{I_u}$ (we omit this straightforward but tedious computation).

Let $t\in\mathscr{F}\mathscr{I}n_X$ represent an idempotent of $\mathscr{F}\mathscr{I}_X$. Then $g(t)$ is the identity of $\mathscr{F}\mathscr{G}_X$, i.e. $g(t)$ is an empty word. If t is a canonical word, then $R(t)=g(t)$, i.e. t has the empty root. Thus, a canonical word represents an idempotent of $\mathscr{F}\mathscr{I}_X$ if and only it it has the empty root.

Let $u=(c_1c_1^{-1})\dots(c_pc_p^{-1})$ and $t=(d_1d_1^{-1})\dots(d_qd_q^{-1})$ be two canonical words representing idempotents of $\mathscr{F}\mathscr{I}_X$. If $u\equiv t$, i.e. $\bar{u}=\bar{t}$, then $\Delta_{I_u}=\Delta_{I_t}$. It follows that $I_u=I_t$, $\{c_1, \dots, c_p\}=\{d_1, \dots, d_q\}$. In particular, $p=q$.

Since $w\equiv v$, $(a_1a_1^{-1})\dots(a_ma_m^{-1})(aa^{-1})\equiv(b_1b_1^{-1})\dots(b_nb_n^{-1})(aa^{-1})$ (we have already proved that $a=b$). However, the latter words need not be canonical. If they are canonical, then $\{a_1, \dots, a_m, a\}=\{b_1, \dots, b_n, a\}$, therefore, $\{a_1, \dots, a_m\}=\{b_1, \dots, b_n\}$.

If a is a beginning of one of the words $\{a_1, \dots, a_m\}$, then $(a_1a_1^{-1})\dots(a_ma_m^{-1})(aa^{-1})\equiv$ $\equiv(a_1a_1^{-1})\dots(a_ma_m^{-1})$. Suppose the word $(b_1b_1^{-1})\dots(b_nb_n^{-1})(aa^{-1})$ is canonical. Then $\{a_1, \dots, a_m\}=\{b_1, \dots, b_n, a\}$. Let a be a beginning of a_i and $a_i=b_j$. Then a is a beginning of b_j, a contradiction. Therefore, a_i is a beginning of a which contradicts the supposition that $(a_1a_1^{-1})\dots(a_ma_m^{-1})a$ is canonical. Thus, a is a beginning of one of the words $\{b_1, \dots, b_n\}$ and $(b_1b_1^{-1})\dots(b_nb_n^{-1})(aa^{-1})\equiv(b_1b_1^{-1})\dots(b_nb_n^{-1})$. It follows that $\{a_1, \dots, a_m\}=\{b_1, \dots, b_n\}$. Lemma 4 is proved.

Let $\mathscr{C}_X$ be the set of all canonical words. Then $\mathscr{C}_X$ is a cross-section of $\mathscr{F}\mathscr{I}_X$, i.e. every element of $\mathscr{F}\mathscr{I}_X$ is represented by a uniquely determined canonical word. Thus, there exists a natural bijection of $\mathscr{F}\mathscr{I}_X$ onto $\mathscr{C}_X$. We have proved

THEOREM 2. *Let $\mathscr{C}_X$ be the set of all canonical words over the alphabet $X\cup X^{-1}$. For $w, v\in\mathscr{C}_X$ define $w=v$ if and only if $R(w)=R(v)$ and $P(w)$ possesses the same components as $P(v)$. Define $w\cdot v=\mathscr{C}(wv)$, $w^{-1}=\mathscr{C}(w^{-1})$. Then $\mathscr{C}_X$ is a free inverse semigroup isomorphic to $\mathscr{F}\mathscr{I}_X$.*

REMARK. Let $w=(a_1a_1^{-1})\dots(a_ma_m^{-1})a$ and $v=(b_1b_1^{-1})\dots(b_nb_n^{-1})b$ be canonical words. Then

$$w^{-1}=\big(g(a^{-1}a_1)g(a_1^{-1}a)\big)\dots\big(g(a^{-1}a_m)g(a_m^{-1}a)\big)a^{-1}\in\mathscr{C}_X$$

is the inverse for w in $\mathscr{C}_X$.

$$w\cdot v\equiv(a_1a_1^{-1})\dots(a_ma_m^{-1})\big(g(ab_1)g(b_1^{-1}a^{-1})\big)\dots\big(g(ab_n)g(b_n^{-1}a^{-1})\big)(aa^{-1})g(ab).$$

To obtain a canonical form of $w\cdot v$ one needs to delete those factors $(a_ia_i^{-1})$ and $\big(g(ab_i)g(b_i^{-1}a^{-1})\big)$, (aa^{-1}) whose components are beginnings of the other components.

As corollaries to Theorem 2 we obtain some properties of free inverse semigroups. Now we identify $\mathscr{F}\mathscr{I}_X$ with $\mathscr{C}_X$ and consider the elements of $\mathscr{F}\mathscr{I}_X$ as canonical words.

COROLLARY 1. *A canonical word w is an idempotent of $\mathscr{F}\mathscr{I}_X$ if and only if $R(w)$ is an empty word, i.e. if $P(w)=w$.*

The proof is incorporated in the proof of Lemma 4.

COROLLARY 2. *R is the maximum group homomorphism of $\mathscr{F}\mathscr{I}_X$, it maps $\mathscr{F}\mathscr{I}_X$ onto the free group $\mathscr{F}\mathscr{G}_X$; if 1 is the identity of $\mathscr{F}\mathscr{G}_X$ then R^{-1} (1) is the set of all idempotents of $\mathscr{F}\mathscr{I}_X$.*

PROOF. The same as for Corollary 1.

COROLLARY 3 ([9]). *The semilattice $E(\mathscr{F}\mathscr{I}_X)$ of all idempotents of $\mathscr{F}\mathscr{I}_X$ is isomorphic to the semilattice E of all closed subsets of nonempty reduced words.*

PROOF. The isomorphism between $E(\mathscr{F}\mathscr{I}_X)$ and E maps every idempotent $w\in(\mathscr{F}\mathscr{I}_X)$ onto the closed set consisting of all nonempty beginnings of all components of $P(w)$.

Let $\leqq$ denote the canonical (natural) order relation on $\mathscr{F}\mathscr{I}_X$. The same symbol $\leqq$ will denote the canonical order of the inverse semigroup $\mathscr{F}\mathscr{I}_X^1$ which is the free inverse semigroup $\mathscr{F}\mathscr{I}_X$ with identity adjoined. If $w\leqq v$, then w is called a *minorant* of v and v is called a *majorant* of w. For every $w=(a_1a_1^{-1})\dots(a_na_n^{-1})a\in\mathscr{C}_X$ let $W(w)=n$ denote the *weight* of w.

COROLLARY 4. *For every $w, v\in\mathscr{F}\mathscr{I}_X$ $w\leqq v$ if and only if $R(w)=R(v)$ and each component of $P(v)$ is a beginning of a (necessarily uniquely defined) component of $P(w)$. In particular, $W(v)\leqq W(w)$.*

PROOF. Let $w\leqq v$. Then $w=uv$ for an idempotent $u\in\mathscr{F}\mathscr{I}_X$. By Corollary 1, $R(w)=R(uv)=R(v)$. Each component of $P(v)$ is a beginning of a component of

$P(w)$ or of a component of $P(u)$, the latter being the case, the component of $P(u)$ containing a component of $P(v)$ as a beginning should be a beginning of a component of $P(w)$. Two different components of $P(v)$ are not one a beginning of the other, therefore, they cannot be beginnings of the same component of $P(w)$. It follows that $P(v)$ cannot possess more components than $P(w)$, i.e. $W(v) \leqq W(w)$.

Now let $R(w)=R(v)$ and each component of $P(v)$ be a beginning of a component of $P(w)$. By a straightforward computation we obtain $w=ww^{-1}v$, i.e. $w \leqq v$.

COROLLARY 5. *Majorants of idempotents of a free inverse semigroup are idempotents.*

COROLLARY 6. *Every element* w *of* $\mathcal{F}\mathcal{I}_X$ *possesses no more than* $(|a_1|+1)\dots$ $\dots(|a_n|+1)$ *different majorants if* $w \notin E(\mathcal{F}\mathcal{I}_X)$ *and no more than* $(|a_1|+1)\dots(|a_n|+1)-1$ *different majorants if* w *is an idempotent. In particular, as an ordered set,* $\mathcal{F}\mathcal{I}_X$ *satisfies the ascending chain condition.*

COROLLARY 7. *Free inverse semigroups satisfy the ascending chain condition for principal right ideals.*

PROOF. The latter condition is equivalent to the ascending chain condition for the semilattice $E(\mathcal{F}\mathcal{I}_X)$.

Corollary 7 has been obtained independently by H. E. SCHEIBLICH.

COROLLARY 8 ([14]). *Every element of* $\mathcal{F}\mathcal{I}_X^1$ *has a uniquely defined maximal majorant (namely, if* $w \in \mathcal{F}\mathcal{I}_X^1$, *then* $R(w)$ *is the maximal majorant of* w*).* $\mathcal{F}\mathcal{I}_X$ *and* $\mathcal{F}\mathcal{I}_X^1$ *are generated by their maximal elements.*

Let $\mathcal{F}\mathcal{G}_X$ be prefix ordered (i.e. for $w, v \in \mathcal{F}\mathcal{G}_X$ $w \leqq v$ means that w is a beginning of v). Then $\mathcal{F}\mathcal{G}_X$ is a tree semilattice.

COROLLARY 9. $E(\mathcal{F}\mathcal{I}_X^1)$ *is a free semilattice over a partially ordered set dual to* $\mathcal{F}\mathcal{G}_X$.

PROOF. By Corollary 3, $E(\mathcal{F}\mathcal{I}_X^1)$ is isomorphic to E^1 which is the set of all closed subsets including an empty subset. Corollary 8 follows by Theorem 4.2 from [6].

COROLLARY 10. *An element* w *of* $\mathcal{F}\mathcal{I}_X$ *is maximal if and only if* $W(w)=0$.

Let σ denote the smallest group congruence on $\mathcal{F}\mathcal{I}_X^1$. It is known [13] that $w \equiv v(\sigma)$ if and only if w and v have a common minorant. It follows that $w \equiv v(\sigma) \leftrightarrow$ $\leftrightarrow R(w)=R(v)$. In particular, every σ-class contains the largest element: if w belongs to a σ-class then $R(w)$ is the largest element of this σ-class [14]. Thus, $\mathcal{F}\mathcal{I}_X^1$ is an F-inverse semigroup in the sense of [4]. This fact has been proved independently by L. O'CARROLL.

COROLLARY 11. *Every* σ*-class of* $\mathcal{F}\mathcal{I}_X^1$ *is a distributive lattice relative to* $\leqq$. *In particular,* $E(\mathcal{F}\mathcal{I}_X^1)$ *is a distributive lattice. Moreover, for every* $u, v, w \in \mathcal{F}\mathcal{I}_X^1$ *such that* $u \equiv v(\sigma)$ *we have* $(u \vee v)w = uw \vee vw$ *and* $w(u \vee v) = wu \vee wv$. *Here* $\vee$ *denotes the operation of forming the least upper bound. Elements from different* σ*-classes are incomparable relative to* $\leqq$.

PROOF. If $w, v \in \mathcal{F}\mathcal{I}_X^1$ and $w \leqq v$ or $v \leqq w$ then, by Corollary 4, $R(w)=R(v)$, i.e. $w \equiv v(\sigma)$. Therefore, two elements from different σ-classes cannot be comparable relative to $\leqq$. Now let $w \equiv v(\sigma)$. Then $R(w)=R(v)$ is a common majorant of w, v. However, two elements of an inverse semigroup which possess a common majorant, possess also the greatest lower bound [13], i.e. $w \wedge v$ exists. Let $\{a_1, \dots, a_n\}$ be a list of all components of $P(w)$. Let c_i denote the longest beginning of a_i which is also a beginning of some component of $P(v)$. Clearly, such c_i always exists (c_i may be empty). Let $u=C((c_1 c_1^{-1}) \dots (c_n c_n^{-1}) R(w))$. By Corollary 4, $w \leqq u$ and $v \leqq u$. Suppose now $w \leqq t$ and $v \leqq t$ for some $t=(d_1 d_1^{-1}) \dots (d_k d_k^{-1}) a$. By Corollary 4, every d_i is a beginning of some $a_{j(i)}$ and of some component of $P(v)$. It follows that d_i is a beginning of $c_{j(i)}$. By Corollary 4, $u \leqq t$, i.e. $u=v \vee w$. Therefore, every σ-class of $\mathcal{F}\mathcal{I}_X^1$ is a lattice.

Let A_a denote the σ-class containing a word $a \in \mathcal{F}\mathcal{G}_X$. Then a is the largest element of the lattice A_a. It is well known that the set A_a of all minorants of a is order-isomorphic to anyone of the ordered sets $A_{aa^{-1}}$, $A_{a^{-1}}$, $A_{a^{-1}a}$ (here $A_{aa^{-1}}$ denotes the set of all minorants of $aa^{-1} \in \mathcal{F}\mathcal{I}_X^1$). Clearly, the lattice $A_{aa^{-1}}$ is a principal ideal of the lattice $A_1=E(\mathcal{F}\mathcal{I}_X^1)$, therefore, the lattice A_a is distributive if A_1 is. Clearly, the distributivity of A_1 follows from the identities $(u \vee v)w=uw \vee vw$ and $w(u \vee v)=wu \vee wv$ for all $u, v, w \in \mathcal{F}\mathcal{I}_X^1$, $u \equiv v(\sigma)$. On the other hand, these identities follow from distributivity of A_1 [11]. We give an independent proof of this fact here.

Clearly, $u^{-1}u \vee v^{-1}v \leqq (u \vee v)^{-1}(u \vee v)$. Since $u=uu^{-1}u \leqq (u \vee v)u^{-1}u \leqq (u \vee v)(u^{-1}u \vee v^{-1}v)$ and, analogously, $v \leqq (u \vee v)(u^{-1}u \vee v^{-1}v)$, we obtain $u \vee v \leqq (u \vee v)(u^{-1}u \vee v^{-1}v)$, whence,

$$(u \vee v)^{-1}(u \vee v) \leqq (u \vee v)^{-1}(u \vee v)(u^{-1}u \vee v^{-1}v) \leqq u^{-1}u \vee v^{-1}v.$$

Thus,

$$(u \vee v)^{-1}(u \vee v) = u^{-1}u \vee v^{-1}v.$$

Since $uw \leqq (u \vee v)w$ and $vw \leqq (u \vee v)w$, we obtain $uw \vee vw \leqq (u \vee v)w$. Using distributivity of A_1, we obtain

$$\big((u \vee v)w\big)^{-1}(u \vee v)w = w^{-1}(u \vee v)^{-1}(u \vee v)w = w^{-1}(u^{-1}u \vee v^{-1}v)w =$$

$$= w^{-1}\big(ww^{-1}(u^{-1}u \vee v^{-1}v)ww^{-1}\big)w = w^{-1}(ww^{-1}u^{-1}uww^{-1} \vee ww^{-1}v^{-1}vww^{-1})w \leqq$$

$$\leqq w^{-1}\big(w(w^{-1}u^{-1}uw \vee w^{-1}v^{-1}vw)w^{-1}\big)w \leqq$$

$$\leqq w^{-1}u^{-1}uw \vee w^{-1}v^{-1}vw = (uw \vee vw)^{-1}(uw \vee vw).$$

If for two elements g and h of an inverse semigroup $g \leqq h$ and $hh^{-1} \leqq gg^{-1}$ hold, then

$$h = hh^{-1}h \leqq gg^{-1}h = g, \text{ i.e., } g = h.$$

Therefore, $(u \vee v)w = uw \vee vw$. Now

$$w(u \vee v) = \big((u \vee v)^{-1}w^{-1}\big)^{-1} =$$

$$= \big((u^{-1} \vee v^{-1})w^{-1}\big)^{-1} = (u^{-1}w^{-1} \vee v^{-1}w^{-1})^{-1} = wu \vee wv.$$

COROLLARY 12. *An element w of $\mathcal{F}\mathcal{I}_X$ is an idempotent if and only if* $ww^{-1} = w^{-1}w$.

PROOF. The "only if" part is trivial. To prove the "if" part suppose

$$ww^{-1} = w^{-1}w \text{ and } w = (a_1a_1^{-1}) \dots (a_na_n^{-1})a \in \mathscr{C}_X.$$

Then

$$ww^{-1} \equiv (a_1a_1^{-1}) \dots (a_na_n^{-1})(aa^{-1})$$

and

$$w^{-1}w \equiv a^{-1}(a_1a_1^{-1}) \dots (a_na_n^{-1})a \equiv$$

$$\equiv \big(g(a^{-1}a_1)g(a_1^{-1}a)\big) \dots \big(g(a^{-1}a_n)g(a_n^{-1}a)\big)(a^{-1}a).$$

It may be verified by straightforward computation that the latter word is canonical if and only if all the words $\{a_1, \dots, a_n, a\}$ begin with the same letter; otherwise, the canonical equivalent of the latter word is $\big(g(a^{-1}a_1)g(a_1^{-1}a)\big)\dots\big(g(a^{-1}a_n)g(a_n^{-1}a)\big)$.

Case 1. Let the words $\{a_1, \dots, a_n, a\}$ begin with the same letter. Then $W(ww^{-1}) = W(w^{-1}w) = n+1$, therefore, the words $\{a_1, \dots, a_n, a\}$ are the components of ww^{-1} and the words $\{g(a^{-1}a_1), \dots, g(a^{-1}a_n), a^{-1}\}$ are the components of $w^{-1}w$. Thus, the two sets of components coincide. If $a^{-1} = a$ then $g(a^2)$ is an empty word; it follows that a is empty and w is an idempotent.

Now let $a^{-1} = a_i$. Then $g(a^{-1}a_i) = g(a^{-2}) = a_j$ for some j, $g(a^{-3}) = g(a^{-1}a_j) = a_k$ for some k etc. After a finite number of steps we obtain $g(a^{-p}) = g(a^{-1}a_q) = a$, i.e. $g(a^{p+1})$ is an empty word. Therefore, a is empty and w is an idempotent.

Case 2. Let the words $\{a_1, \dots, a_n, a\}$ do not begin with the same letter. Then the components of $w^{-1}w$ are $\{g(a^{-1}a_1), \dots, g(a^{-1}a_n)\}$ and $W(ww^{-1}) = W(w^{-1}w) = n$. Therefore, a is a beginning of some of the words $\{a_1, \dots, a_n\}$, say, of the word a_i, and the components of ww^{-1} are $\{a_1, \dots, a_n\}$. Now $a_i = ab$ for a nonempty word b, it follows that $ab = a_i = g(a^{-1}a_j)$, i.e. $g(a^2b) = a_j$. Therefore $g(a^2b) = a_j = g(a^{-1}a_k)$ for some k. It follows that $g(a^3b) = g(aa^{-1}a_k) = g(a_k) = a_k = g(a^{-1}a_p)$ for some p. Proceeding along these lines we obtain after a finite number of steps that $g(a^qb) = g(a^mb)$ for different q and m, i.e., in case $q > m$, $g(a^{q-m}b) = g(b) = b$. It follows that $g(a^{q-m})$ is an empty word, i.e. a is empty and w is an indempotent.

COROLLARY 13. *$\mathscr{F}\mathscr{I}_X$ does not contain nontrivial subgroups.*

PROOF. Suppose G is a nontrivial subgroup of $\mathscr{F}\mathscr{I}_X$ and $w \in G$, w is not an idempotent. Then ww^{-1} and $w^{-1}w$ are the identity of G, therefore $ww^{-1} = w^{-1}w$ and, by Corollary 12, w is an idempotent, a contradiction.

COROLLARY 14 ([7]). *The Green equivalence $\mathscr{H}$ on $\mathscr{F}\mathscr{I}_X$ is the identical equivalence.*

PROOF. Suppose $w \equiv v(\mathscr{H})$. Then $ww^{-1} = vv^{-1}$ and $w^{-1}w = v^{-1}v$. Now

$$(wv^{-1})(wv^{-1})^{-1} = wv^{-1}vw^{-1} = ww^{-1}ww^{-1} = ww^{-1} =$$

$$= vv^{-1} = vv^{-1}vv^{-1} = vw^{-1}wv^{-1} = (wv^{-1})^{-1}(wv^{-1}).$$

By Corollary 12, wv^{-1} is an idempotent. Therefore, $wv^{-1} = (wv^{-1})^{-1} = vw^{-1}$. We obtain

$$w = ww^{-1}ww^{-1}w = wv^{-1}vw^{-1}w = (wv^{-1})(wv^{-1})^{-1}w =$$

$$= (wv^{-1})^{-1}w = vw^{-1}w = vv^{-1}v = v.$$

Thus, $\mathscr{H}$ is the identity on $\mathscr{F}\mathscr{I}_X$.

We could find the other Green equivalences on $\mathscr{F}\mathscr{I}_X$, however, we omit their description which is a matter of simple computations. Notice that equivalence classes of all Green equivalences on $\mathscr{F}\mathscr{I}_X$ are finite. In particular, $\mathscr{F}\mathscr{I}_X$ does not contain a bicyclic subsemigroup (such a subsemigroup is included into a single $\mathscr{D}$-class, the latter class should be infinite, a contradiction). Every nonidempotent element of $\mathscr{F}\mathscr{I}_X$ generates an infinite subsemigroup of $\mathscr{F}\mathscr{I}_X$ (since the homomorphic image of such an element in $\mathscr{F}\mathscr{G}_X$ is not identity and generates an infinite subsemigroup of $\mathscr{F}\mathscr{G}_X$). Therefore, $\mathscr{F}\mathscr{I}_X$ does not contain nontrivial Brandt subsemigroups.

Since $\mathscr{F}\mathscr{I}_X$ does not contain bicyclic subsemigroups, the Green equivalences $\mathscr{D}$ and $\mathscr{J}$ coincide on $\mathscr{F}\mathscr{I}_X$. In effect, $\mathscr{D}\subset\mathscr{J}$ in any semigroup. Let an inverse semigroup S do not contain bicyclic subsemigroups. If $s, t\in S$ and $s\equiv t(\mathscr{J})$, then $t=xsy$ and $s=utv$ for some $u, v, x, y\in S$. Therefore, $s=uxsyv$. Let $w=ss^{-1}uxss^{-1}$. It is easy to compute that $ww^{-1}=ss^{-1}$ and $w^{-1}w\leqq ss^{-1}$. If $w^{-1}w<ss^{-1}$ then $w^{-1}w<ww^{-1}$ and the element w generates a bicyclic inverse subsemigroup of S, a contradiction. Therefore, $w^{-1}w=ss^{-1}$. Analogously, $zz^{-1}=z^{-1}z=s^{-1}s$ for $z=s^{-1}syvs^{-1}s$. It follows that $s\equiv xs(\mathscr{L})$ and $xs\equiv xsy(\mathscr{R})$, whence $s\equiv xsy(\mathscr{D})$, i.e. $s\equiv t(\mathscr{D})$. Therefore, $\mathscr{J}\subset\mathscr{D}$, i.e. $\mathscr{D}=\mathscr{J}$.

Note added on January 31, 1973. After the paper had been submitted for publication, we received the following relevant papers [15—17]. In [15] a new construction for $\mathscr{F}\mathscr{I}_X$ (in terms of "birooted word trees") is given. It is proved also that $\mathscr{F}\mathscr{I}_X$ is Hopfian if X is finite, it is residually finite and completely semisimple. In [16] the first part of our Corollary 11 is proved, there are given new proofs for a number of other results on $\mathscr{F}\mathscr{I}_X$ (e.g. those from [7, 14]), it is proved also that $\mathscr{F}\mathscr{I}_X$ is Hopfian (i.e. endomorphisms onto are automorphisms) if X is finite. In [17] a construction for $\mathscr{F}\mathscr{I}_X$ is given which is rather alike to ours. Of course, all the constructions for $\mathscr{F}\mathscr{I}_X$ (namely, those of [9], [15], [17] and from this paper) could be deduced one from another. E.g., a construction quite similar to ours has been actually deduced from that of [15] in [18]. Every construction has merits and drawbacks of its own. E.g., the Green relations on $\mathscr{F}\mathscr{I}_X$ seem to have the simplest expressions when the constructions [9] and [15] are used. The fact that the word problem for $\mathscr{F}\mathscr{I}_X$ is soluble (first proved in [15]) follows immediately from our construction (since an algorithm transforming every word to a canonical form is given).

References

[1] C. Eberhart and J. Selden, One-parameter inverse semigroups, *Trans. Amer. Math. Soc.*, **168** (1972), 53—66.

[2] Л. М. Глускин, Об инверсных полугруппах, *Записки механико-математического факультета Харьковского государственного университета и Харьковского математического общества*, **28** (1961), серия 4, 103—110.

[3] M. Hall, Jr., *The theory of groups*, Macmillan (New York, 1959).

[4] R. McFadden and L. O'Carroll, *F*-inverse semigroups, *Proc. London Math., Soc.*, **22** (1971), 652—666.

[5] W. D. Munn, Uniform semilattices and bisimple inverse semigroups, *Quarterly J. Math. Oxford*, (2) **17** (1966), 151—159.

[6] A. Horn and N. Kimura, The category of semilattices, *Algebra Univ.*, **1** (1971), 26—38

[7] N. R. Reilly, Free generators in free inverse semigroups (preprint; 21 pp.).

[8] H. E. Scheiblich, A characterization of a free elementary inverse semigroup, *Semigroup Forum*, **2** (1971), 76—79.

[9] H. E. Scheiblich, Free inverse semigroups, *Semigroup Forum*, **4** (1972), 351—359.

[10] Б. М. Шайн, К теории обобщенных групп, *Доклады АН СССР*, **153** (1963), 296—299.
[11] Б. М. Шайн, *Алгебры отношений*, докторская диссертация (Саратов, 1965).
[12] Б. М. Шайн, Симметрическая полугруппа преобразований покрывается своими инверсными подполугруппами, *Acta Math. Acad. Sci. Hungar.*, **22** (1971), 163—170.
[13] В. В. Вагнер, Теория обобщенных груд и обобщенных групп, *Матем. сб.*, **32** (1953), 545—632.
[14] В. В. Вагнер, Обобщенные груды и обобщенные группы с транзитивным отношением совместности, *Ученые записки Саратовского государственного университета, Выпуск механико-математический*, **70** (1961), 25—39.
[15] W. D. Munn, Free inverse semigroups, Preprint (8 pp.).
[16] L. O'Carroll, On free inverse semigroups, Preprint (13 pp.).
[17] G. B. Preston, Free inverse semigroups, Preprint (22 pp.).
[18] W. D. Munn, Further remarks on free inverse semigroups (private communication).

(Received October 3, 1972)

410 600 Saratov
ul. Vavilova 2, kv. 111
USSR